Using scalar invariants

Malcolm MacCallum

Queen Mary University of London

18 December 2025

What scalar invariants exist?

Christoffel proved in 1869 that scalars constructed from the metric and its derivatives must be functions of the metric itself and the Riemann tensor and its covariant derivatives.

The first examples to spring to mind are **scalar polynomial invariants**, SPIs, such as $R_{ab}R^{ab}$ or $C_{abcd}C^{cdef}C_{ef}{}^{ab}$. Often when people just say "invariant" they mean "s.p. invariant".

What scalar invariants exist?

Christoffel proved in 1869 that scalars constructed from the metric and its derivatives must be functions of the metric itself and the Riemann tensor and its covariant derivatives.

The first examples to spring to mind are **scalar polynomial invariants**, SPIs, such as $R_{ab}R^{ab}$ or $C_{abcd}C^{cdef}C_{ef}{}^{ab}$. Often when people just say "invariant" they mean "s.p. invariant".

Indeed that occurs in the title of one of Seno's papers:
"On the existence of horizons in spacetimes with vanishing curvature invariants" (JHEP, 2003(11), 046)

However, these are not the only, and in my view not the best, choice.

Cartan invariants

An important alternative is to use ideas due to Élie Cartan, as follows. As a side benefit, in my view an important one, Cartan invariants require less calculation, in general.

Cartan invariants

An important alternative is to use ideas due to Élie Cartan, as follows. As a side benefit, in my view an important one, Cartan invariants require less calculation, in general.

Let $F(\mathcal{M})$ denote a "suitable" frame bundle over a spacetime $\mathcal{M}$ (i.e. take the set of all frames at each point) and $\mathcal{R}^q$ be the set $\{R_{abcd},\, R_{abcd;f}\,,\, \ldots, R_{abcd;f_1 f_2 \cdots f_q}\}$ of the components of the Riemann tensor and its covariant derivatives up to the qth in a frame.

Cartan invariants

An important alternative is to use ideas due to Élie Cartan, as follows. As a side benefit, in my view an important one, Cartan invariants require less calculation, in general.

Let $F(\mathcal{M})$ denote a "suitable" frame bundle over a spacetime $\mathcal{M}$ (i.e. take the set of all frames at each point) and $\mathcal{R}^q$ be the set $\{R_{abcd},\ R_{abcd;f},\ \ldots,\ R_{abcd;f_1 f_2 \cdots f_q}\}$ of the components of the Riemann tensor and its covariant derivatives up to the qth in a frame.

Choose from $F(\mathcal{M})$ in a canonical and invariant way, e.g. use the principal null directions of the Weyl tensor. The resulting $\{R_{abcd},\ R_{abcd;f},\ \ldots,\ R_{abcd;f_1 f_2 \cdots f_q}\}$ are called the **Cartan invariants**. They are scalars, because the frames are invariantly defined, e.g. $R_{ijkl} a^i b^j c^k d^l$ where **a**, **b**, **c** and **d** are basis vectors of the chosen frame. This is like characterizing a symmetric bilinear map (matrix) by its eigenvalues.

Calculating invariants

The SPIs are at least quadratic in curvature tensor components and the contractions involved may hide very large numbers of individual terms. It is therefore very useful to adopt a method that reduces the number of terms.

Calculating invariants

The SPIs are at least quadratic in curvature tensor components and the contractions involved may hide very large numbers of individual terms. It is therefore very useful to adopt a method that reduces the number of terms.

Example: consider multiplying two univariate polynomials of degree 10. Each has 11 terms. The end result has degree 20, 21 terms.

Calculating invariants

The SPIs are at least quadratic in curvature tensor components and the contractions involved may hide very large numbers of individual terms. It is therefore very useful to adopt a method that reduces the number of terms.

Example: consider multiplying two univariate polynomials of degree 10. Each has 11 terms. The end result has degree 20, 21 terms.

A simple alternative is to use Cartan invariants instead, with the advantage they are linear in the curvature components. In principle this can be messy because of the nature of the formulae for solution of quartics, but in practice it is often manageable.

Are SPIs good enough to characterize spacetimes?

Until 2009, I would have said definitely not. Note that *pp* waves and flat space both have all scalar polynomial invariants, of all orders, equal to zero. In fact all vacuum type N and III metrics with $\rho = 0$ have this property (Pravda and Bicak 2001 and earlier). There are also metrics which have equal non-zero s.p. invariants (e.g. Siklos 1996, Pravda 1999, Hervik 2004).

These ambiguities are associated with the indefiniteness of the metric and the non-compactness of the Lorentz group (Schmidt 1998).

Are s.p. invariants good enough? Yes, usually!

Coley et al (CQG 26, 025013 (2009)) gave an argument that all spacetimes are completely characterized by their SPIs, **except for spacetimes in the Kundt class**. Their proof seemed to me to have a gap and I have not yet been able to see if it is filled by their later work. Nevertheless the result seems correct.

Are s.p. invariants good enough? Yes, usually!

Coley et al (CQG 26, 025013 (2009)) gave an argument that all spacetimes are completely characterized by their SPIs, **except for spacetimes in the Kundt class**. Their proof seemed to me to have a gap and I have not yet been able to see if it is filled by their later work. Nevertheless the result seems correct.

In particular they have followed up by a substantial number of papers, which I will not have time to discuss today. These consider not only 4D spacetimes, but higher dimensional spaces and various signatures.

Classification of singularities

One can characterize singularities by the behaviour of the Riemann tensor components along a curve of finite length ending at the singularity in a parallelly-propagated frame, or in other frames.

- **Quasiregular singularities**. In these cases the Riemann tensor components remain bounded along the curve.

Classification of singularities

One can characterize singularities by the behaviour of the Riemann tensor components along a curve of finite length ending at the singularity in a parallelly-propagated frame, or in other frames.

- **Quasiregular singularities**. In these cases the Riemann tensor components remain bounded along the curve.
- **Intermediate curvature singularities** or "whimper singularities" (Ellis and King 1974) For these the components are bounded in some frame(s) but unbounded in a parallely-propagated frame.

Classification of singularities

One can characterize singularities by the behaviour of the Riemann tensor components along a curve of finite length ending at the singularity in a parallelly-propagated frame, or in other frames.

- **Quasiregular singularities**. In these cases the Riemann tensor components remain bounded along the curve.
- **Intermediate curvature singularities** or “whimper singularities” (Ellis and King 1974) For these the components are bounded in some frame(s) but unbounded in a parallely-propagated frame.
- **Essential Curvature singularities**. Here the Riemann tensor components are unbounded in all tetrads.

Classification of singularities

One can characterize singularities by the behaviour of the Riemann tensor components along a curve of finite length ending at the singularity in a parallelly-propagated frame, or in other frames.

- **Quasiregular singularities**. In these cases the Riemann tensor components remain bounded along the curve.
- **Intermediate curvature singularities** or “whimper singularities” (Ellis and King 1974) For these the components are bounded in some frame(s) but unbounded in a parallely-propagated frame.
- **Essential Curvature singularities**. Here the Riemann tensor components are unbounded in all tetrads.

Classification of singularities

One can characterize singularities by the behaviour of the Riemann tensor components along a curve of finite length ending at the singularity in a parallelly-propagated frame, or in other frames.

- **Quasiregular singularities**. In these cases the Riemann tensor components remain bounded along the curve.
- **Intermediate curvature singularities** or "whimper singularities" (Ellis and King 1974) For these the components are bounded in some frame(s) but unbounded in a parallely-propagated frame.
- **Essential Curvature singularities**. Here the Riemann tensor components are unbounded in all tetrads.

Quasiregular singularities are locally extendible in the sense that every curve ending at one has a neighbourhood isometric to a neighbourhood in a regular complete spacetime. Such singularities are common on the axes in axisymmetric solutions, where they may be described as struts between particles or bodies.

SPIs and singularities

I used my own choice of names yesterday. In the literature, the 'essential' case are called SP curvature singularities, because an SPI is unbounded along the incomplete curve as it reaches its end. (To avoid the long circumlocution, I'll say 'at the singularity', although we don't know how to locate the singularities themselves.) The 'intermediate' class are then called non-SP curvature singularities.

SPIs and singularities

I used my own choice of names yesterday. In the literature, the 'essential' case are called SP curvature singularities, because an SPI is unbounded along the incomplete curve as it reaches its end. (To avoid the long circumlocution, I'll say 'at the singularity', although we don't know how to locate the singularities themselves.) The 'intermediate' class are then called non-SP curvature singularities.

Note that no SPI blows up in the non-SP case. (People sometimes assume SPIs must be unbounded at singularities.)

SPIs and singularities

I used my own choice of names yesterday. In the literature, the 'essential' case are called SP curvature singularities, because an SPI is unbounded along the incomplete curve as it reaches its end. (To avoid the long circumlocution, I'll say 'at the singularity', although we don't know how to locate the singularities themselves.) The 'intermediate' class are then called non-SP curvature singularities.

Note that no SPI blows up in the non-SP case. (People sometimes assume SPIs must be unbounded at singularities.)

In the non-SP curve the existence of unboundedness in some frame and boundedness in parallally-propagated frames must be because the Lorentz transformation between a frame with bounded components and the parallelly-propagated frame has components that are unbounded along the curve. The frame with bounded components could be a frame defining Cartan invariants.

Applying the criteria

The nasty part is that we cannot usually calculate the geodesics explicitly. So what is usually done is calculate invariants in some frame, not at all shown to be a frame parallely propagated along an incomplete geodesic, and see where they become unbounded, and then argue that causal curves going through that region will see unbounded Riemann components.

We can for example calculate some set of Cartan invariants.

Cartan invariants and singularities

Since all SPIs can be written as polynomials in Cartan invariants, at least one Cartan invariant must be unbounded at an SP singularity in any frame. So we anyway do not have to calculate the parallelly-propagated frame to check.

Cartan invariants and singularities

Since all SPIs can be written as polynomials in Cartan invariants, at least one Cartan invariant must be unbounded at an SP singularity in any frame. So we anyway do not have to calculate the parallelly-propagated frame to check.

If all Cartan invariants are bounded, so are all SPIs and we can only have a quasi-regular singularity.

Cartan invariants and singularities

Since all SPIs can be written as polynomials in Cartan invariants, at least one Cartan invariant must be unbounded at an SP singularity in any frame. So we anyway do not have to calculate the parallelly-propagated frame to check.

If all Cartan invariants are bounded, so are all SPIs and we can only have a quasi-regular singularity.

The awkward case is when we find unbounded Cartan invariants. Then we do have a curvature singularity but it's not trivial to tell if it's SP or non-SP. We can of course in principle calculate the SPIs.

Cartan invariants and singularities

Since all SPIs can be written as polynomials in Cartan invariants, at least one Cartan invariant must be unbounded at an SP singularity in any frame. So we anyway do not have to calculate the parallelly-propagated frame to check.

If all Cartan invariants are bounded, so are all SPIs and we can only have a quasi-regular singularity.

The awkward case is when we find unbounded Cartan invariants. Then we do have a curvature singularity but it's not trivial to tell if it's SP or non-SP. We can of course in principle calculate the SPIs.

To do better we wuld need to know more about how the SPIs fix the Cartan invariants. Then assuming the Coley et al result is correct, we might be able to just check if the solution is the Kundt class, and if not we could deduce that we have an SP singularity. If we are in the Kundt class it's straightforward.

Singularities

Unboundedness of an SPI implies a singularity. However:

An “infinite” Riemann tensor does not imply a singularity. Geodesics can be continued across a delta function curvature modelling a thin shell or an impulsive gravitational wave.

Singularities

Unboundedness of an SPI implies a singularity. However:

An "infinite" Riemann tensor does not imply a singularity. Geodesics can be continued across a delta function curvature modelling a thin shell or an impulsive gravitational wave.

An invariant blowing up does not imply a singularity. For example (MacCallum 2006) the invariant $1/R_{abcd;e}R^{abcd;e}$ blows up at the Schwarzschild horizon, but the horizon is not singular.

Singularities

Unboundedness of an SPI implies a singularity. However:

An "infinite" Riemann tensor does not imply a singularity. Geodesics can be continued across a delta function curvature modelling a thin shell or an impulsive gravitational wave.

An invariant blowing up does not imply a singularity. For example (MacCallum 2006) the invariant $1/R_{abcd;e}R^{abcd;e}$ blows up at the Schwarzschild horizon, but the horizon is not singular.

At non-SP singularities Riemann tensor components are OK. But an invariant involving first derivatives is unbounded (Siklos 1978). An example was studied by Podolsky and Belan (2004).

Question: when does blow up of higher derivative invariants imply a singularity?

A conjecture on singularities

These arguments lead to a conjecture as follows.

Geodesic continuation needs a C^{2-} metric. In invariantly-defined frames the connection coefficients are typically expressible as ratios of first derivative Cartan invariants to zeroth derivative ones. We know that there are "intermediate" or "whimper" singularities where SPIs of the Riemann tensor do not blow up, while SPIs of the first derivatives of the Riemann tensor do. Hence:

A conjecture on singularities

These arguments lead to a conjecture as follows.

Geodesic continuation needs a C^{2-} metric. In invariantly-defined frames the connection coefficients are typically expressible as ratios of first derivative Cartan invariants to zeroth derivative ones. We know that there are “intermediate” or “whimper” singularities where SPIs of the Riemann tensor do not blow up, while SPIs of the first derivatives of the Riemann tensor do. Hence:

Conjecture: Spacetime singularities are either locally extendible or at least one Cartan invariant in $\mathcal{R}^1$ has an infinite limit along any curve approaching the singularity.

“Directional singularities”

Another application of invariants is to “directional” singularities, where, a singular point apparently has directionally dependent limits. Szekeres and Scott showed that the directional singularity of the Curzon metric hid more extended regions at whose boundary the original coordinates broke down. My student Taylor showed (2005) that such cases could be appropriately “unravelled” by using level surfaces of Cartan invariants to define new coordinates. Lake (2003) used the Weyl tensor SPIs to show that the Kerr singularity was not directional.

“Directional singularities”

Another application of invariants is to “directional” singularities, where, a singular point apparently has directionally dependent limits. Szekeres and Scott showed that the directional singularity of the Curzon metric hid more extended regions at whose boundary the original coordinates broke down. My student Taylor showed (2005) that such cases could be appropriately “unravelled” by using level surfaces of Cartan invariants to define new coordinates. Lake (2003) used the Weyl tensor SPIs to show that the Kerr singularity was not directional.

“Kinematic singularities”

Here the world lines of a perfect fluid with w sufficiently large cannot be extended beyond a finite time although density and pressure remain finite. These were found in spatially-homogeneous cosmologies, where the fluid accelerates so that asymptotically it has the speed of light.

"Kinematic singularities"

Here the world lines of a perfect fluid with w sufficiently large cannot be extended beyond a finite time although density and pressure remain finite. These were found in spatially-homogeneous cosmologies, where the fluid accelerates so that asymptotically it has the speed of light.

Here the geodesics orthogonal to the surfaces of homogeneity are complete and the Ricci tensor vanishes as one follows the fluid congruence: either the Weyl tensor or the kinematic variables of the congruence may become unbounded. In the latter case, if $\mu + p$ is not 0 and has a non-zero limit, there can be unbounded scalars in the covariant derivatives of the curvature, though not in the curvature itself.

“Kinematic singularities”

Here the world lines of a perfect fluid with w sufficiently large cannot be extended beyond a finite time although density and pressure remain finite. These were found in spatially-homogeneous cosmologies, where the fluid accelerates so that asymptotically it has the speed of light.

Here the geodesics orthogonal to the surfaces of homogeneity are complete and the Ricci tensor vanishes as one follows the fluid congruence: either the Weyl tensor or the kinematic variables of the congruence may become unbounded. In the latter case, if $\mu + p$ is not 0 and has a non-zero limit, there can be unbounded scalars in the covariant derivatives of the curvature, though not in the curvature itself.

In the example I studied with Coley and others, given an integer p, the Cartan (and hence s.p.) scalars can be finite up to the p-th derivative, but not the $(p + 1)$-th.

Newman-Penrose method

The cosmic censorship hypothesis (proved in some circumstances) says that singlarities cannot be 'naked', i.e. must be hidden behind horizons. So I'll now talk about horizons. But to do so, I need a very brief introduction to the Newman-Penrose technique.

Newman-Penrose method

The cosmic censorship hypothesis (proved in some circumstances) says that singlarities cannot be 'naked', i.e. must be hidden behind horizons. So I'll now talk about horizons. But to do so, I need a very brief introduction to the Newman-Penrose technique.

I'll leave aside the spinorial version and think of the null tetrad version. A null tetrad uses two real null vectors and two complex conjugate vectors, $\{l^a, n^a, m^a \bar{m}^a\}$ such that $l^a l_a = n^a n_a = m^a m_a = \bar{m}^a \bar{m}_a = 0$ and $-l_a n^a = 1 = m^a \bar{m}_a$. (The spacetime stays real.) In this frame each connection component is parametrized by a Greek letter. It turns out the Weyl tensor is represented by 5 complex quantities Ψ_0 to Ψ_4. If we make a so-called null rotation:

$$\hat{l} = l, \quad \hat{n} = n + zm + \bar{z}\bar{m} - \frac{1}{2} z\bar{z} l, \quad \hat{m} = m - zl$$

then the "principal null directions" are given by the roots of

$$\Sigma(\frac{4}{i})\Psi_i z^i = 0$$

Black hole and other horizons

Karlhede et al (1982) first noted that $R_{abcd;e}R^{abcd;e} = 0$ at the Schwarzschild horizon (so a prudent space traveller might monitor that). Skea in his thesis noted that this is not true for other horizons (a point rediscussed by Saa (2007) for higher-dimensional static cases: he also found points where $R_{abcd;e}R^{abcd;e} = 0$ which are not horizons). Lake (2003) continued the work on Kerr by considering first derivative invariants, and found their vanishing characterized the horizons.

Black hole and other horizons

Karlhede et al (1982) first noted that $R_{abcd;e}R^{abcd;e} = 0$ at the Schwarzschild horizon (so a prudent space traveller might monitor that). Skea in his thesis noted that this is not true for other horizons (a point rediscussed by Saa (2007) for higher-dimensional static cases: he also found points where $R_{abcd;e}R^{abcd;e} = 0$ which are not horizons). Lake (2003) continued the work on Kerr by considering first derivative invariants, and found their vanishing characterized the horizons.

Moffat and Toth (2014) considered the relation of the "Karlhede invariant" (i.e. $R_{abcd;e}R^{abcd;e}$) to discussions of a "firewall" at the horizon.

In 2006 I suggested a test for horizons in Petrov type D spacetimes using the NP variable ρ in the canonical frame. This has a well-understood geometric interpretation: since the horizon is a marginally trapped surface, the outgoing null vector ℓ must be surface-forming and non-expanding implying that $\rho = 0$ there. For these metrics ρ is a ratio of Cartan invariants.

Black hole and other horizons (cont)

More recently Lake and Abdelqader (2015) have given an invariant characterization of the Kerr horizon, and, prompted by that, Page and Shoom (2015) have given a more general one for stationary black holes.

Black hole and other horizons (cont)

More recently Lake and Abdelqader (2015) have given an invariant characterization of the Kerr horizon, and, prompted by that, Page and Shoom (2015) have given a more general one for stationary black holes. In a paper last year, with Coley, McNutt and some students, we showed this can be more neatly rewritten in Cartan invariants, and found simple invariants giving the ergosphere also for the Kerr case. I want now to briefly describe those results.

In the published version we considered the following examples in 4 and 5 dimensions:
the Kerr-Newman-NUT-(Anti)-de Sitter metric in 4D
the Reissner-Nordström-(Anti)-de Sitter metric in 5D
the Kerr-NUT-(Anti)-de Sitter metric in 5D. In the arXiv version we also considered:
the Tangherlini metric (analogue of Schwarzschild in 5D)
the simply rotating Myers-Perry metric (an analogue of Kerr in 5D)
and wrote up separately some subcases of the other metrics.

The SPIs used

The SPIs used by the Abdelqader and Lake and Page and Shoom papers can be denoted by $I_1, \ldots, I_7$, and are:

$$I_1 = C^{abcd} C_{abcd}, \quad I_2 = C^{*abcd} C_{abcd}, \quad I_3 = C^{abcd;e} C_{abcd;e}, \tag{1}$$
$$I_4 = C^{*abcd;e} C_{abcd;e}, \quad I_5 = (I_1)_{;a}(I_1)^{;a}, \quad I_6 = (I_2)_{;a}(I_2)^{;a}, \quad I_7 = (I_1)_{;a}(I_2)^{;a},$$

where C_{abcd} is the Weyl tensor and C^*_{abcd} is its dual, and a semicolon denotes a covariant derivative.

Note that in a vacuum I_3 is the Karlhede invariant. From the invariants above, Abdelqader and Lake then defined the dimensionless invariants

$$Q_1 = \frac{(I_1^2 - I_2^2)(I_5 - I_6) + 4 I_1 I_2 I_7}{3\sqrt{3}(I_1^2 + I_2^2)^{\frac{9}{4}}}, \tag{2}$$
$$Q_2 = \frac{I_5 I_6 - I_7^2}{27(I_1^2 + I_2^2)^{\frac{5}{2}}}, \quad Q_3 = \frac{I_5 + I_6}{6\sqrt{3}(I_1^2 + I_2^2)^{\frac{5}{4}}}$$

where I_1 to I_7 are given by (1).

The 4D main example

The 4D Kerr-Newman-NUT-(Anti)-de Sitter metric is given by

$$\begin{aligned} ds^2 &= -\frac{Q}{R^2}\left[dt - \left(a\sin^2\theta + 4l\sin^2\frac{\theta}{2}\right)d\phi\right]^2 + \frac{R^2}{Q}dr^2 \\ &\quad + \frac{P}{R^2}\left[adt - \left(r^2 + (a+l)^2\right)d\phi\right]^2 + \frac{R^2}{P}\sin^2\theta d\theta^2 \qquad (3) \end{aligned}$$

where $R \equiv R(r,\theta)$, $P \equiv P(\theta)$ and $Q \equiv Q(r)$ are functions of $\cos\theta$ and r, containing the parameters $m, e, g, a, l,$ and Λ which are, respectively, mass, the electric and magnetic charges, a rotation parameter, a NUT parameter in a de Sitter or anti-de Sitter background, and the cosmological constant:

$$\begin{aligned} R^2 &= r^2 + (l + a\cos\theta)^2 && (4) \\ P &= \sin^2\theta(1 + (3l + a\cos\theta)(l + a\cos\theta)\Lambda/3), && (5) \\ Q &= (a^2 - l^2 + e^2 + g^2) - 2mr + r^2 - \Lambda[(3l^2 + a^2)r^2 + r^4]/3. && (6) \end{aligned}$$

Cartan approach

We would like to have an extended Cartan invariant that detects the event horizon. Looking at $\nabla\Psi_{20'}$ we find that

$$\rho = \mu = -\frac{1}{\sqrt{2}} \frac{\sqrt{Q}[r - i(a\cos\theta + l)]}{R^3}. \tag{7}$$

Computing the roots of $Q(r)$ for arbitrary a, l, m, e, g and Λ is not a pleasant task. However, for this extended Cartan invariant we do not need to compute them, as it is clear that the zeros of ρ are exactly the zeros of $Q(r)$.

The ergosurface can be detected by combining ρ with another Cartan invariant,

$$\tau = \pi = \frac{1}{\sqrt{2}} \frac{a\sqrt{P}[r - i(a\cos\theta + l)]}{R^3}, \tag{8}$$

to produce the following extended Cartan invariant:

$$\rho^2 - \tau^2 = \frac{(Q - a^2 P)[r - i(a\cos\theta + l)]^2}{2R^6}. \tag{9}$$

Relation to SPIs

We can now easily compute the Q_i which are

$$Q_1 = \frac{2\mathcal{R}[(\bar{\Psi}_2^2(\nabla\Psi_2.\nabla\Psi_2)]}{9(\Psi_2\bar{\Psi}_2)^{5/2}}, \quad Q_2 = \frac{-2||\nabla\bar{\Psi}_2 \wedge \nabla\Psi_2||^2}{18^2(\Psi_2\bar{\Psi}_2)^3}, \qquad (10)$$

$$Q_3 = \frac{\nabla\Psi_2.\nabla\bar{\Psi}_2}{18(\Psi_2\bar{\Psi}_2)^{3/2}},$$

where $\mathcal{R}$ denotes the real part.

Q_2 vanishes at the horizon, on the axis and at points where $1 + (3l + a\cos\theta)(l + a\cos\theta)\Lambda/3 = 0$. (However, from (7), ρ vanishes only on the horizon and at the origin even when there is a Maxwell field.)

Since the numerator of Q_1 equals (9), Q_1 will detect the ergosurface for the Kerr solution.

Main References

Abdelqader M, Lake K (2015) Invariant characterization of the Kerr spacetime: Locating the horizon and measuring the mass and spin of rotating black holes using curvature invariants. Phys Rev D 91:084017, arXiv:1412.8757

Brooks D, Chavy-Waddy PC, Coley AA, Forget A, Gregoris D, MacCallum MAH, McNutt DD (2018) Cartan invariants as event horizon detectors. Gen Relativ Gravit 50:37 arXiv:1709.03362

MacCallum MAH (2006) On singularities, horizons, invariants, and the results of Antoci, Liebscher and Mihich (GRG 38, 15 (2006) and earlier). Gen Rel Grav 38:1887–1899, arXiv:gr-qc/0608033

Page DN, Shoom AA (2015) Local invariants vanishing on stationary horizons: A diagnostic for locating black holes. Phys Rev Lett 114:141102, arXiv:1510.03510

An example

In [?] the author derives a Kerr-Schild solution based on a Hopf fibering in the form

$$ds^2 = -2dudv + dx^2 + dy^2 + Hk_\mu k_\nu dx^\mu dx^\nu \tag{11}$$

in coordinates (u, v, x, y), with constants b and N and

$$label Haradak_\mu = \left(1, \frac{x^2+y^2}{2(u^2+b^2)}, -\frac{ux+by}{u^2+b^2}, -\frac{uy-bx}{u^2+b^2}\right), H = \frac{Nu}{u^2+b^2}. \tag{12}$$

Harada also shows it is of Petrov type D. Since all Petrov type D vacuum solutions are known [?, ?] this must be a known solution, not new as claimed. The issue is to identify which of the known solutions it is.

In the exact solutions book, the result of [?] is cited. It shows that Kerr-Schild spacetimes with a geodesic, diverging, and shearfree k_μ can be written (swapping the names u and v) as

$$\mathrm{d}s^2 = 2(\mathrm{d}\zeta\,\mathrm{d}\bar{\zeta} - \mathrm{d}u\,\mathrm{d}v) - 2S(\overline{Y}\mathrm{d}\zeta + Y\mathrm{d}\bar{\zeta} + Y\overline{Y}\mathrm{d}u + \mathrm{d}v)^2 \tag{13}$$

with $\zeta = x + iy$. This is of the same form as (**??**) if $Y = -\zeta/(u + ib)$, and

Identifying the Harada solution

The solution (**??**) is easily checked by CLASSI to be a vacuum Petrov type D solution but the classification routines say it has 4 Killing vectors, not just the two found by Harada. Using CLASSI's facilities

The known Petrov D solutions were the first solutions to be clssified using the technique just discussed: see [**?**, **?**]. The files for all of then are freely available.

Temporary page!

LaTeX was unable to guess the total number of pages correctly.
was some unprocessed data that should have been added to the
this extra page has been added to receive it.

If you rerun the document (without altering it) this surplus page
away, because LaTeX now knows how many pages to expect for t
document.