Exercise 4.2

1. **Factorize each of the following expressions:**

(i) $4x^4 - 81y^4$ (ii) $a^4 - 64b^4$

(iii) $x^4 + 4x^2 + 4$ (iv) $x^4 - 14x^2 + 1$

(v) $x^4 - 30x^2y^2 + 9y^4$ (vi) $x^4 + 11x^2y^2 + y^4$

Solution: Let's factorize each expression:

(i) $\mathbf{4x^4 - 81y^4}$

$$4x^4 - 81y^4 = (2x^2)^2 - (9y^2)^2$$
$$= (2x^2 + 9y^2)(2x^2 - 9y^2)$$

Thus, the factorized form is:

$$(2x^2 + 9y^2)(2x^2 - 9y^2)$$

(ii) $\mathbf{a^4 - 64b^4}$

Step 1: Recognize the expression as the sum of squares

$$a^4 - 64b^4 = (a^2)^2 - (8b^2)^2$$

Step 2: Apply the sum of squares identity

$$a^4 - 64b^4 = (a^2 + 8b^2)(a^2 - 8b^2)$$

Final Answer:

$$a^4 - 64b^4 = (a^2 + 8b^2)(a^2 - 8b^2)$$

(iii) $\mathbf{x^4 + 4x^2 + 4}$

Let $z = x^2$. The expression becomes:

$$z^2 + 4z + 4$$

Now factor the quadratic:

$$z^2 + 4z + 4 = (z + 2)^2$$

Substitute back $z = x^2$:

$$(x^2 + 2)^2$$

Thus, the factorized form is:

$$(x^2 + 2)^2$$

(iv) $\mathbf{x^4 - 14x^2 + 1}$

Let $z = x^2$. The expression becomes:

$$z^2 - 14z + 1$$

Factor the quadratic:

$$z^2 - 14z + 1 = (z - 7)^2 - 48$$

We can further write this as a difference of squares:

$$(z-7)^2-(4\sqrt{3})^2=(z-7+4\sqrt{3})(z-7-4\sqrt{3})$$

Substituting back $z=x^2$, we get:

$$(x^2-7+4\sqrt{3})(x^2-7-4\sqrt{3})$$

This is the factorized form.

(v) $x^4-30x^2y^2+9y^4$

Step 1: Recognize the expression as a quadratic trinomial

$$x^4-30x^2y^2+9y^4=(x^2)^2-30(x^2)(y^2)+(3y^2)^2$$

Step 2: Factorize the quadratic expression

$$x^4-30x^2y^2+9y^4=(x^2-27y^2)(x^2-3y^2)$$

Final Answer:

$$x^4-30x^2y^2+9y^4=(x^2-27y^2)(x^2-3y^2)$$

(vi) $x^4+11x^2y^2+y^4$

Step 1: Recognize the expression as a quadratic trinomial

$$x^4+11x^2y^2+y^4=(x^2)^2+11(x^2)(y^2)+(y^2)^2$$

Step 2: Factorize the quadratic expression

$$x^4+11x^2y^2+y^4=(x^2+9y^2)(x^2+2y^2)$$

Final Answer:

$$x^4+11x^2y^2+y^4=(x^2+9y^2)(x^2+2y^2)$$

2. Factorize each of the following expressions:

(i) $(x+1)(x+2)(x+3)(x+4)+1$

(ii) $(x+2)(x-7)(x-4)(x-1)+17$

(iii) $(2x^2+7x+3)(2x^2+7x+5)+1$

(iv) $(3x^2+5x+3)(3x^2+5x+5)-3$

(v) $(x+1)(x+2)(x+3)(x+6)-3x^2$

(vi) $(x+1)(x-1)(x+2)(x-2)+5x^2$

(i) $(x+1)(x+2)(x+3)(x+4)+1$

Sol: Rearranging the term:

$$=(x+1)(x+4)(x+2)(x+3)+1$$
$$=(x^2+5x+4)(x^2+5x+6)+1$$

Put $x^2+5x=y$

$$=(y+4)(y+6)+1$$
$$=y^2+10y+24+1$$

$= y^2 + 10y + 25$
$= (y + 5)^2$
$= (x^2 + 5x + 5)^2$

(ii) $(x + 2)(x - 7)(x - 4)(x - 1) + 17$

$= (x^2 - 5x - 14)(x^2 - 5x + 4) - 17$

Put $x^2 - 5x = y$

$= (y - 14)(y + 4) + 17$
$= y^2 - 10y - 56 + 17$
$= y^2 - 10y - 39$
$= y^2 - 13y + 3y - 39$
$= y^2 - 13y + 3y - 39$
$= y(y - 13) + 3(y - 13)$
$= (y + 3)(y - 3)$
$= (x^2 - 5x + 3)(x^2 - 5x - 13)$

(iii) $(2x^2 + 7x + 3)(2x^2 + 7x + 5) + 1$

Put $2x^2 + 7x = y$

$= (y + 3)(y + 5) + 1$
$= y^2 + 8x + 15 + 1$
$= y^2 + 8x + 16$
$= (y + 4)^2$
$= (2x^2 + 7x + 4)^2$

(iv) $(3x^2 + 5x + 3)(3x^2 + 5x + 5) - 3$

Put $3x^2 + 5x = y$

$= (y + 3)(y + 5) - 3$
$= y^2 + 8x + 15 - 3$
$= y^2 + 8x + 12$
$= y^2 + 6y + 2y + 12$
$= y(y + 6) + 2(y + 6)$
$= (y + 2)(y + 6)$
$= (3x^2 + 5x + 2)(3x^2 + 5x + 6)$

(v) $(x + 1)(x + 2)(x + 3)(x + 6) - 3x^2$

$= (x^2 + 4x + 6)(x^2 + 8x + 6)$

$$= (x + 1)(x + 6)(x + 2)(x + 3) - 3x^2$$
$$= (x^2 + 7x + 6)(x^2 + 5x + 6) - 3x^2$$

Put $x^2 + 6 = y$

$$= (y + 7x)(y + 5x) - 3x^2$$
$$= y^2 + 12xy + 35x^2 - 3x^2$$
$$= y^2 + 12xy + 32x^2$$
$$= y^2 + 8xy + 4xy + 32x^2$$
$$= y(4 + 8x) + 4x(y + 8x)$$
$$= (y + 4x)(y + 8x)$$
$$= (x^2 + 4x + 6)(x^2 + 8x + 6)$$

(vi) $\boldsymbol{(x + 1)(x - 1)(x + 2)(x - 2) + 5x^2}$

Re-arranging the terms

$$\text{Exp} = (x + 1)(x + 2)(x - 1)(x + 2) + 5x^2$$
$$= (x^2 + 2x + x + 2)(x^2 - 2x - x + 2) + 5x^2$$
$$= (x^2 + 3x + 2)(x^2 - 3x + 2) + 5x^2$$

Put $y = (x^2 + 2)$

$$\text{Exp} = (y + 3x)(y - 3x) + 5x^2$$
$$= y^2 - 9x^2 + 5x^2$$
$$= y^2 - 4x^2$$
$$= (y + 2x)(y - 2x)$$

Replace the value of y

$$= (x^2 + 2 + 2x)(x^2 + 2x - 2x)$$
$$= (x^2 + 2x + 2)(x^2 - 2x + 2)$$

3. Factorize:

(i) $\boldsymbol{8x^3 + 12x^2 + 6x + 1}$

(ii) $\boldsymbol{27a^3 + 108a^2b + 144ab^2 + 64b^3}$

(iii) $\boldsymbol{x^3 + 18x^2y + 108xy^2 + 216y^3}$

(iv) $\boldsymbol{8x^3 - 125y^3 + 150xy^2 - 60x^2y}$

Solution: Let's factorize the given expressions step by step.

(i) $\boldsymbol{8x^3 + 12x^2 + 6x + 1}$

We can group the terms in pairs to look for common factors:

$$8x^3 + 12x^2 + 6x + 1 = (8x^3 + 12x^2) + (6x + 1)$$

Factor out the common terms in each group:

$$= 4x^2(2x + 3) + 1(6x + 1)$$

Now, observe that we cannot factor this expression any further by grouping. So, the factorized form is:

$$(4x^2 + 1)(2x + 3)$$

(ii) $\mathbf{27a^3 + 108a^2b + 144ab^2 + 64b^3}$

First, notice that all the terms have a common factor of 1, so let's group the terms:

$$27a^3 + 108a^2b + 144ab^2 + 64b^3$$
$$= (27a^3 + 108a^2b) + (144ab^2 + 64b^3)$$

Factor out the greatest common factors in each group:

$$= 9a^2(3a + 12b) + 16b^2(9a + 4b)$$

Next, we can factor further:

$$= (3a + 4b)(9a^2 + 36ab + 16b^2)$$

Thus, the factorized form is:

$$(3a + 4b)(9a^2 + 36ab + 16b^2)$$

(iii) $\mathbf{x^3 + 18x^2y + 108xy^2 + 216y^3}$

Observe the coefficients of the terms, and factor out the greatest common factor (GCF) first:

$$x^3 + 18x^2y + 108xy^2 + 216y^3$$
$$= x^3 + 3 \cdot 6x^2y + 3 \cdot 36xy^2 + 3 \cdot 72y^3$$

Now, factor out $x + 3y$ from the terms:

$$= (x + 6y)^3$$

Thus, the factorized form is:

$$(x + 6y)^3$$

(iv) $\mathbf{8x^3 - 125y^3 + 150xy^2 - 60x^2y}$

$$= (2x)^3 - 3 \times 4x^2 \times 5y - 3.2x \times 25y^2 - (5y)^3$$
$$= (2x - 5y)^3$$

4. **Factorize:**

(i) $125a^3 - 1$ (ii) $64x^3 + 125$

(iii) $x^6 - 27$ (iv) $1000a^3 + 1$

(v) $343x^3 + 216$ (vi) $27 - 512y^3$

Solution: Let's factorize each expression step by step:

(i) $125a^3 - 1$

This is a difference of cubes, since $125a^3 = (5a)^3$ and $1 = 1^3$.

The difference of cubes formula is:

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Using this formula:

$$125a^3 - 1 = (5a - 1)((5a)^2 + (5a)(1) + 1^2)$$

Simplifying:

$$= (5a - 1)(25a^2 + 5a + 1)$$

Thus, the factorized form is:

$$(5a - 1)(25a^2 + 5a + 1)$$

(ii) $64x^3 + 125$

This is a sum of cubes, since $64x^3 = (4x)^3$ and $125 = 5^3$. The sum of cubes formula is:

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

Using this formula:

$$64x^3 + 125 = (4x + 5)((4x)^2 - (4x)(5) + 5^2)$$

Simplifying:

$$= (4x + 5)(16x^2 - 20x + 25)$$

Thus, the factorized form is:

$$(4x + 5)(16x^2 - 20x + 25)$$

(iii) $x^6 - 27$

This is a difference of cubes, since $x^6 = (x^2)^3$ and $27 = 3^3$. The difference of cubes formula is:

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Using this formula:

$$x^6 - 27 = (x^2 - 3)((x^2)^2 + (x^2)(3) + 3^2)$$

Simplifying:

$$= (x^2 - 3)(x^4 + 3x^2 + 9)$$

Thus, the factorized form is:

$$(x^2 - 3)(x^4 + 3x^2 + 9)$$

(iv) $\mathbf{1000a^3 + 1}$

This is a sum of cubes, since $1000a^3 = (10a)^3$ and $1 = 1^3$. The sum of cubes formula is:

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

Using this formula:

$$1000a^3 + 1 = (10a + 1)((10a)^2 - (10a)(1) + 1^2)$$

Simplifying:

$$= (10a + 1)(100a^2 - 10a + 1)$$

Thus, the factorized form is:

$$(10a + 1)(100a^2 - 10a + 1)$$

(v) $\mathbf{343x^3 + 216}$

This is a sum of cubes, since $343x^3 = (7x)^3$ and $216 = 6^3$. The sum of cubes formula is:

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

Using this formula:

$$343x^3 + 216 = (7x + 6)((7x)^2 - (7x)(6) + 6^2)$$

Simplifying:

$$= (7x + 6)(49x^2 - 42x + 36)$$

Thus, the factorized form is:

$$(7x + 6)(49x^2 - 42x + 36)$$

(vi) $\mathbf{27 - 512y^3}$

This is a difference of cubes, since $27 = 3^3$ and $512y^3 = (8y)^3$. The difference of cubes formula is:

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Using this formula:

$$27 - 512y^3 = (3 - 8y)((3)^2 + (3)(8y) + (8y)^2)$$

Simplifying:

$$= (3 - 8y)(9 + 24y + 64y^2)$$

Thus, the factorized form is:

$$(3 - 8y)(9 + 24y + 64y^2)$$

4.3 Highest Common Factor (HCF) and Least Common Multiple (LCM) of Algebraic Expressions

4.3.1 Highest Common Factor (HCF)

The HCF of algebraic expressions refers to the greatest algebraic expression that divides two or more algebraic expressions without leaving a remainder.

We can find HCF of given expressions by the following two methods:

(a) By factorization (b) By division

(a) *HCF by Factorization Method*

Example 19: Find the HCF of $6x^2y, 9xy^2$

Solution:

$$6x^2y = 2 \times 3 \times x \times x \times y$$

$$9xy^2 = 3 \times 3 \times x \times y \times y$$

$\therefore$ HCF $= 3 \times x \times y$ (Product of common factors)

$$= 3xy$$

Example 20: Find the HCF by factorization method

$$x^2 - 27, x^2 + 6x - 27, x^2 - 9$$

Solution:

$$x^3 - 27 = x^3 - 3^3$$

$$= (x - 3)[(x)^2 + (3)(x) + (3)^2]$$

$$= (x - 3)(x^2 + 3x + 9)$$

$$x^2 + 6x - 27 = x^2 + 9x - 3x - 27$$

$$= x(x + 9) - 3(x + 9)$$

$$= (x + 9)(x - 3)$$

$$x^2 - 9 = x^2 - 3^2$$

$$= (x - 3)(x + 3)$$

Hence, HCF $= x - 3$

(b) HCF by Division Method

Example 21: Find HCF of $6x^3 - 17x^2 - 5x + 6$ and $6x^3 - 5x^2 - 3x + 2$ by using division method.

Solution:

$$\begin{array}{r|l} & \quad 1 \\ 6x^3 - 17x^2 - 15x + 6 & 6x^3 - 5x^2 - 3x + 2 \\ & \underline{{}_{-}6x^3 \mp 17x^2 \mp 5x \pm 6} \\ & \quad 12x^2 + 2x - 4 \end{array}$$

Here, $12x^2 + 2x - 4 = 2(6x^2 + x - 2)$

2 is not common in both the given polynomials, so we ignore it and consider only $6x^2 + x - 2$.

$$\begin{array}{r|l} & \quad x - 3 \\ 6x^2 + x - 2 & 6x^3 - 17x^2 - 5x + 6 \\ & \underline{{}_{-}6x^3 \pm x^2 \mp 2x} \\ & \quad -18x^2 - 3x + 6 \\ & \quad \underline{{}_{+}^{-}18x^2 \mp 3x \pm 6} \\ & \qquad 0 \end{array}$$

Hence, HCF $= 6x^2 + x - 2$

4.3.2 Least Common Multiple (LCM)

The LCM of two or more algebraic expressions is the smallest expression that is divisible by each of the given expressions.

To find the LCM by factorization, we use the formula.

LCM = Common factors × Non-common factors

Example 22: Find the LCM of $4x^2y$, $8x^3y^2$.

Solution: $4x^2y = 2 \times 2 \times x \times x \times y$

$8x^3y^2 = 2 \times 2 \times 2 \times x \times x \times x \times y \times y$

Common factors $= 2 \times 2 \times x \times x \times y = 4x^2y$

Non-common factors $= 2 \times x \times y = 2xy$

LCM = Common factors × Non-common factors

$= 4x^2y \times 2xy$

$= 8x^3y^2$

Example 23: Find the LCM of the polynomials

$x^2 - 3x + 2$, $x^2 - 1$ and $x^2 - 5x + 4$.

Solution: As $x^2 - 3x + 2 = x^2 - 2x - x + 2$

$$= x(x-2) - 1(x-2)$$

$$= (x-2)(x-1)$$

And $x^2 - 1 = (x-1)(x+1)$

$$x^2 - 5x + 4 = x^2 - 4x - x + 4$$

$$= x(x-4) - 1(x-4)$$

$$= (x-4)(x-1)$$

Common factors $= x - 1$

Non-common factors $= (x+1)(x-2)(x-4)$

LCM = Common factors × Non-common factors

$$= (x-1) \times (x+1)(x-2)(x-4)$$

$$= (x-1)(x+1)(x-2)(x-4)$$

4.3.3 Relationship between LCM and HCF

The relationship between LCM and HCF can be expressed as follows:

$$\text{LCM} \times \text{HCF} = p(x) \times q(x)$$

Where, $p(x) = 1^{st}$ polynomial

$q(x) = 2^{nd}$ polynomial

Example 24: LCM and HCF of two polynomials are $x^3 - 10x^2 + 11x + 70$ and $x - 7$. If one of the polynomials is $x^2 - 12x + 35$, find the other polynomial.

Solution: Given that: LCM $= x^3 - 10x^2 + 11x + 70$

HCF $= x - 7$

$p(x) = x^2 - 12x + 35$

As we know that: $q(x) = \dfrac{\text{LCM} \times \text{HCF}}{p(x)}$

$$= \frac{(x^3 - 10x^2 + 11x + 70)(x-7)}{x^2 - 12x + 35}$$

$$\begin{array}{r|l} & x + 2 \\ \hline x^2 - 12x + 35 & x^3 - 10x^2 + 11x + 70 \\ & \underline{\mp x^3 \pm 12x^2 \mp 35x} \\ & 2x^2 - 24x + 70 \\ & \underline{\mp 2x^2 \pm 24x \mp 70} \\ & 0 \end{array}$$