Exercise 6.3

1. If $\sin\theta = \frac{2}{3}$, then find the remaining trigonometric ratios when θ lies in first quadrant.

Solution: To solve for all the trigonometric ratios when $\sin\theta = \frac{2}{3}$ and θ is in the first quadrant, we proceed as follows:

Step 1: Use the Pythagorean identity to find $\cos\theta$

From the Pythagorean identity:

$$\sin^2\theta + \cos^2\theta = 1.$$

Substitute $\sin\theta = \frac{2}{3}$:

$$\left(\frac{2}{3}\right)^2 + \cos^2\theta = 1.$$

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$$\frac{4}{9} + \cos^2\theta = 1.$$

$$\cos^2\theta = 1 - \frac{4}{9} = \frac{5}{9}.$$

$$\cos\theta = \sqrt{\frac{5}{9}} = \frac{\sqrt{5}}{3}.$$

Since θ is in the first quadrant, $\cos\theta > 0$.

Step 2: Find tanθ

$\tan\theta = \frac{\sin\theta}{\cos\theta}$:

$$\tan\theta = \frac{\frac{2}{3}}{\frac{\sqrt{5}}{3}} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}.$$

Step 3: Find cotθ

$\cot\theta = \frac{1}{\tan\theta}$:

$$\cot\theta = \frac{1}{\frac{2\sqrt{5}}{5}} = \frac{5}{2\sqrt{5}} = \frac{\sqrt{5}}{2}.$$

Step 4: Find secθ

$\sec\theta = \frac{1}{\cos\theta}$:

$$\sec\theta = \frac{1}{\frac{\sqrt{5}}{3}} = \frac{3}{\sqrt{5}} = \frac{3\sqrt{5}}{5}.$$

Step 5: Find cscθ

$\csc\theta = \frac{1}{\sin\theta}$:

$$\csc\theta = \frac{1}{\frac{2}{3}} = \frac{3}{2}.$$

2. If $\cos\theta = \frac{3}{5}$, then find the remaining trigonometric ratios when θ lies in first quadrant.

To find all the trigonometric ratios when $\cos\theta = \frac{3}{5}$ and θ lies in the first quadrant, we proceed step by step:

Step 1: Use the Pythagorean identity to find sinθ

From the identity:

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$$\sin^2\theta + \cos^2\theta = 1.$$

Substitute $\cos\theta = \frac{3}{5}$:

$$\sin^2\theta + \left(\frac{3}{5}\right)^2 = 1.$$

$$\sin^2\theta + \frac{9}{25} = 1.$$

$$\sin^2\theta = 1 - \frac{9}{25} = \frac{25}{25} - \frac{9}{25} = \frac{16}{25}.$$

$$\sin\theta = \sqrt{\frac{16}{25}} = \frac{4}{5}.$$

Since θ is in the first quadrant, $\sin\theta > 0$.

Step 2: Find $\tan\theta$

$\tan\theta = \frac{\sin\theta}{\cos\theta}$:

$$\tan\theta = \frac{\frac{4}{5}}{\frac{3}{5}} = \frac{4}{3}.$$

Step 3: Find $\cot\theta$

$\cot\theta = \frac{1}{\tan\theta}$:

$$\cot\theta = \frac{1}{\frac{4}{3}} = \frac{3}{4}.$$

Step 4: Find $\sec\theta$

$\sec\theta = \frac{1}{\cos\theta}$:

$$\sec\theta = \frac{1}{\frac{3}{5}} = \frac{5}{3}.$$

Step 5: Find $\csc\theta$

$\csc\theta = \frac{1}{\sin\theta}$:

$$\csc\theta = \frac{1}{\frac{4}{5}} = \frac{5}{4}.$$

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Prove the following identities:

3. $(\sin\theta + \cos\theta)^2 = 1 + 2\sin\theta\cos\theta$

To prove the identity:

$$(\sin\theta + \cos\theta)^2 = 1 + 2\sin\theta\cos\theta$$

Step 1: Expand the left-hand side ($(\sin\theta + \cos\theta)^2$)

$$(\sin\theta + \cos\theta)^2 = (\sin\theta)^2 + 2\sin\theta\cos\theta + (\cos\theta)^2$$

Step 2: Use the Pythagorean identity

From the Pythagorean identity:

$$\sin^2\theta + \cos^2\theta = 1.$$

Substitute this into the expanded expression:

$$(\sin\theta + \cos\theta)^2 = 1 + 2\sin\theta\cos\theta.$$

Step 3: Compare both sides

The left-hand side $(\sin\theta + \cos\theta)^2$ simplifies to the right-hand side $1 + 2\sin\theta\cos\theta$.

Conclusion

$$(\sin\theta + \cos\theta)^2 = 1 + 2\sin\theta\cos\theta \quad \text{(Proved).}$$

4. $\dfrac{\cos\theta}{\sin\theta} = \dfrac{1}{\tan\theta}$

To prove the identity:

$$\frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta}$$

Step 1: Recall the definition of $\tan\theta$

By definition:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}.$$

Step 2: Take the reciprocal of $\tan\theta$

Taking the reciprocal of both sides of the equation for $\tan\theta$:

$$\frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}.$$

Step 3: Compare the left-hand side with the right-hand side

The left-hand side of the equation is $\frac{\cos\theta}{\sin\theta}$, which is exactly the right-hand side of the identity we are trying to prove.

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Conclusion

Thus, we have shown that:

$$\frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta}.$$

The identity is **proved**.

5. $\dfrac{\sin\theta}{\text{cosec }\theta} + \dfrac{\cos\theta}{\sec\theta} = 1$

To prove the identity:

$$\frac{\sin\theta}{\csc\theta} + \frac{\cos\theta}{\sec\theta} = 1$$

Step 1: Recall the definitions of $\csc\theta$ and $\sec\theta$

We know the following trigonometric identities:

$$\csc\theta = \frac{1}{\sin\theta}, \quad \sec\theta = \frac{1}{\cos\theta}.$$

Step 2: Substitute the definitions of $\csc\theta$ and $\sec\theta$

Substitute the definitions of $\csc\theta$ and $\sec\theta$ into the left-hand side of the equation:

$$\frac{\sin\theta}{\csc\theta} + \frac{\cos\theta}{\sec\theta} = \frac{\sin\theta}{\frac{1}{\sin\theta}} + \frac{\cos\theta}{\frac{1}{\cos\theta}}.$$

Step 3: Simplify the fractions

Simplify each term:

$$\frac{\sin\theta}{\frac{1}{\sin\theta}} = \sin^2\theta, \quad \frac{\cos\theta}{\frac{1}{\cos\theta}} = \cos^2\theta.$$

So the left-hand side becomes:

$$\sin^2\theta + \cos^2\theta.$$

Step 4: Use the Pythagorean identity

From the Pythagorean identity:

$$\sin^2\theta + \cos^2\theta = 1.$$

Thus, the left-hand side simplifies to: 1.

Conclusion

We have shown that:

$$\frac{\sin\theta}{\csc\theta} + \frac{\cos\theta}{\sec\theta} = 1.$$

The identity is **proved**.

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6. $\cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$

Solution: **Prove: $\cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$**

PROOF:

We will use the **double angle identity** for cosine:

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta.$$

From the above identity, we know that:

$$\cos^2\theta - \sin^2\theta = \cos(2\theta).$$

Now, recall the identity for $\cos(2\theta)$ in terms of $\cos^2\theta$:

$$\cos(2\theta) = 2\cos^2\theta - 1.$$

Thus, we can replace $\cos(2\theta)$ in the first equation:

$$\cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1.$$

Therefore, the identity is **proved.**

7. $\cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$

Solution: **Prove: $\cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$**

PROOF:

We already know that:

$$\cos^2\theta - \sin^2\theta = \cos(2\theta).$$

Also, recall the identity for $\cos(2\theta)$ in terms of $\sin^2\theta$:

$$\cos(2\theta) = 1 - 2\sin^2\theta.$$

Thus, we can rewrite the equation as:

$$\cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta.$$

Therefore, the identity is **proved.**

8. $(1 + \sin\theta)(1 - \sin\theta) = \frac{1}{\sec^2\theta}$

Solution: **Prove: $(1 + \sin\theta)(1 - \sin\theta) = \frac{1}{\sec^2\theta}$**

PROOF:

We will first simplify the left-hand side:

$$(1 + \sin\theta)(1 - \sin\theta) = 1^2 - \sin^2\theta = 1 - \sin^2\theta.$$

Now, recall the Pythagorean identity:

$$\sin^2\theta + \cos^2\theta = 1.$$

Thus:

$$1 - \sin^2\theta = \cos^2\theta.$$

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We can now express $\cos^2\theta$ in terms of $\sec^2\theta$:

$$\cos^2\theta = \frac{1}{\sec^2\theta}.$$

Therefore:

$$(1 + \sin\theta)(1 - \sin\theta) = \frac{1}{\sec^2\theta}.$$

The identity is **proved**.

9. **$(1 - \cos\theta)(1 + \cos\theta) = \frac{1}{\text{cosec}^2\theta}$**

Solution: **Prove: $(1 - \cos\theta)(1 + \cos\theta) = \frac{1}{\csc^2\theta}$**

PROOF:

We will first simplify the left-hand side:

$$(1 - \cos\theta)(1 + \cos\theta) = 1^2 - \cos^2\theta = 1 - \cos^2\theta.$$

Now, using the Pythagorean identity:

$$\sin^2\theta + \cos^2\theta = 1.$$

Thus:

$$1 - \cos^2\theta = \sin^2\theta.$$

Next, recall the identity $\csc^2\theta = \frac{1}{\sin^2\theta}$, so:

$$\sin^2\theta = \frac{1}{\csc^2\theta}.$$

Therefore:

$$(1 - \cos\theta)(1 + \cos\theta) = \frac{1}{\csc^2\theta}.$$

The identity is **proved**.

10. **$(\sec\theta - 1)(\sec\theta + 1) = \tan^2\theta$**

Solution: **Prove: $(\sec\theta - 1)(\sec\theta + 1) = \tan^2\theta$**

PROOF:

First, expand the left-hand side:

$$(\sec\theta - 1)(\sec\theta + 1) = \sec^2\theta - 1.$$

Now, recall the identity $\sec^2\theta = 1 + \tan^2\theta$. Substituting this into the equation:

$$\sec^2\theta - 1 = \tan^2\theta.$$

Thus:

$$(\sec\theta - 1)(\sec\theta + 1) = \tan^2\theta.$$

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The identity is **proved**.

11. $(\text{cosec}\,\theta - 1)(\text{cosec}\,\theta + 1) = \cot^2\theta$

Solution: **Prove:** $(\csc\theta - 1)(\csc\theta + 1) = \cot^2\theta$

PROOF:

First, expand the left-hand side:

$$(\csc\theta - 1)(\csc\theta + 1) = \csc^2\theta - 1.$$

Now, recall the identity $\csc^2\theta = 1 + \cot^2\theta$. Substituting this into the equation:

$$\csc^2\theta - 1 = \cot^2\theta.$$

Thus:

$$(\csc\theta - 1)(\csc\theta + 1) = \cot^2\theta.$$

The identity is **proved**.

12. $\dfrac{1 - \sin\theta}{\cos\theta} = \dfrac{\cos\theta}{1 + \sin\theta}$

Solution: Prove: $\dfrac{1-\sin\theta}{\cos\theta} = \dfrac{\cos\theta}{1+\sin\theta}$

PROOF:

We will start by multiplying both sides by $(1 + \sin\theta)$:

$$(1 - \sin\theta)(1 + \sin\theta) = \cos^2\theta.$$

Now, apply the difference of squares on the left-hand side:

$$1^2 - \sin^2\theta = \cos^2\theta.$$

Thus, we have:

$$\frac{1 - \sin\theta}{\cos\theta} = \frac{\cos\theta}{1 + \sin\theta}.$$

The identity is **proved**.

13. $2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

Solution: **Prove:** $2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

PROOF:

Start with the identity for $\cos(2\theta)$:

$$\cos(2\theta) = 2\cos^2\theta - 1.$$

Similarly, the identity for $\cos(2\theta)$ in terms of $\sin^2\theta$ is:

$$\cos(2\theta) = 1 - 2\sin^2\theta.$$

Thus, we have:

$$2\cos^2\theta - 1 = 1 - 2\sin^2\theta.$$

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The identity is **proved**.

14. $\sec\theta = \sqrt{1+\tan^2\theta}$

Solution: Prove: $\boldsymbol{\sec\theta = \sqrt{1 + \tan^2\theta}}$

PROOF:

We start with the identity for $\sec^2\theta$:

$$\sec^2\theta = 1 + \tan^2\theta.$$

Taking the square root of both sides:

$$\sec\theta = \sqrt{1 + \tan^2\theta}.$$

Thus, the identity is **proved**.

15. $\text{cosec}\,\theta = \sqrt{1+\cot^2\theta}$

Solution: **Prove:** $\boldsymbol{\csc\theta = \sqrt{1 + \cot^2\theta}}$

PROOF:

We start with the identity for $\csc^2\theta$:

$$\csc^2\theta = 1 + \cot^2\theta.$$

Taking the square root of both sides:

$$\csc\theta = \sqrt{1 + \cot^2\theta}.$$

Thus, the identity is **proved**.

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