Exercise 2.1

1. Find the discriminant of the following given quadratic equations.

(i) $2x^2 + 3x - 1 = 0$

Solution: $2x^2 + 3x - 1 = 0$

$Here\ a = 2, b = 3, c = -1$

$Disc = b^2 - 4ac$

$= (3)^2 - 4(2)(-1)$

$= 9 + 8 = 17$

(ii) $6x^2 - 8x + 3 = 0$

Solution: $6x^2 - 8x + 3 = 0$

$Here\ a = 6, b = -8, c = 3$

$Disc = b^2 - 4ac$

$= (-8)^2 - 4(6)(3)$

$= 64 - 72 = -8$

(iii) $9x^2 - 30x + 25 = 0$

Solution: $9x^2 - 30x + 25 = 0$

$Here\ a = 9, b = -30, c = 25$

$Disc = b^2 - 4ac$

$= (-30)^2 - 4(9)(25)$

$= 900 - 900 = 0$

(iv) $4x^2 - 7x - 2 = 0$

Solution: $4x^2 - 7x - 2 = 0$

$Here\ a = 4, b = -7, c = -2$

$Disc = b^2 - 4ac$

$= (-7)^2 - 4(4)(-2)$

$= 49 + 32 = 81$

2. Find the nature of roots of the following given quadratic equations and verify the result by solving the equation:

(i) $x^2 - 23x + 120 = 0$

Solution: $x^2 - 23x + 120 = 0$

Here $a = 1, b = -23, c = 120$

$Disc = b^2 - 4ac$

$= (-23)^2 - 4(1)(120)$

$= 529 - 480 = 49 = (7)^2 > 0$

and is a(Perfect square) therefore

the roots are real rational and unequal

Verification: $x^2 - 23x + 120 = 0$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$x = \frac{-(23) \pm \sqrt{(-23)^2-4(1)(120)}}{2(1)}$$

$$x = \frac{23 \pm \sqrt{529-480}}{2}$$

$$x = \frac{23 \pm \sqrt{49}}{2}$$

$$x = \frac{23 \pm 7}{2}$$

Either $x = \frac{23+7}{2}$ or $x = \frac{23-7}{2}$

$x = \frac{30}{2}$ or $= \frac{16}{2}$

$x = 15$ or $x = 8$

Evidently the roots are real, rational and unequal.

(ii) $2x^2 + 3x + 7 = 0$

Solution: $2x^2 + 3x + 7 = 0$

Here $a = 2, b = 3, c = 7$

$Disc = b^2 - 4ac$

$= (3)^2 - 4(2)(7)$

$9 - 56 = -47 < 0$

As disc is negative, therefore, the roots are imaginary and unequal

Verification: $2x^2 + 3x + 7 = 0$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$x = \frac{-3 \pm \sqrt{(3)^2-4(2)(7)}}{2(2)}$$

$$x = \frac{-3 \pm \sqrt{9-56}}{4}$$

$$x = \frac{-3 \pm \sqrt{-47}}{4}$$

Evidently the roots are imaginary and unequal.

(iii) $16x^2 + 24x + 9 = 0$

Solution: $16x^2 + 24x + 9 = 0$

Here $a = 16, b = -24, c = 9$

Disc $= b^2 - 4ac$

$= (-24)^2 - 4(16)(9)$

$= 576 - 576 = 0$

As disc is zero, therefore the roots are real (rational) and equal.

Verification: $16x^2 + 24x + 9 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(24) \pm \sqrt{(-24)^2 - 4(16)(9)}}{2(16)}$$

$$x = \frac{24 \pm \sqrt{576-576}}{32}$$

$$x = \frac{24 \pm 0}{32}$$

$$x = \frac{24 \pm 0}{32}$$

$$x = \frac{24}{32}$$

$$x = \frac{3}{4}$$

Evidently the roots are real and unequal

(iv) $3x^2 + 7x - 13 = 0$

Solution: $3x^2 + 7x - 13 = 0$

Here $a = 3, b = 7, c = -13$

Disc $= b^2 - 4ac$

$= (7)^2 - 4(3)(-13)$

$49 + 156 = 205 > 0$ *not a Perfect square*

Therefore the roots are real, irrational and unequal.

Verification: $3x^2 + 7x - 13 = 0$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$x = \frac{-7 \pm \sqrt{(7)^2 - 4(3)(-13)}}{2(3)}$$

$$x = \frac{-7 \pm \sqrt{49 + 156}}{6}$$

$$x = \frac{-7 \pm \sqrt{205}}{6}$$

Evidently, the roots are real, irrational and unequal.

3. **For what value of k, the expression.**

$k^2x^2 + 2(k+1)x + 4$ *is perfect square.*

Solution: Let $k^2x^2 + 2(k+1)(x+4) = 0$

Here $a = k^2, b = 2(k+1), c = 4$

$Disc = b^2 - 4ac$

$= [2(k+1)]^2 - 4(k)^2(4)$

$= 4(k^2 + 2k + 1) - 16k^2$

$= 4k^2 + 8k + 4 - 16k^2$

$= -12k^2 + 8k + 4$

Given expression is a perfect square therefore the roots are rational and equal. Therefore

$Disc = 0$

$-12k^2 + 8k + 4 = 0$

$12k^2 - 8k - 4 = 0$ (Dividing by 4 we get)

$3k^2 - 2k - 1 = 0$

$3k^2 - 3k + k - 1 = 0$

$(k-1)(3k+1) = 0$

$(k-1) = 0$ *or* $3k + 1 = 0$

$3k = -1$

Thus $k = 1$ $\quad k = -\frac{1}{3}$

4. **Find the value of k, if the roots of the following equation are equal.**

(i) $(2k-1)x^2 + 3kx + 3 = 0$

Solution: $(2k-1)x^2 + 3kx + 3 = 0$

Here $a = 2k - 1, b = 3k, c = 3$

For equal roots Disc. must be zero therefore

$Disc = b^2 - 4ac = 0$

$(3k)^2 - 4(2k-1)(3) = 0$

$9k^2 - 24k + 12 = 0$ *Dividing both sided by 3*

$3k^2 - 8k + 4 = 0$

$3k^2 - 2k - 6k + 4 = 0$

$5(3k-2) - 2(3k-2) = 0$

$(k-2)(3k-2) = 0$

Either $k - 2 = 0$ *or* $3k - 2 = 0$

$k = 2$ $\qquad\qquad$ $3k = 2$

or $\quad k = \frac{2}{3}$

(ii) $x^2 + 2(k+2)x + (3k+4) = 0$

Solution: $x^2 + 2(k+2)x + (3k+4) = 0$

$Here\ a = 1, b = 2(k+2), c = 3k+4$

For equal roots Disc. must be zero therefore

$Disc = b^2 - 4ac = 0$

$[2(k+2)]^2 - 4(1)(3k+4) = 0$

$4(k^2 + 4k + 4) - 12k - 16 = 0$

$4k^2 + 16l + 16 - 12k - 16 = 0$

$4k^2 + 4k = 0$

$4k(k+1) = 0$

$4k = 0 \quad or \quad k + 1 = 0$

$k = 0 \quad or \quad k = -1$

(iii) $(3k+2)x^2 - 5(k+1)x + (2k+3) = 0$

$Here\ a = 3k+2, b = -5(k+1), c = 2k+3$

For equal roots Disc. must be zero therefore

$Disc = b^2 - 4ac = 0$

$[-5(k+1)]^2 - 4(3k+2)(2k+3) = 0$

$25(k^2 + 2k + 1) - 4(6k^2 + 13k + 6) = 0$

$25k^2 + 50k + 25 - 24k^2 - 52k - 24 = 0$

$k^2 - 2k + 1 = 0$

$(k-1)^2 = 0$

$k - 1 = 0$

$k = 1$

5. Show that the equation $x^2 + (mx+c)^2 = a^2$ has equal roots, if $c^2 = a^2(1+m^2)$

Solution: $x^2 + (mx+c)^2 = a^2$

$x^2 + m^2x^2 + 2mcx + c^2 = a^2$

$x^2 + m^2x^2 + 2mcx + c^2 - a^2 = 0$

$(1+m^2)^2x^2 + 2mcx + (c^2 - a^2) = 0$

Disc = must be zero for equal roots

$Here\ a = 1 + m^2,\ b = 2mc,\ c = c^2 - a^2$

$Disc: b^2 - 4ac = 0$

By Putting values of a, b, c, we get

$(2mc)^2 - 4(1+m^2)(c^2 - a^2) = 0$

$4m^2c^2 - 4(c^2 - a^2 + m^2c^2 - m^2a^2 = 0$

$4m^2c^2 - 4c^2 + 4a^2 - 4m^2c^2 + 4m^2a^2 = 0$

$$-4c^2 = -4m^2a^2 - 4a^2 \text{ (Dividing by } -4)$$
$$c^2 = m^2a^2 + a^2$$
$$c^2 = a^2(m^2 + 1)$$
$$c^2 = a^2(m^2 + 1)$$
$$c^2 = a^2(1 + m^2) \qquad \textit{Hence the result}$$

(<u>OR Second Method</u>)

Solution:

$$x^2 + (mx + c)^2 = a^2$$
$$x^2 + m^2x^2 + 2mcx + c^2 = a^2$$
$$x^2 + m^2x^2 + 2mcx + c^2 - a^2 = 0$$
$$(1 + m^2)^2x^2 + 2mcx + (c^2 - a^2) = 0$$
$$Disc = \text{must be zero for equal roots}$$
$$\text{Here } a = 1 + m^2,\ b = 2mc,\ c = c^2 - a^2$$

Discriminant = $b^2 - 4ac$

Disc $= (2mc)^2 - 4(1+m^2)(c^2 - a^2)$

$= 4m^2c^2 - 4(c^2 - a^2 + m^2c^2 - m^2a^2)$

$= 4m^2c^2 - 4c^2 - 4a^2 - 4m^2c^2 + 4m^2a^2$

$= 4a^2 - 4c^2 + 4m^2a^2 = 4(a^2 - c^2 + m^2a^2)$

Given that $c^2 = a^2(1+m^2)$ therefore

Disc $= 4[a^2 - a^2(1+m^2) + m^2a^2]$

$= 4(a^2 - a^2 - a^2m^2 + m^2a^2) = 0$

As disc = 0 therefore roots are equal if $c^2 = a^2(1+m^2)$

6. Find the condition that the roots of the equation $(mx + c)^2 - 4ac$ are equal.

Solution: $(mx + c)^2 - 4ax = 0$

$$m^2x^2 + 2cmx + c^2 - 4ax = 0$$
$$m^2x^2 + 2cmx - 4ax + c^2 = 0$$
$$m^2x^2 + (2cm - 4a)x + c^2 = 0$$
$$\text{The roots will be equal when its } Disc = 0$$
$$Disc = b^2 - 4ac = 0$$
$$\text{Here, } a = m^2, b = 2cm - 4a,\ c = c^2$$
$$(2cm - 4a)^2 - 4(m^2)(c^2) = 0$$
$$4c^2m^2 + 16a^2 - 16cma - 4c^2m^2 = 0$$
$$16a^2 - 16cma = 0$$
$$16a(a - cm) = 0$$
$$a - cm = 0 \quad as\ 16a \neq 0$$

therefore $a = cm$ is the required condition

7. If the roots of the equation
$(c^2 - ab)x^2 - 2(a^2 - bc)x + (b^2 - ac) = 0$ are equal, then a=0 or $a^3 + b^3 + c^3 = 3abc$

Solution:

$$(c^2 - ab)x^2 - 2(a^2 - bc)x + (b^2 - ac) = 0$$

The roots will be equal when its Disc $= 0$

Therefore; $Disc = b^2 - 4ac = 0$

$$[-2(a^2 - bc)]^2 - 4(c^2 - ab)(b^2 - ac) = 0$$

$$4(a^4 - 2a^2bc + b^2c^2) - 4(b^2c^2 + ac^3 - ab^3 + a^2bc) = 0$$

(*Dividing both sides by* 4, *we get*)

$$a^4 - 2a^2bc + b^2c^2 - b^2c^2 + ac^3 + ab^3 + a^2bc = 0$$

$$a^4 - 3a^2bc + ac^3 + ab^2 = 0$$

$$a(a^3 - 3abc + c^3 + b^3) = 0$$

Now, $a = 0$ *or* $a^3 - 3abc + c^3 + b^3 = 0$

$$a^3 + b^3 + c^3 = 3abc$$

Hence proved

8. Show that the roots of the following equations are rational.

(i) $a(b - c)x^2 + b(c - a)x + c(a - b) = 0$

Solution:

$$a(b - c)x^2 + b(c - a)x + c(a - b) = 0$$

The roots will be rational, if Disc. is a perfect square.

$Disc = b^2 - 4ac = 0$

$Disc = [b(c - a)]^2 - 4a(b - c)(c)(a - b)$

$= b^2(c^2 + a^2 - 2ac) - 4ac(ab - b^2 - ax + bc)$

$= b^2c^2 + a^2b^2 - 2ab^2c - 4a^2bc + 4ab^2c + 4a^2c^2 - 4abc^2$

$= a^2b^2 + b^2c^2 + 4a^2c^2 + 2ab^2c + 4ab^2c - 4a^2bc - 4abc^2$

$= (ab)^2 + (bc)^2 + (-2ac)^2 + 2(ab)(bc) + 2(bc)(-2ac) + 2(-2ac)(ab)$

$= (ab + bc - 2ac)^2$ *which is a Perfect square.*

Hence, the roots are rational.

(ii) $(a + 2b)x^2 + 2(a + b + c)x + (a + 2c) = 0$

Solution:

$$(a + 2b)x^2 + 2(a + b + c)x + (a + 2c) = 0$$

The roots will be rational, if Disc is a perfect square.

$Disc = b^2 - 4ac = 0$

$Disc = [2(a + b + c)]^2 - 4(a + 2b)(a + 2c)$

$= 4(a + b + c)^2 - 4(a + 2b)(a + 2c)$

$= 4(a^2 + b^2 + c^2 + 2ab + 2bc + 2ca) - 4(a^2 + 2ca + 2ab\text{-}4bc)$

$= 4[a^2 + b^2 + c^2 + 2ab + 2bc + 2ca - a^2 - 2ca - 2ab - 4bc]$

$= 4[b^2 + c^2 - 2bc]$

$= 4(b - c)^2$

$= [2(b - c)]^2$ *which is a perfect square.*

Hence, the roots are rational.

9. For all values of k, prove that the roots of the equation. $x^2 - 2\left(k + \frac{1}{5}\right)x + 4 = -0, k \neq 0$ are real.

Solution:

$x^2 - 2\left(k + \frac{1}{k}\right)x + 4 = 0$

For real roots Disc. must be positive.

Here $a = 1, b = -2\left(k + \frac{1}{k}\right), c = 4$

$Disc = b^2 - 4ac$

$= \left[-2\left(k + \frac{1}{k}\right)\right]^2 - 4(1)(4)$

$= 4\left(k + \frac{1}{k}\right)^2 - 16$

$= 4\left(k^2 + \frac{1}{k^2} + 2\right) - 16$

$= 4k^2 + \frac{4}{k^2} + 8 - 16$

$= 4k^2 + \frac{4}{k^2} - 8$

$= 4\left[k^2 + \frac{1}{k^2} - 2\right]$

$= 4\left[k - \frac{1}{k}\right]^2$

$= \left[2\left(k - \frac{1}{k}\right)\right]^2 > 0$ *which is positive for all values of* x

Hence the roots are real

10. Show that the roots of the equation. $(b - c)x^2 + (c - a)x + (a - b) = 0$ *are real*

Solution:

$(b - c)x^2 + (c - a)x + (a - b) = 0$

For real roots Disc must be positive

$Disc = b^2 - 4ac$

$= (c - a)^2 - 4(b - c)(a - b)$

$= c^2 + a^2 - 2ca - 4(ab - b^2 - ca + bc)$
$= c^2 + a^2 - 2ac - 4ab + 4b^2 + 4ca - 4bc$
$= c^2 + a^2 + 4b^2 + 4ca - 4bc$
$= c^2 + a^2 + 4b^2 - 2ca - 4bc - 4ab$
$= [(c)^2 + (a)^2 + (2b)^2 - 2(c)(a) - 2(a)(2b) - 2(c)(2b)]$
$= (c + a - 2b)^2 > 0$ *which is always positive.*
Hence, the roots are real.

Cube roots of Unity and their Properties

Q1. Find the cube roots of unity?

OR

Find the cube roots of one?

Solution: Let x be a cube root of unity

$\therefore \quad x = \sqrt[3]{1} = (1)^{\frac{1}{3}}$

$x^3 = 1 \quad \Rightarrow \quad x^3 - 1 = 0$

$(x - 1)(x^2 + x + 1) = 0$

Either $x - 1 = 0 \quad \Rightarrow \quad x = 1$

or $\quad x^2 + x + 1 = 0$

$\therefore \quad x = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3}i}{2} \qquad (\because \sqrt{-1} = i)$

Thus the three cube roots of unity are:

$1, \frac{-1+\sqrt{3}i}{2}$ and $\frac{-1-\sqrt{3}i}{2}$

Note: We know that the numbers containing i are called complex numbers. So $\omega = \frac{-1+\sqrt{3}i}{2}$ and $\omega^2 = \frac{-1-\sqrt{3}i}{2}$ are called complex or imaginary cube roots of unity.

Q2. Prove that each of the complex cube roots of unity is the square of the other.

Proof: The complex cube roots of unity are

$$\frac{-1 + \sqrt{-3}}{2} \; and \; \frac{-1 - \sqrt{-3}}{2}$$

We prove that

$$\left(\frac{-1+\sqrt{-3}}{2}\right)^2 = \frac{-1-\sqrt{-3}}{2} \text{ and } \left(\frac{-1-\sqrt{-3}}{2}\right)^2 = \frac{-1+\sqrt{-3}}{2}.$$

$$\left(\frac{-1+\sqrt{-3}}{2}\right)^2 = \frac{1+(-3)-2\sqrt{-3}}{4}$$

$$= \frac{-2 - 2\sqrt{-3}}{4}$$

$$= \frac{2(-1-\sqrt{-3})}{4}$$
$$= \frac{-1-\sqrt{-3}}{2}$$

also

$$\left(\frac{-1-\sqrt{-3}}{2}\right)^2 = \frac{1+(-3)+2\sqrt{-3}}{4}$$
$$= \frac{-2+2\sqrt{-3}}{4}$$
$$= \frac{2(-1+\sqrt{-3})}{4}$$
$$= \frac{-1+\sqrt{-3}}{2}$$

Thus, each of the complex cube root of unity is the square of the other, that is,
if

$\omega = \frac{-1+\sqrt{-3}}{2}$; then $\omega^2 = \frac{-1-\sqrt{-3}}{2}$ *and if* $\omega = \frac{-1+\sqrt{-3}}{2}$, then $\omega^2 = \frac{-1-\sqrt{-3}}{2}$

Q3. Prove that the product of three cube roots of unity is one.

Proof: Three cube roots of unity are

1, $\frac{-1+\sqrt{-3}}{2}$ and $\frac{-1-\sqrt{-3}}{2}$

The product of cube roots of unity

$$(1)\left(\frac{-1+\sqrt{-3}}{2}\right)\left(\frac{-1-\sqrt{-3}}{2}\right)$$

$= \frac{(-1)^2-(\sqrt{-3})^2}{4} = \frac{1-(-3)}{4} = \frac{1+3}{4} = \frac{4}{4} = 1$

i.e., $(1)(\omega)(\omega^2) = 1$ *or* $\omega^3 = 1$

Remember that:

$\omega^4 = \omega^3.\omega = 1.\omega = \omega$

Q4. Prove that each complex cube root of unity is reciprocal of the other.

Proof: We know that $\omega^3 = 1$

$\Rightarrow \omega.\omega^2 = 1, so$

$\omega = \frac{1}{\omega^2}$ or $\omega^2 = \frac{1}{\omega}$

Thus, each complex cube root of unity is reciprocal of the other.

Q5. Prove that the sum of all the cube roots of unity is zero.

i.e., $1+\omega+\omega^2=0$

Proof: The cube roots of unity are

$$1, \frac{-1+\sqrt{-3}}{2} \text{ and } \frac{-1-\sqrt{-3}}{2}$$

If $\omega=\frac{-1+\sqrt{-3}}{2}$, then $\omega^2=\frac{-1-\sqrt{-3}}{2}$

The sum of all the roots $=1+\omega+\omega^2$

$$=1+\frac{-1+\sqrt{-3}}{2}+\frac{-1-\sqrt{-3}}{2}$$

$$=\frac{2-1+\sqrt{-3}-1-\sqrt{-3}}{2}=\frac{0}{2}=0$$

Thus, $1+\omega+\omega^2=0$

We can easily deduce the following results, that is,

(i) $1+\omega^2=-\omega$

(ii) $1+\omega=-\omega^2$

(iii) $\omega+\omega^2=-1$