Exercise 3.4

1. Prove that $a : b = c : d, if$

(i) $\frac{4a+5b}{4a-5b} = \frac{4c+5d}{4c-5d}$

Solution: $\frac{4a+5b}{4a-5b} = \frac{4c+5d}{4c-5d}$

By componendo − dividendo theorem

$$\frac{(4a+5b)+(4a-5b)}{(4a+5b)-(4a-5b)} = \frac{(4c+5d)+(4c-5d)}{(4c+5d)-(4c-5d)}$$

$$\frac{4a+5b+4a-5b}{4a+5b-4a+5b} = \frac{4c+5d+4c-5d}{4c+5d-4c+5d}$$

$$\frac{8a}{10b} = \frac{8c}{10d}$$

$$\frac{a}{b} = \frac{c}{d}$$

Hence $= a : b = c : d$ *proved*

(ii) $\frac{2a+9b}{2a-9b} = \frac{2c+9d}{2c-9d}$

Solution: $\frac{2a+9b}{2a-9b} = \frac{2c+9d}{2c-9d}$

By componendo − dividendo theorem

$$\frac{(2a+9b)+(2a-9b)}{(2a+9b)-(2a-9b)} = \frac{(2c+9d)+(2c-9d)}{(2c+9d)-(2c-9d)}$$

$$\frac{2a+9b+2a-9b}{2a+9b-2a+9b} = \frac{2c+9d+2c-9d}{2c+9d-2c+9d}$$

$$\frac{4a}{18b} = \frac{4c}{18d}$$

$$\frac{a}{b} = \frac{c}{d}$$

Hence $= a : b = c : d$ *proved*

(iii) $\frac{ac^2+bd^2}{ac^2-bd^2} = \frac{c^3+d^3}{c^3-d^3}$

Solution: $\frac{ac^2+bd^2}{ac^2-bd^2} = \frac{c^3+d^3}{c^3-d^3}$

By componendo − dividendo theorem

$$\frac{(ac^2+bd^2)+(ac^2-bd^2)}{(ac^2+bd^2)-(ac^2-bd^2)} = \frac{(c^3+d^3)+(c^3-d^3)}{(c^3+d^3)-(c^3-d^3)}$$

$$\frac{ac^2+bd^2+ac^2-bd^2}{ac^2+bd^2-ac^2} = \frac{c^3+d^3+c^3-d^3}{c^3+d^3-c^3+d^3}$$

$$\frac{2ac^2}{2bd^2} = \frac{2c^3}{2d^3} \quad (Multiplying\ by\ \frac{d^2}{c^2})$$

$$\frac{a}{b} = \frac{c}{d}$$

Hence $= a : b = c : d$ *proved*

(iv) $\frac{a^2c+b^2d}{a^2c-b^2d} = \frac{ac^2+bd^2}{ac^2-bd^2}$

Solution: $\frac{a^2c+b^2d}{a^2c-b^2d} = \frac{ac^2+bd^2}{ac^2-bd^2}$

By componendo – dividendo theorem

$$\frac{(a^2c+b^2d)+(a^2c-b^2d)}{(a^2c+b^2d)-(a^2c-b^2d)} = \frac{(ac^2+bd^2)+(ac^2-bd^2)}{(ac^2+bd^2)-(ac^2-bd^2)}$$

$$\frac{a^2c+b^2d+a^2c-b^2d}{a^2c+b^2d-a^2c+b^2d} = \frac{a^2c+bd^2+ac^2-bd^2}{ac^2+bd^2-ac^2+bd^2}$$

$$\frac{2a^2c}{2b^2d} = \frac{2ac^2}{2bd^2}$$

$$\frac{a^2c}{b^2d} = \frac{ac^2}{bd^2}$$

$$\frac{a \times ac}{b \times bd} = \frac{(ac)c}{(bd)d}$$

$$\frac{a}{b} = \frac{c}{d}$$

Hence $= a : b = c : d$ *proved*

(v) $pa + qb : pa - qb = pc + qd : pc - qd$

Solution: $pa + qb : pa - qb = pc + qd : pc - qd$

$$\frac{pa + pb}{pa - pb} = \frac{pc + qd}{pc - qd}$$

By componendo – dividendo theorem

$$\frac{(pa+qb)+(pq-qb)}{(pa+qb)-(pa-qb)} = \frac{(pa+qb)+(pa-qb)}{(pa+qb)-(pa-qb)}$$

$$\frac{pa+qb+pa-qb}{pa+qb-pa+qb} = \frac{pc+qd+pc-qd}{pc+qd-pc+qd}$$

$$\frac{2pa}{2qb} = \frac{2pc}{2qd}$$

$$\frac{pa}{qb} = \frac{pc}{qd}$$

$$\frac{a}{b} = \frac{c}{d}$$

Hence $= a : b = c : d$ *proved*

(vi) $\frac{a+b+c+d}{a+b-c-d} = \frac{a-b+c-d}{a-b-c+d}$

Solution: $\frac{a+b+c+d}{a+b-c-d} = \frac{a-b+c-d}{a-b-c+d}$

By componendo – dividendo theorem

$$\frac{(a+b+c+d)+(a+b+c-d)}{(a+b+c+d)-(a+b-c-d)} = \frac{(a-b+c-d)+(a-b-c+d)}{(a-b+c-d)-(a-b-c+d)}$$

$$\frac{a+b+c+d+a+b-c-d}{a+b+c+d-a-b+c+d} = \frac{a-b+c-d+a-b-c+d}{a-b+c-d-a+b+c-d}$$

$$\frac{2a++2b}{2c+2d} = \frac{2a-2b}{2c-2d}$$

$$\frac{a+b}{c+d} = \frac{a-b}{c-d}$$

Again by componendo – dividendo theorem

$$\frac{a+b+a-b}{a+b-a+b} = \frac{c+d+c-d}{c+d-c+d}$$

$$\frac{2a}{2b} = \frac{2c}{2d}$$

$$\frac{a}{b} = \frac{c}{d}$$

Hence $= a : b = c : d$ *proved*

(vii) $\frac{2a+3b+2c+3d}{2a+3b-2c-3d} = \frac{2a-3b+2c-3d}{2a-3b-2c+3d}$

Solution: $\frac{2a+3b+2c+3d}{2a+3b-2c-3d} = \frac{2a-3b+2c-3d}{2a-3b-2c+3d}$

By componendo – dividendo theorem

$$\frac{2a+3b+2c+3d+2a+3b-2c-3d}{2a+3d+2c+3d-2a-3b+2c-3d} = \frac{2a-3b+2c-3d+2a-3b-2c+3d}{2a-3b+2c-3d-2a+3b-2c-3d}$$

$$\frac{4a+6b}{4c+6d}=\frac{4a-6b}{4c-6b}$$

$$\frac{2(2a+3b)}{2(2c+3d)}=\frac{2(2a-3b)}{2(2c-3d)}$$

$$\frac{2a+3b}{2c+3d}=\frac{2a-3b}{2c-3d}$$

Again by componendo − dividendo theorem

$$\frac{2a+3b+2a-3b}{2a+3b-2a+3b}=\frac{2c+3d+2c-3d}{2c+3d-2c+3d}$$

$$\frac{4a}{6b}=\frac{4c}{6d}$$

$$\frac{a}{b}=\frac{c}{d}$$

Hence $= a : b = c : d$ *proved*

(viii) $\frac{a^2+b^2}{a^2-b^2}=\frac{ac+bd}{ac-bd}$

Solution: $\frac{a^2+b^2}{a^2-b^2}=\frac{ac+bd}{ac-bd}$

By componendo − dividendo theorem

$$\frac{a^2+b^2+a^2-b^2}{a^2+b^2-a^2+b^2}=\frac{ac+bd+ac-bd}{ac+bd-ac+bd}$$

$$\frac{2a^2}{2b^2}=\frac{2ac}{2bd}$$

$$\frac{a^2}{b^2}=\frac{ac}{bd}$$

$$\frac{a\times a}{b\times b}=\frac{a\times c}{b\times d}$$

$$\frac{a}{b}=\frac{c}{d}$$

Hence $= a : b = c : d$ *proved*

2. Using theorem of componendo-dividendo.

(i) Find the value of $\frac{x+2y}{x-2y}+\frac{x+2z}{x-2z}$, *if* $x=\frac{4yz}{y+z}$

Solution: $x=\frac{4yz}{y+z}$

$$x=\frac{2y(2z)}{y+z}$$

$$\frac{x}{2y}=\frac{2z}{y+z}$$

By componendo – dividendo theorem

$$\frac{x+2y}{x-2y}=\frac{2z+y+z}{2z-y-z}$$

$$\frac{x+2y}{x-2y}=\frac{3z+y}{z-y} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

Now $x=\frac{4zy}{y+z}$

$$x=\frac{2z(2y)}{y+z}$$

$$\frac{x}{2z}=\frac{2y}{y+z}$$

Again by componendo – dividendo theorem

$$\frac{x+2z}{x-2z}=\frac{2y+y+z}{2y-y-z}$$

$$\frac{x+2y}{x-2y}=\frac{3y+z}{y-z} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (ii)$$

By adding (i) *and* (ii), *we get*

$$\frac{x+2y}{x-2y}+\frac{x+2y}{x-2y}=\frac{3z+y}{z-y}+\frac{3y+z}{y-z}$$

$$=\frac{3z+y}{z-y}-\frac{3y+z}{z-y}$$

$$=\frac{3z+y-3y-z}{z-y}$$

$$=\frac{2z-2y}{z-y}$$

$$=\frac{2(z-y)}{(z-y)}$$

$$=2$$

(ii) **Find the value of** $\frac{m+5n}{m-5n}+\frac{m+5[}{m-5p}$, **if** $m=\frac{10np}{n+p}$

Solution: $m=\frac{10np}{n+p}$

$$m=\frac{(5n)(2p)}{n+p}$$

$$\frac{m}{5n} = \frac{2p}{n+p}$$

By componendo – dividendo theorem

$$\frac{m+5n}{m-5n} = \frac{2p+n+p}{2p-n-p}$$

$$= \frac{3p+n}{p-n} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

Now $m = \dfrac{10np}{n+p}$

$$m = \frac{(2n)(5p)}{n+p}$$

$$\frac{m}{5p} = \frac{n}{n+p}$$

Agaom by componendo – dividendo theorem

$$\frac{m+5p}{m-5p} = \frac{2n+n+p}{2n-n-p}$$

$$= \frac{3n+p}{n-p} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

By adding (i)*and* (ii), *we get*

$$\frac{m+5n}{m-5n} + \frac{m+5p}{m-5p} = \frac{3p+n}{p-n} + \frac{3n+p}{3n-p}$$

$$= \frac{3p+n}{p-n} - \frac{3p+n}{p-n}$$

$$= \frac{3p+n-3n-p}{p-n}$$

$$= \frac{2p-2n}{p-n}$$

$$= \frac{2(p-n)}{(p-n)}$$

$$= 2$$

(iii) **Find the value of** $\dfrac{x-6a}{x+6a} - \dfrac{x+6b}{x-6b}$, **if** $x = \dfrac{12ab}{a-b}$

Solution: $x = \dfrac{12ab}{a-b}$

$$x = \frac{(6a)(2b)}{a-b}$$

$$\frac{x}{6a} = \frac{2b}{a-b}$$

By componendo – dividendo theorem

$$\frac{x-a}{x+6a}=\frac{2b-a+b}{2b+a-b}$$

$$=\frac{3b-a}{b+a}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(i)$$

Now, $x=\frac{12ab}{a-b}$

$$\frac{x}{6b}=\frac{2a}{a-b}$$

Again by componendo – dividendo theorem

$$\frac{x+6b}{x-6a}=\frac{2a+a-b}{2a-a+b}$$

$$=\frac{3a-b}{a+b}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(ii)$$

By subtracting (ii)from (i), *we get*

$$\frac{x-6a}{x+6a}-\frac{x+6b}{x-6b}=\frac{3b-a}{b+a}-\frac{3a-b}{a+b}$$

$$=\frac{3b-a-3a+b}{a+b}$$

$$=\frac{4b-4a}{a+b}$$

$$\frac{x-6a}{x+6a}-\frac{x-6b}{x+6b}=\frac{4(b-a)}{a+b}$$

(iv) Find the value of $\frac{x-3y}{x+3y}-\frac{x+3z}{x-3z}$, **if** $x=\frac{3yz}{y-z}$

Solution: $x=\frac{3yz}{y-z}$

$$\frac{x}{3y}=\frac{z}{y-z}$$

By componendo – dividendo theorem

$$\frac{x-3y}{x+3y}=\frac{z-y+z}{z+y-z}$$

$$=\frac{2z-y}{y}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(i)$$

Now $x=\frac{3yz}{y-z}$

$$\frac{x}{3z}=\frac{y}{y-z}$$

Again by componendo – dividendo theorem

$$\frac{x+3z}{x-3y}=\frac{y+y-z}{y-y+z}$$

$$=\frac{2y-z}{z} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (ii)$$

By subtracting (ii)from (i), *we get*

$$\frac{x-3y}{x+3y}-\frac{x+3z}{x-3z}=\frac{2z-y}{y}-\frac{2y-z}{z}$$

$$=\frac{z(2z-y)-y(2y-z)}{yz}$$

$$=\frac{2z^2-yz-2y^2+yz}{yz}$$

$$=\frac{2z^2-2y^2}{yz}$$

$$=\frac{2(z^2-y^2)}{yz}$$

(v) **Find the value of** $\frac{s-3p}{x+3p}+\frac{s+3q}{s-3q}$, **if** $s=\frac{6pq}{P-q}$.

Solution: $s=\frac{6pq}{p-q}$

$$s=\frac{(3p)(2q)}{p-q}$$

$$\frac{s}{3p}=\frac{2q}{p-q}$$

By componendo – dividendo theorem

$$\frac{s-3p}{s+3p}=\frac{2q-p+q}{2q+p-q}$$

$$=\frac{3q-p}{q+p} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

Now $s=\frac{6pq}{p-q}$

$$\frac{s}{3q}=\frac{2p}{p-q}$$

Again by componendo – dividendo theorem

$$\frac{s+3q}{s-3q}=\frac{2p+q-q}{2p-p+q}$$

$$=\frac{3p-q}{p+q} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

By adding (i)*and* (ii), *we get*

$$\frac{s-3p}{s+3p}+\frac{s+3p}{s-3p}=\frac{3q-p}{q+p}+\frac{3p-q}{p+q}$$

$$=\frac{3q-p+3p-q}{p+q}$$

$$=\frac{2p+2q}{p+q}$$

$$=\frac{2(p+q)}{(p+q)}$$

$$=2$$

(vi) **Solve** $\frac{(x-2)^2-(x-4)^2}{(x-2)^2+(x-4)^2}=\frac{12}{13}$

Solution: *By componendo – dividendo theorem*

$$\frac{(x-2)^2-(x-4)^2+(x-2)^2+(x-4)^2}{(x-2)^2-(x-4)^2-(x-2)^2-(x-4)^2}=\frac{12+13}{12-13}$$

$$\frac{2(x-2)^2}{-2(x-4)^2}=\frac{25}{-1}$$

$$\frac{(x-2)^2}{(x-4)^2}=25$$

Taking square root on both sides, we get

$$\frac{x-2}{x-4}=\pm 5$$

when $\frac{x-2}{x-4}=5$	when $\frac{x-2}{x-4}=-5$
$x-2=5x-20$	$x-2=-5x+20$
$x-5x=-20+2$	$x+5x=20+2$
$-4x=-18$	$6x=22$
$x=\frac{-18}{-4}$	$x=\frac{22}{6}$
$=\frac{9}{2}$	$x=\frac{11}{3}$

$$Solution\ set = \left\{\frac{9}{2}, \frac{11}{3}\right\}$$

(vii) Solve $\frac{\sqrt{x^2+2}+\sqrt{x^2-2}}{\sqrt{x^2+2}-\sqrt{x^2-2}} = 2$

Solution: *By componendo – dividendo theorem*

$$\frac{\sqrt{x^2+2}+\sqrt{x^2-2}+\sqrt{x^2+2}-\sqrt{x^2-2}}{\sqrt{x^2+2}} = \frac{2+1}{2-1}$$

$$\frac{2\sqrt{x^2+2}}{2\sqrt{x^2-2}} = \frac{3}{1}$$

$$\frac{\sqrt{x^2+2}}{\sqrt{x^2-2}} = 3$$

$$\frac{x^2+2}{x^2-2} = 9 \qquad (by\ squaring)$$

$$x^2+2 = 9(x^2-2)$$

$$x^2+2 = 9x^2-18$$

$$x^2+2 = -18-2$$

$$-8x^2 = -20$$

$$x^2 = -\frac{20}{-8}$$

$$x^2 = \frac{5}{2}$$

$$x = \pm\sqrt{\frac{5}{2}} \quad (extraneous\ root)$$

$$Hence\ solution\ set = \{\}\ or\ \emptyset$$

(viii) Solve $\frac{\sqrt{x^2+8p^2}-\sqrt{x^2-p^2}}{\sqrt{x^2+8p^2}+\sqrt{x^2-p^2}} = \frac{1}{3}$

By componendo – dividendo theorem

$$\frac{\sqrt{x^2+8p^2}-\sqrt{x^2-p^2}+\sqrt{x^2+8p^2}+\sqrt{x^2-p^2}}{\sqrt{x^2+8p^2}-\sqrt{x^2-p^2}-\sqrt{x^2+8p^2}-\sqrt{x^2-p^2}} = \frac{1+3}{1-3}$$

$$\frac{2\sqrt{x^2+8p^2}}{-2\sqrt{x^2-p^2}} = \frac{4}{-2}$$

$$\frac{\sqrt{x^2+8p^2}}{\sqrt{x^2-p^2}} = 2\ (by\ squaring)$$

$$\frac{x^2+8p^2}{x^2-p^2} = 4$$

$$x^2+8p^2 = 4(x^2-p^2)$$

$$x^2+8p^2 = 4x^2-4p^2$$

$$x^2 - 4x^2 = -4p^2 - 8p^2$$
$$-3x^2 = -12p^2$$
$$x^2 = 4p^2$$

(Taking square root on both sides)

$$x = \pm 2p$$

Hence solution set $= \{2p, -2p\}$

(ix) **Solve** $\frac{(x+5)^3-(x-3)^3}{(x+5)^3+(x-3)^3} = \frac{13}{14}$

Solution: *By componendo – dividendo theorem*

$$\frac{(x+5)^3-(x-3)^3+(x+5)^3+(x-3)^3}{(x+5)^3-(x-3)^3-(x+5)^2-(x-3)^3} = \frac{13+14}{13-14}$$

$$\frac{2(x+5)^3}{-2(x-3)^3} = \frac{27}{1}$$

$$\frac{(x+5)^3}{(x-3)^3} = 27$$

$$\left(\frac{x+5}{x-3}\right)^3 = (3)^3$$

Taking cube root on both sides

$$\frac{x+5}{x-3} = 3$$
$$x + 5 = 3(x - 3)$$
$$x + 5 = 3x - 9$$
$$x - 3x = -9 - 5$$
$$-2x = -14$$
$$x = \frac{-14}{-2}$$
$$x = 7$$

Hence solution set $= \{7\}$