Exercise 4.4

Resolve into partial fractions.

1. $\dfrac{x^3}{(x^2+4)^2}$

Solution: Let $\dfrac{x^3}{(x^2+4)^2} = \dfrac{Ax+B}{x^2+4} = \dfrac{Cx+D}{(x^2+4)^2} \ldots\ldots\ldots(\text{i})$

$Multiplying\ both\ sides\ by\ (x^2+4)^2,\ we\ get$

$x^3 = (Ax+B)(x^2+4),\ we\ get$

$x^3 = Ax^3 + 4Ax + Bx^2 + Bx^2 + 4B + Cx + D$

$x^3 = Ax^3 + Bx^2 + (4A+C)x + (4B+D)$

$by\ comparing\ co-efficients\ of\ x^3\ in\ (\text{ii}), \text{we get}$

$1 = A$

$\boldsymbol{A = 1}$

$by\ comparing\ co-efficients\ of\ x^2\ in\ (\text{ii}), we\ get$

$0 = B$

$\boldsymbol{B = 0}$

$by\ comparing\ co-efficient\ x\ in\ (\text{ii}), \text{we get}$

$0 = 4A + C$

$Put\ A = 1\ in\ it, we\ get$

$0 = 4(1) + C$

$\boldsymbol{C = -4}$

$by\ comparing\ constants\ terms\ in\ (ii), we\ get$

$0 = 4B + D$

$Put\ B = 0\ in\ it, we\ get$

$0 = 0 + D$

$D = 0$

$Puttting\ values\ of\ A, B, C, D\ in\ (\text{i}), \text{we get}$

$$= \frac{x+0}{x^2+4} + \frac{-4x+0}{(x^2+4)^2}$$

$$= \frac{x}{x^2+4} - \frac{4x}{(x^2+4)^2}\ are\ partial\ fractions.$$

$$Hence,\ \frac{x^3}{(x^2+4)^2} = \frac{x}{x^2+4} - \frac{4x}{(x^2+4)^2}$$

2. $\dfrac{x^4+3x^2+x+1}{(x+1)(x^2+1)^2}$

Solution: *Let* $\dfrac{x^4+3x^2+x+1}{(x+1)(x^2+1)^2} = \dfrac{A}{x+1} + \dfrac{Bx+C}{(x^2+1)} + \dfrac{Dx+E}{(x^2+1)^2} \dots\dots (K)$

Multiplying both sides by $(x+1)(x^2+1)^2$, *we get*

$$x^4+3x^2+x+1 = A(x^2+1)^2 + (Bx+C)(x+1)(x^2+1) + (Dx+E)(x+1) \dots\dots\dots\dots (L)$$

$$x^4+3x^2+x+1 = A(x^4+2x^2+1) + (Bx+C)(x^3+x+x^2+1) + Dx^2+Dx+Ex+E$$

$$x^4+3x^2+x+1 = Ax^4+2Ax^2+A+Bx^4+Bx^2+Bx^3+Bx+Cx^3+Cx+Cx^2+C+Dx^2+Dx+Ex+E$$

$$x^4+3x^2+x+1 = (A+B)x^4+(B+C)x^3+(2A+B+C+D)x^2+(B+C+D+E)x+A+C+E$$

Put $x=-1$ *in* (L), *we get*

$(1)^4+3(-1)^2+(-1)+1 = A((-1)^2+1)\cdot$

$1+3(1)-1+1 = A(2)^2$

$4 = 4A$

$\therefore A = 1$

by comparing co – efficients of x^4, x^3, x^2 *and constant terms*

$A+B=1 \dots\dots\dots\dots\dots (\text{i})$

$B+C=0 \dots\dots\dots\dots\dots (\text{ii})$

$2A+B+C+D=3 \dots\dots (\text{iii})$

$A+C+E=1 \dots\dots\dots\dots (\text{iv})$

Put $A=1$ *in* (i), *we get*

$1+B=1$

$B=0$

Put $B=0$ *in* (ii), we get

$0+C=0$

$C=0$

Put values of A, B, C, D, E *in* (K), *we get*

Therefore, partial fraction are

$$\frac{1}{x+1} + \frac{0+0}{x^2+1} + \frac{x+0}{(x^2+1)^2}$$

$$= \frac{1}{x+1} + \frac{x}{(x^2+1)^2}$$

Hence, $\dfrac{x^4+3x^2+x+1}{(x+1)(x^2+1)^2} = \dfrac{1}{x+1} + \dfrac{x}{(x^2+1)^2}$

$2(1)+B+C+D=3$
$2(1)+0+0+D=3$
$2+0+0+D=3$
$2+D=3$
$D=3-2$
$D=1$

3. $\dfrac{x^2}{(x+1)(x^2+1)^2}$

Solution: Let $\dfrac{x^2}{(x+1)(x^2+1)^2} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+1} + \dfrac{Dx+E}{(x^2+1)^2} \dots\dots (i)$

Multiplying both sides by $(x+1)(x^2+1)^2$, *we get*

$$x^2 = A(x^2+1)^2 + (Bx+C)(x+1)(x^2+1) + (Dx+E)(x+1) \dots\dots\dots (ii)$$

$$x^2 = A(x^4+2x^2+1) + (Bx+C)(x^3+x+x^2+1) + (Dx^2+Dx+Ex+E)$$

$$x^2 = Ax^4 + 2Ax^2 + A + Bx^4 + Bx^2 + Bx^3 + Bx + Cx^3 + Cx + Cx^2 + C + Dx^2 + Dx + Ex + E$$

$$x^2 = (A+B)x^4 + (B+C)x^3 + (2A+B+C+D)x^2 + (B+C+D+E)x + A+C+E$$

Put $x = -1$ *in (ii), we get*

$(-1)^2 = A((-1)^2)+1)^2$

$1 = A(1+1)^2$

$1 = A(2)^2$

$1 = 4A$

$A = \frac{1}{4}$

by comparing the co efficients of x^4, x^3, x^2, x *and constant terms.*

$A + B = 0 \dots\dots\dots\dots\dots\dots (iii)$

$B + C = 0 \dots\dots\dots\dots\dots\dots (iv)$

$2A + B + C + D = 1 \dots\dots (v)$

$B + C + D + E = 0 \dots\dots (vi)$

$A + C + D + E = 0 \dots\dots (vii)$

Put $A = \frac{1}{4}$ *in (iii), we get*

$\frac{1}{4} + B = 0$

$B = -\frac{1}{4}$ *in (iv), we get*

Put $B = -\frac{1}{4}$ *in (iv), we get*

$-\frac{1}{4} + C = 0$

$C = \frac{1}{4}$

Put values of A, C *in (vii), we get*

$\frac{1}{4} + \frac{1}{4} + E = 0$

$E = -\frac{1}{4} - \frac{1}{4}$

$E = -\frac{1}{2}$

Put values of B, C in (vi), we get

$D + E = 0$

$D = -E$

Put $E = -\frac{1}{2}$ in it, we get

$D = -\left(-\frac{1}{2}\right)$

$D = +\frac{1}{2}$

Putting values of A, B, C, D, E in (i), we get

$$\frac{1}{4(x+1)} + \frac{-\frac{1}{4}x + \frac{1}{4}}{(x^2+1)} + \frac{\frac{1}{2}x - \frac{1}{2}}{(x^2+1)^2}$$

$$\frac{1}{4(x+1)} - \frac{x-1}{4(x^2+1)} + \frac{x-1}{2(x^2+1)^2} \quad (Partial\ fractions)$$

Hence, $\frac{x^2}{(x+1)(x^2+1)^2} = \frac{1}{4(x+1)} - \frac{x-1}{4(x^2+1)} + \frac{x-1}{2(x^2+1)^2}$

4. $\frac{x^2}{(x-1)(x^2+1)^2}$

Solution: Let $\frac{x^2}{(x-1)(x^2+1)^2} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2} \ldots\ldots (i)$

Multiplying both sides by $(x-1)(x^2+1)^2$, we get

$x^2 = A(x^2+1)^2 + (Bx+C)(x-1)(x^2+1) + (Dx+E)(x-1) \ldots\ldots\ldots (ii)$

$x^2 = A(x^4+2x^2+1) + (Bx+C)(x^3-x-x^2-1) + (Dx^2-Dx+Ex-E)$

$x^2 = Ax^4 + 2Ax^2 + A + Bx^4 - Bx^2 - Bx^3 - Bx + Cx^3 - Cx - Cx^2 - C + Dx^2 - Dx + Ex - E = 0$

$x^2 = (A+B)x^4 - (B-C)x^3 + (2A-B-C+D)x^2 - (B+C+D+E) + (A-C-E)$

by Comparing co-efficients of x^4, x^3, x^2, x and constant terms.

$0 = (A+B) \ldots\ldots\ldots\ldots (iii)$ x^4

$0 = -B + C \ldots\ldots\ldots\ldots (iv)$ x^3

$1 = 2A - B - C + D \ldots\ldots (v)$ x^2

$0 = -B - C - D + E \ldots\ldots (vi)$ x

$0 = A - C - E \ldots\ldots\ldots\ldots (vii)$ constant

Put $x - 1 = 0$ i.e; $x = 1$ in (ii), we get

$(1)^2 = A((1)^2+1)^2$

$1 = A(1+1)^2$

$1 = A(2)^2$

$1 = 4A$

$A = \frac{1}{4}$

$\frac{1}{4} + B = 0$ *from (iii), we get*

$B = -\frac{1}{4}$

$-\left(-\frac{1}{4}\right) + C = 0$ *from (iv), we get*

$\frac{1}{4} + C = 0$

$C = -\frac{1}{4}$

Put values of A, B, C in (vii), we get

$\frac{1}{4} + \frac{1}{4} - E = 0$

$E = \frac{2}{4} = \frac{1}{2}$

$0 = \frac{1}{4} + \frac{1}{4} - D + \frac{1}{2}$ *from (vi), we get*

$0 = \frac{1}{2} - D + \frac{1}{2}$

$D = \frac{1}{2} + \frac{1}{2} = 1$

Putting values of A, B, C, D, E, in (i), we get

$\frac{1}{4(x-1)} + \frac{-\frac{1}{4}x - \frac{1}{4}}{x^2+1} + \frac{x + \frac{1}{2}}{(x^2+1)^2}$ Same L.C.M goes down

$= \frac{1}{4(x-1)} - \frac{x+1}{4x^2+1} + \frac{2x+1}{2(x^2+1)^2}$ *are partial feractions*

$$\textbf{Hence,} \frac{x^2}{(x-1)(x^2+1)^2} = \frac{1}{4(x-1)} - \frac{x+1}{4(x^2+1)} + \frac{2x+1}{2(x^2+1)^2}$$

5. $\frac{x^4}{(x^2+2)^2}$

Solution: $\frac{x^4}{(x^2+2)^2}$

$$= \frac{x^4}{x^4 + 4x^2 + 4}$$

By long division, we have

$$x^4+4x^2+4\,\overline{)\,x^4\,}\;1$$
$$\underline{\pm x^4 \pm 4x^2 \pm 4}$$
$$-4x^2-4$$
$$-(4x^2+4)$$

$$= 1-\frac{4x^2+4}{x^4+4x^2+4}$$
$$= 1-\frac{4x^2+4}{(x^2+2)^2}$$

Now consider $\frac{4x^2+4}{(x^2+2)^2}$

Let $\frac{4x^2+4}{(x^2+2)^2} = \frac{Ax+B}{(x^2+2)} + \frac{Cx+D}{(x^2+2)^2}$ ……..(i)

Multiplying both sides by $(x^2+2)^2$, we get

$4x^2+4 = (Ax+B)(x^2+2)+Cx+D$

$4x^2+4 = Ax^3+2Ax+Bx^2+2B+Cx+D$

$4x^2+4 = Ax^3+Bx^2+(2A+C)+2B+D$

By Comparing

co − efficient of x^3, x^2, x and constant terms.

$A = 0$ ……….(ii) (co − efficient of x^3)

$B = 4$ ……….(iii) (co − efficient of x^2)

$0 = 2A + C$ ….(iv) (co − efficient of x)

$4 = 2B + D$ ….(v) (Constant Terms only)

Put $A = 0$ in (iv), we get

$0 = 0 + C$

$C = 0$

Put $B = 4$ in (v), we get

$4 = 2(4) + D$

$4 = 8 + D$

$D = 4 - 8$

$D = 4$

by Putting values of A, B, C, D in (i), we get

$$\frac{4x^2+4}{(x^2+2)^2} = \frac{0+4}{x^2+2} + \frac{0-4}{(x^2+2)^2}$$

$$\frac{4x^2+4}{(x^2+2)^2} = \frac{4}{x^2+2} - \frac{4}{(x^2+2)^2}$$

Thus, $1-\frac{4x^2+4}{(x^2+2)^2}=1-\frac{4}{x^2+2}+\frac{4}{(x^2+2)^2}$

are the required Partial fractions

6. $\frac{x^5}{(x^2+1)^2}$

Solution: $=\frac{x^5}{(x^2+1)^2}$

$=\frac{x^5}{x^4+2x^2+1}$

By long division, we have

$$\begin{array}{r|l} & x \\ x^4+2x^2+1 & x^5 \\ & \underline{+x^5+2x^3+x} \\ & -2x^3-x \\ & -(4x^3+x) \end{array}$$

$=x-\frac{2x^2+x}{x^4+2x^2+1}$

$=x-\frac{2x^3+x}{(x^2+1)^2}$

$=x-\frac{2x^3+x}{(x^2+1)^2}$

Now consider

$=\frac{2x^2+x}{(x^2+1)^2}$

let $\frac{2x^2+x}{(x^2+1)^2}=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{(x^2+1)^2}$ ……… (K)

Multiplying both sides by $(x^2+1)^2$ *, we get*

$2x^3+x=(Ax+B)(x^2+1)+Cx+D$

$2x^3+x=Ax^3+Ax+Bx^2+B+Cx+D$

$2x^3+x=Ax^3+Bx^2+(A+C)x+B+D$

By comparing co-efficients of x^3, x^2, x *and constant terms.*

$\boxed{A=2}$ ……….. (i)

$B=0$ ……….. (ii)

$A+C=1$ ……. (iii)

$B+D=0$ ……. (iv)

Put $A=2$ *in (iii)*

$2 + C = 1$

$\boldsymbol{C = -1}$

Put $B = 0$ *in* (iv)

$0 + D = 0$

$D = 0$

Putting values of A, B, C, D *in* (K)

$$\frac{2x^3 + x}{(x^2 + 1)^2} = \frac{2x + 0}{x^2 + 1} + \frac{-1x + 0}{(x^2 + 1)^2}$$

$$\frac{2x^3 + x}{(x^2 + 1)^2} = \frac{2x}{x^2 + 1} - \frac{x}{(x^2 + 1)^2}$$

Thus, $$x - \frac{2x^3 + x}{(x^2 + 1)^2} = x - \left(\frac{2x}{x^2 + 1} - \frac{x}{(x^2 + 1)^2}\right)$$

$$= x - \frac{2x}{x^2 + 1} + \frac{x}{(x^2 + 1)^2}$$

are the required Partial fractions