Exercise 1.4

1. Find the three cube roots of:

i) 8

Let Z be cube root of 8 then $z=(8)^{1/3}$

$\Rightarrow z^3=8$

$z^3-8=0$

$z^3-2^3=0$

$(z-2)(z^2+2z+4)=0$

$z-2=0 \quad , z^2+2z+4=0$

$z=2 \quad , z^2+2z+4=0$

$z=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$z=\dfrac{-2\pm\sqrt{2^2-4(1)(4)}}{2(1)}=\dfrac{-2\pm\sqrt{4-16}}{2}$

$z=\dfrac{-2\pm\sqrt{-12}}{2}=\dfrac{-2\pm\sqrt{4(-3)}}{2}$

$z=\dfrac{-2\pm2\sqrt{-3}}{2}=2\left(\dfrac{-1\pm\sqrt{-3}}{2}\right)$

Hence

$z=2 , z=2\left(\dfrac{-1+\sqrt{-3}}{2}\right), z=2\left(\dfrac{-1-\sqrt{-3}}{2}\right)$

$z=2 , z=2\omega, z=2\omega^2$

S.S= $\{2,2\omega,2\omega^2\}$

ii) −8

Sol: Let Z be cube root of −8 then $z=(-8)^{1/3}$

$\Rightarrow z^3=-8$

$z^3+8=0$

$z^3+2^3=0$

$(z+2)(z^2-2z+4)=0$

$z+2=0 \quad , z^2-2z+4=0$

$z=-2 \quad , z^2-2z+4=0$

$z=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}=\dfrac{-(-2)\pm\sqrt{(-2)^2-4(1)(4)}}{2(1)}$

$z=\dfrac{-(-2)\pm\sqrt{4-16}}{2}=\dfrac{-(-2)\pm\sqrt{-12}}{2}$

$z=\dfrac{-(-2)\pm\sqrt{4(-3)}}{2}=\dfrac{-(-2)\pm2\sqrt{-3}}{2}$

$z=-2\left(\dfrac{-1\mp\sqrt{-3}}{2}\right)$

Hence

$z=-2 , z=-2\left(\dfrac{-1-\sqrt{-3}}{2}\right), z=-2\left(\dfrac{-1+\sqrt{-3}}{2}\right)$

$z=-2 , z=-2\omega, z=-2\omega^2$

S.S.= $\{-2,-2\omega,-2\omega^2\}$

iii) −27

Sol: Let Z be cube root of −27 then $z=(-27)^{1/3}$

$\Rightarrow z^3=-27$

$z^3+27=0$

$z^3+3^3=0$

$(z+3)(z^2-3z+9)=0$

$z+3=0 \quad , z^2-3z+9=0$

$z=-3 \quad , z^2-3z+9=0$

$z=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}=\dfrac{-(-3)\pm\sqrt{(-3)^2-4(1)(9)}}{2(1)}$

$z=\dfrac{-(-3)\pm\sqrt{9-36}}{2}=\dfrac{-(-3)\pm\sqrt{-27}}{2}$

$z=\dfrac{-(-3)\pm\sqrt{9(-3)}}{2}=\dfrac{-(-3)\pm3\sqrt{-3}}{2}$

$z=-3\left(\dfrac{-1\mp\sqrt{-3}}{2}\right)$

Hence

$z=-3 , z=-3\left(\dfrac{-1-\sqrt{-3}}{2}\right), z=-3\left(\dfrac{-1+\sqrt{-3}}{2}\right)$

$z=-3 , z=-3\omega, z=-3\omega^2$

S.S.= $\{-3,-3\omega,-3\omega^2\}$

iv) 64

Sol: Let Z be cube root of 64 then $z=(64)^{\frac{1}{3}}$

$\Rightarrow z^3=64$

$z^3-64=0$

$z^3-4^3=0$

$(z-4)(z^2+4z+16)=0$

$z-4=0 \quad , z^2+4z+16=0$

$z=4 \quad , z^2+4z+16=0$

$$z=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-4\pm\sqrt{4^2-4(1)(16)}}{2(1)}$$

$$z=\frac{-4\pm\sqrt{16-64}}{2}=\frac{-4\pm\sqrt{-48}}{2}$$

$$z=\frac{-4\pm\sqrt{16(-3)}}{2}=\frac{-4\pm4\sqrt{-3}}{2}=4\left(\frac{-1\pm\sqrt{-3}}{2}\right)$$

Hence

$$z=4\ ,z=4\left(\frac{-1+\sqrt{-3}}{2}\right),z=4\left(\frac{-1-\sqrt{-3}}{2}\right)$$

$z=4\ ,z=4\omega, z=4\omega^2$

S.S.$=\{4,4\omega,4\omega^2\}$

v) -125

Sol: Let Z be cube root of -125 then $z=(-125)^{1/3}$

$z^3=-125$

$z^3+125=0$

$z^3+5^3=0$

$(z+5)(z^2-5z+25)=0$

$z+5=0 \quad , z^2-5z+25=0$

$z=-5 \quad , z^2-5z+25=0$

$$z=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-(-5)\pm\sqrt{(-5)^2-4(1)(25)}}{2(1)}$$

$$z=\frac{-(-5)\pm\sqrt{25-100}}{2}=\frac{-(-5)\pm\sqrt{-75}}{2}$$

$$z=\frac{-(-5)\pm\sqrt{25(-3)}}{2}=\frac{-(-5)\pm5\sqrt{-3}}{2}$$

$$z=-5\left(\frac{-1\mp\sqrt{-3}}{2}\right)$$

Hence

$$z=-5\ ,z=-5\left(\frac{-1-\sqrt{-3}}{2}\right),z=-5\left(\frac{-1+\sqrt{-3}}{2}\right)$$

$z=-5\ ,z=-5\omega, z=-5\omega^2$

S.S.$=\{-5,-5\omega,-5\omega^2\}$

2. Find the four fourth roots of $16, 81, 625$. Also show that their sum is zero in each case.

i) **16**

Sol: Let Z be fourth root of 16 then $z=(16)^{1/4}$

$\Rightarrow z^4=16$

$z^4-16=0$

$(z^2)^2-(4)^2=0$

$(z^2-4)(z^2+4)=0$

$z^2-4=0 \quad , \quad z^2+4=0$

$z^2=4 \quad , \quad z^2=-4$

$z=\pm2 \quad , \quad z=\pm2i$

S.S.$=\{2,-2,2i,-2i\}$

Sum of roots $=2+(-2)+2i+(-2i)$

$=\cancel{2}-\cancel{2}+\cancel{2i}-\cancel{2i}=0$

ii) **81**

Sol: Let Z be fourth root of 81 then $z=(81)^{1/4}$

$\Rightarrow z^4=81$

$z^4-81=0$

$(z^2)^2-(9)^2=0$

$(z^2-9)(z^2+9)=0$

$z^2-9=0, z^2+9=0$

$z^2=9, z^2=-9$

$z=\pm3\ , z=\pm3i$

S.S.$=\{3,-3,3i,-3i\}$

Sum of roots $=3+(-3)+3i+(-3i)$

$=\cancel{3}-\cancel{3}+\cancel{3i}-\cancel{3i}=0$

iii) **625**

Sol: Let Z be fourth root of 625 then $z=(625)^{\frac{1}{4}}$

$z^4=625$

$z^4-625=0$

$(z^2)^2-(25)^2=0$

$(z^2-25)(z^2+25)=0$

$z^2-25=0 \quad ,z^2+25=0$

$z^2=25 \quad ,z^2=-25$

$z=\pm5 \quad , \quad z=\pm5i$

S.S.$=\{5,-5,5i,-5i\}$

Sum of roots $=5+(-5)+5i+(-5i)$

$=\cancel{5}-\cancel{5}+\cancel{5i}-\cancel{5i}=0$

3. **If $1, \omega, \omega^2$ are cube roots of unity, show that $1+\omega^n+\omega^{2n}=3$; *where* n is multiple of 3 respectively.**

Sol: Here n is multiple of 3. $\therefore n = 3k\ ; k \in Z^+$

$$\text{L.H.S.} = 1+\omega^n+\omega^{2n}$$
$$= 1+\omega^{3k}+\omega^{2(3k)}$$
$$= 1+\omega^{3k}+\omega^{3(2k)} = 1+(\omega^3)^k+(\omega^3)^{2k}$$
$$= 1+(1)^k+(1)^{2k} \qquad (\omega^3=1)$$
$$= 1+1+1 = 3 = \text{R.H.S.}$$

4. **Evaluate**

i) $$\left(\frac{-1+\sqrt{-3}}{2}\right)^7+\left(\frac{-1-\sqrt{-3}}{2}\right)^7$$

Sol $$\left(\frac{-1+\sqrt{-3}}{2}\right)^7+\left(\frac{-1-\sqrt{-3}}{2}\right)^7$$

$$= \omega^7+(\omega^2)^7 = \omega^7+\omega^{14}$$
$$= \omega^6.\omega+\omega^{12}.\omega^2 = (\omega^3)^2.\omega+(\omega^3)^4.\omega^2 = 1.\omega+1.\omega^2$$
$$= \omega+\omega^2 = -1 \qquad (1+\omega+\omega^2=0 \Rightarrow \omega+\omega^2=-1)$$

ii) $$\left(-1+\sqrt{-3}\right)^5+\left(-1-\sqrt{-3}\right)^5$$

Sol $$\left(-1+\sqrt{-3}\right)^5+\left(-1-\sqrt{-3}\right)^5$$

if $\omega = \dfrac{-1+\sqrt{-3}}{2} \Rightarrow 2\omega = -1+\sqrt{-3}$ and $2\omega^2 = -1-\sqrt{-3}$

We get

$$= (2\omega)^5+(2\omega^2)^5 = 32\omega^5+32\omega^{10} = 32(\omega^3.\omega^2+\omega^9.\omega)$$
$$= 32(1.\omega^2+1.\omega) = 32(\omega^2+\omega) = 32(-1) = -32$$

5. **Show that $(1-\omega+\omega^2)(1-\omega^2+\omega^4)(1-\omega^4+\omega^8)(1-\omega^8+\omega^{16})\ldots to\ 2n\ factors = 2^{2n}$**

Sol: $$\text{L.H.S.} = (1-\omega+\omega^2)(1-\omega^2+\omega^4)(1-\omega^4+\omega^8)(1-\omega^8+\omega^{16})\ldots 2n\ factors$$
$$= (1-\omega+\omega^2)(1-\omega^2+\omega^3.\omega)(1-\omega^3.\omega+\omega^6.\omega^2)(1-\omega^6.\omega^2+\omega^{15}.\omega)\ldots 2n\ factors$$
$$= (1-\omega+\omega^2)(1-\omega^2+1.\omega)(1-1.\omega+1.\omega^2)(1-1.\omega^2+1.\omega)\ldots 2n\ factors$$
$$= (1-\omega+\omega^2)(1-\omega^2+\omega)(1-\omega+\omega^2)(1-\omega^2+\omega)\ldots 2n\ factors$$
$$= \left[(1-\omega+\omega^2)(1-\omega^2+\omega)\right]\left[(1-\omega+\omega^2)(1-\omega^2+\omega)\right]\ldots n\ factors \quad \text{(Combining two like terms)}$$
$$= \left[(1-\omega+\omega^2)(1-\omega^2+\omega)\right]^n \qquad (1+\omega+\omega^2=0 \Rightarrow 1+\omega^2=-\omega \Rightarrow 1+\omega=-\omega^2)$$
$$= \left[(1+\omega^2-\omega)(1+\omega-\omega^2)\right]^n = \left[(-\omega-\omega)(-\omega^2-\omega^2)\right]^n$$
$$= \left[(-2\omega)(-2\omega^2)\right]^n = \left[4\omega^3\right]^n = \left[4(1)\right]^n = [4]^n = \left[2^2\right]^n = 2^{2n} = \text{R.H.S.}$$

6. **Prove that** $\left(\frac{i+\sqrt{3}}{2}\right)^8+\left(\frac{i-\sqrt{3}}{2}\right)^8=-1$

Sol: $\text{L.H.S.}=\left(\frac{i+\sqrt{3}}{2}\right)^8+\left(\frac{i-\sqrt{3}}{2}\right)^8=\left(\frac{i}{i}\cdot\left(\frac{i+\sqrt{3}}{2}\right)\right)^8+\left(\frac{i}{i}\cdot\left(\frac{i-\sqrt{3}}{2}\right)\right)^8=\left(\frac{i^2+i\sqrt{3}}{2i}\right)^8+\left(\frac{i^2-i\sqrt{3}}{2i}\right)^8$

$$=\left(\frac{-1+i\sqrt{3}}{2i}\right)^8+\left(\frac{-1-i\sqrt{3}}{2i}\right)^8=\left(\frac{2\omega}{2i}\right)^8+\left(\frac{2\omega^2}{2i}\right)^8=\left(\frac{\omega}{i}\right)^8+\left(\frac{\omega^2}{i}\right)^8=\frac{\omega^8}{i^8}+\frac{\omega^{16}}{i^8}$$

$$=\frac{1}{i^8}\left(\omega^8+\omega^{16}\right)=\frac{1}{(i^2)^4}\left(\omega^6.\omega^2+\omega^{15}.\omega\right)=\frac{1}{(-1)^4}\left(1.\omega^2+1.\omega\right)=\frac{1}{1}\left(\omega^2+\omega\right)=(-1)=-1=\text{R.H.S.}$$

7. **Evaluate** $\sum_{k=0}^{5}\omega^{2k}$**, *where* ω is an imaginary cube root of unity.**

Sol: $\sum_{k=0}^{5}w^{2k}$

$$=\omega^{2(0)}+\omega^{2(1)}+\omega^{2(2)}+\omega^{2(3)}+\omega^{2(4)}+\omega^{2(5)}$$
$$=\omega^0+\omega^2+\omega^4+\omega^6+\omega^8+\omega^{10}$$
$$=1+\omega^2+\omega^3.\,\omega+1+\omega^6.\,\omega^2+\omega^9.\omega$$
$$=1+\omega^2+1.\omega+1+1.\omega^2+1.\omega \qquad \left(\omega^3=\omega^6=\omega^9=1\right)$$
$$=1+\omega^2+\omega+1+\omega^2+\omega=\left(1+\omega+\omega^2\right)+\left(1+\omega+\omega^2\right)$$
$$=0+0=0 \qquad \left(1+\omega+\omega^2=0\right)$$

8. **If ω is an imaginary cube root of unity, prove that** $\frac{a+b\omega^2+c\omega}{a\omega^2+b\omega+c}=\omega$

Sol: If ω is an imaginary cube root of unity,

$$\omega^3=1$$
$$\omega^2.\omega=1$$
$$\omega^2=\frac{1}{\omega}$$

$$\text{L.H.S.}=\frac{a+b\omega^2+c\omega}{a\omega^2+b\omega+c}$$

$$=\frac{a+b\omega^2+c\omega}{a\left(\frac{1}{\omega}\right)+b\omega+c} \qquad \text{(In denominator replace } \omega^2 \text{ by } \frac{1}{\omega}\text{)}$$

$$=\frac{a+b\omega^2+c\omega}{\frac{a}{\omega}+b\omega+c}=\frac{\left(a+b\omega^2+c\omega\right)}{\frac{\left(a+b\omega^2+c\omega\right)}{\omega}}=\omega=\text{R.H.S.}$$

Alternate Method:

$$\text{L.H.S.}=\frac{a+b\omega^2+c\omega}{a\omega^2+b\omega+c}$$

$$=\frac{\left(a+b\omega^2+c\omega\right)\omega}{\left(a\omega^2+b\omega+c\right)\omega}$$ (Multiply and divide by ω)

$$=\frac{\left(a+b\omega^2+c\omega\right)\omega}{a\omega^3+b\omega^2+c\omega}=\frac{\left(a+b\omega^2+c\omega\right)\omega}{a(1)+b\omega^2+c\omega}=\frac{\left(a+b\omega^2+c\omega\right)\omega}{a+b\omega^2+c\omega}=\omega=R.H.S$$

9. If ω is a cube root of unity, prove that $\frac{a\omega^{12}+b\omega^{17}+c\omega^{19}}{a\omega^{14}+b\omega^{22}+c\omega^{30}}=\omega$

Sol: If ω is an imaginary cube root of unity,

$$\omega^3=1$$

$$\omega^2.\omega=1$$

$$\omega^2=\frac{1}{\omega}$$

> **نوٹ**
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> ω کی power میں وہ نمبر لائیں جو 3 پر تقسیم ہوتا ہے

$$\text{L.H.S.}=\frac{a\omega^{12}+b\omega^{17}+c\omega^{19}}{a\omega^{14}+b\omega^{22}+c\omega^{30}}=\frac{a\omega^{12}+b\omega^{15}.\omega^2+c\omega^{18}.\omega}{a\omega^{12}.\omega^2+b\omega^{21}.\omega+c\omega^{30}}=\frac{a(1)+b(1).\omega^2+c(1).\omega}{a(1).\omega^2+b(1).\omega+c(1)}$$

$$=\frac{a+b\omega^2+c\omega}{a\left(\frac{1}{\omega}\right)+b\omega+c}$$ (In denominator replace ω^2 by $\frac{1}{\omega}$)

$$=\frac{a+b\omega^2+c\omega}{\frac{a}{\omega}+b\omega+c}=\frac{\left(a+b\omega^2+c\omega\right)}{\frac{\left(a+b\omega^2+c\omega\right)}{\omega}}=\omega=\text{R.H.S.}$$