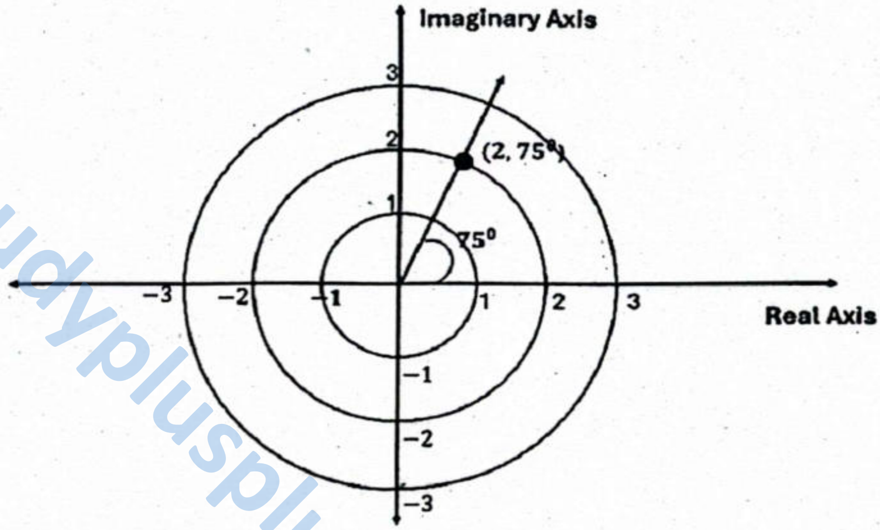
1. Plot the following Points:

i) $(2, 75^\circ)$

Sol: Here $r = 2$ and $\theta = 75^\circ$

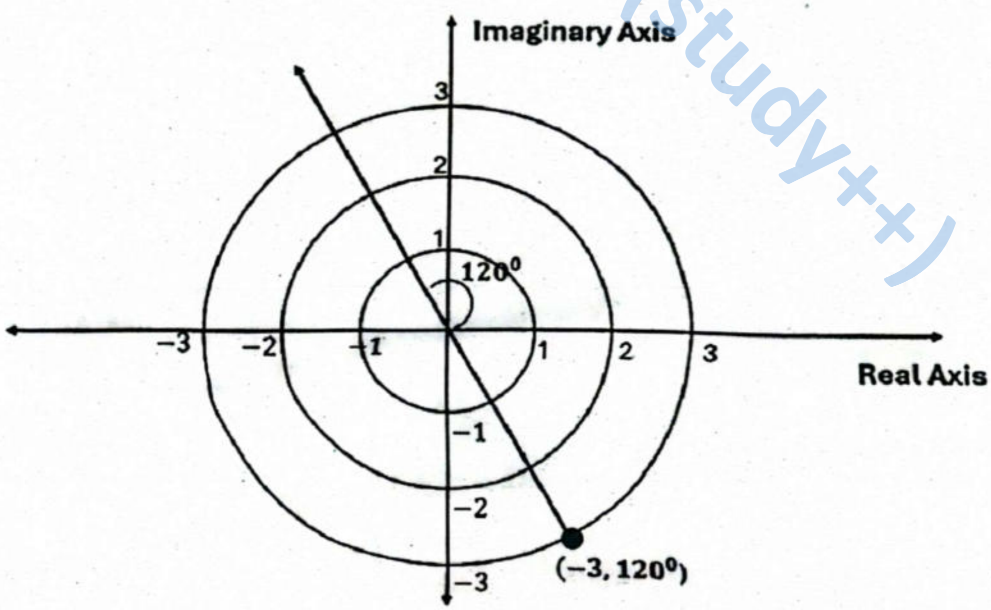
Process: take $r = 2$ on positive x-axis and rotate it till 75° anticlockwise.



ii) $(-3, 120^\circ)$

Sol: Here $r = -3$ and $\theta = 120^\circ$

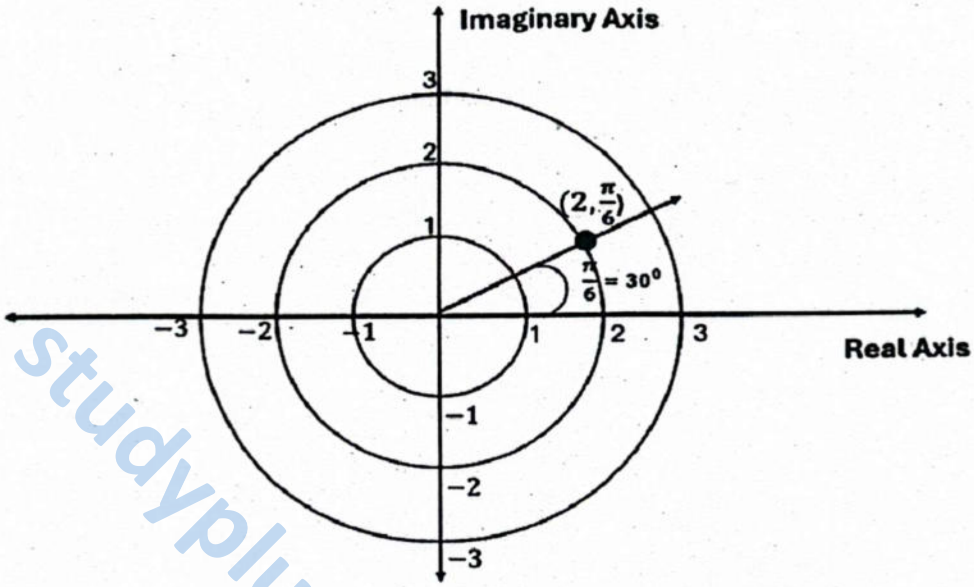
Process: take $r = -3$ at on negative x-axis and rotate further 120° anticlockwise.



iii) $\left(2, \frac{\pi}{6}\right)$

Sol: Here $r = 2$ and $\theta = \frac{\pi}{6} = 30^\circ$

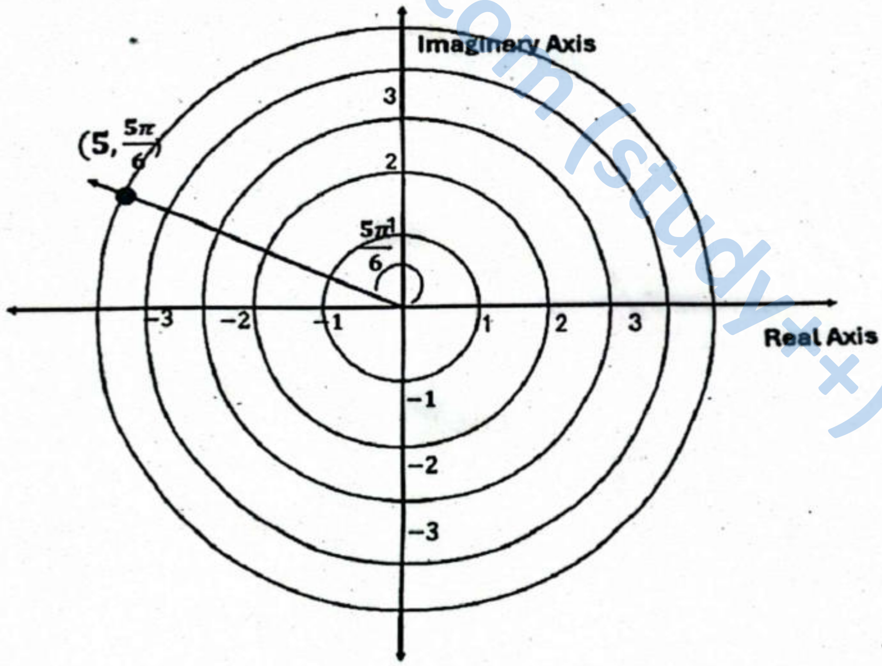
Process: take $r = 2$ on positive x-axis and rotate anticlockwise till $\theta = 30^\circ$



iv) $\left(5, \frac{5\pi}{6}\right)$

Sol: Here $r = 5$ and $\theta = \frac{5\pi}{6} = 150^\circ$

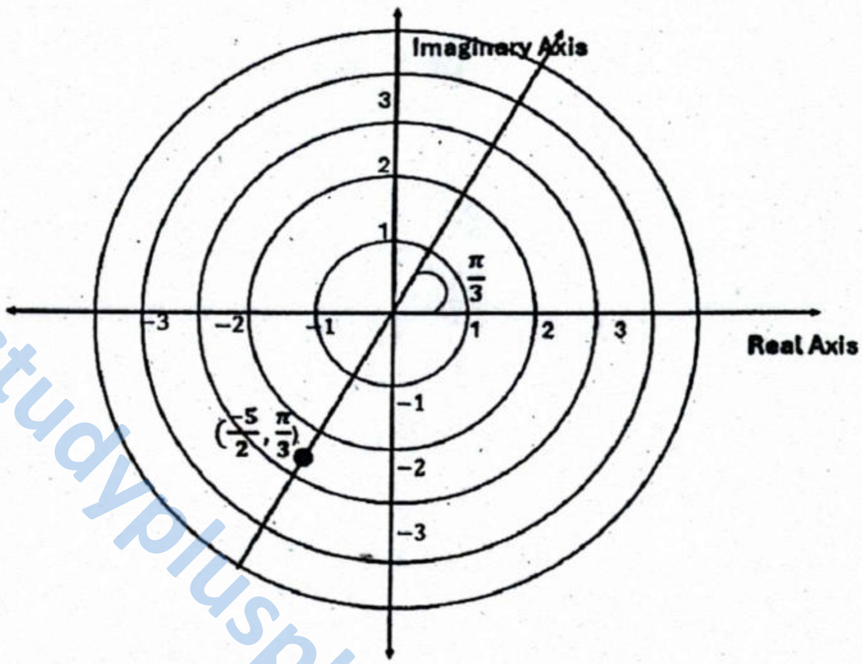
Process: take $r = 5$ on positive x-axis and rotate till $\theta = 150^\circ$ anticlockwise



v) $\left(\frac{-5}{2}, \frac{\pi}{3}\right)$

Sol: Here $r = \frac{-5}{2} = -2.5$ and $\theta = \frac{\pi}{3} = 60°$

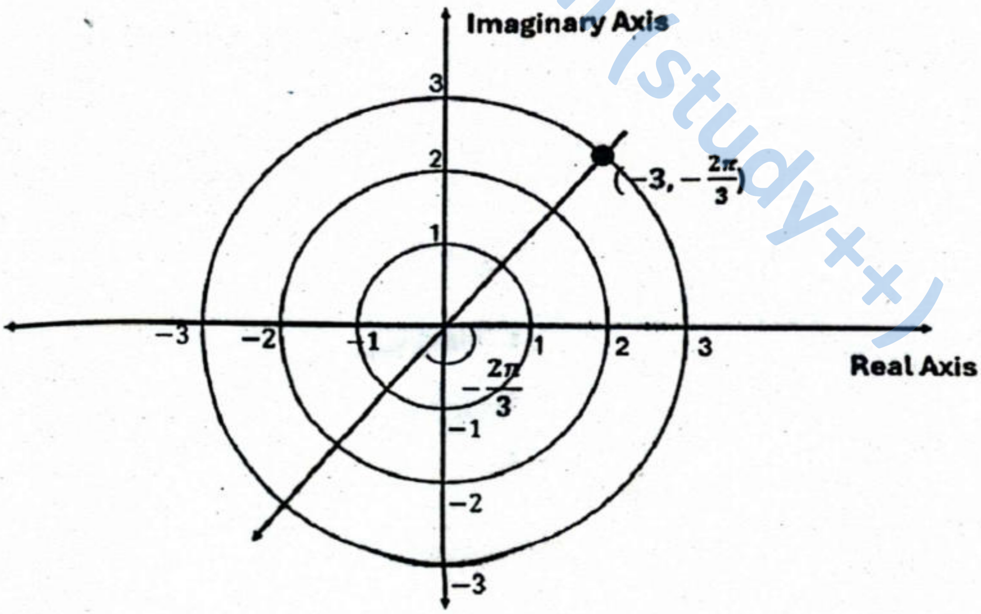
Process: take $r = -2.5$ on negative x-axis and rotate $60°$ anticlockwise



vi) $\left(-3, \frac{-2\pi}{3}\right)$

Sol: Here $r = -3$ and $\theta = \frac{-2\pi}{3} = -120°$

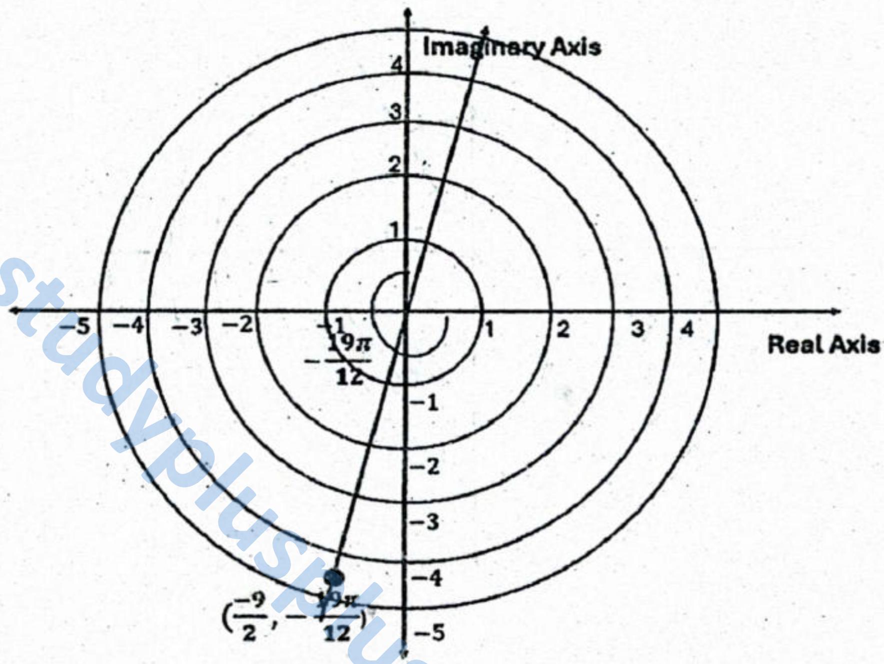
Process: take $r = -3$ on negative x-axis and rotate $120°$ clockwise



vii) $\left(\frac{-9}{2}, \frac{-19\pi}{12}\right)$

Sol: Here $r = \frac{-9}{2} = -4.5$ *and* $\theta = \frac{-19\pi}{12} = \frac{-19(180°)}{12} = -285°$

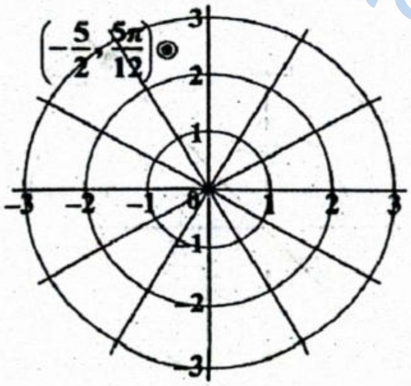
Process: take $r = -4.5$ on negative x-axis and rotate Clockwise till $285°$



viii) $\left(\frac{-5}{2}, \frac{5\pi}{12}\right)$

Sol: Here $r = \frac{-5}{2} = -2.5$ *and* $\theta = \frac{5\pi}{12} = 75°$

Process: take $r = -2.5$ on negative x-axis and rotate $75°$ anticlockwise



2. **Express the following complex numbers in polar form.**

i) $\mathbf{4+3i}$

Sol Here $x=4, y=3$

$$r=\sqrt{x^2+y^2}=r=\sqrt{4^2+3^2}=\sqrt{16+9}=\sqrt{25}=5$$

$$\theta=\tan^{-1}\left(\frac{y}{x}\right)=tan^{-1}\left(\frac{3}{4}\right)\Rightarrow\theta=36.86^\circ$$

(θ is in quad I because x is +ve & y is +ve)

In polar form, $z=r(\cos\theta+i\sin\theta)$

$$z=5(\cos 36.86^\circ+i\sin 36.86^\circ)$$

ii) $\mathbf{1+i}$

Sol Here $x=1, y=1$

$$r=\sqrt{x^2+y^2}=r=\sqrt{1^2+1^2}=\sqrt{1+1}=\sqrt{2}$$

$$\theta=\tan^{-1}\frac{y}{x}=tan^{-1}\left(\frac{1}{1}\right)\Rightarrow\theta=tan^{-1}(1)=\frac{\pi}{4}$$

(θ is in quad I because x is +ve & y is +ve)

In polar form, $z=r(\cos\theta+i\sin\theta)$

$$=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$$

iii) $\mathbf{\frac{1}{2}+\frac{\sqrt{3}}{2}i}$

Sol Here $x=\frac{1}{2}, y=\frac{\sqrt{3}}{2}$

$$r=\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=r=\sqrt{\frac{1}{4}+\frac{3}{4}}=\sqrt{\frac{1+3}{4}}=\sqrt{\frac{4}{4}}=\sqrt{1}=1$$

$$\theta=\tan^{-1}\left(\frac{y}{x}\right)=tan^{-1}\left(\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right)\Rightarrow\theta=tan^{-1}(\sqrt{3})=\frac{\pi}{3}$$

(θ is in quad I because x is +ve & y is +ve)

In polar form, $z=r(\cos\theta+i\sin\theta)$

$$=1\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$$

iv) $\mathbf{-\frac{5}{2}-\frac{5\sqrt{3}}{2}i}$

Sol Here $Here\, x=-\frac{5}{2}, y=-\frac{5\sqrt{3}}{2}$

$$r=\sqrt{\left(-\frac{5}{2}\right)^2+\left(-\frac{5\sqrt{3}}{2}\right)^2}=r=\sqrt{\frac{25}{4}+\frac{75}{4}}=\sqrt{\frac{25+75}{4}}=\sqrt{\frac{100}{4}}=\frac{10}{2}=5$$

$$\theta=\pi+\tan^{-1}\left(\frac{\frac{-5\sqrt{3}}{2}}{\frac{-5}{2}}\right)$$

(θ is in quad III because x is -ve & y is -ve)

$$\theta=\pi+\tan^{-1}\left(\sqrt{3}\right)=\pi+\frac{\pi}{3}=\frac{3\pi+\pi}{3}=\frac{4\pi}{3}$$

In polar form, $z=r\left(\cos\theta+i\sin\theta\right)$

$$=5\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right)$$

v) $\frac{1-i}{1+i}$

Sol: $=\frac{1-i}{1+i}\cdot\frac{1-i}{1-i}=\frac{(1-i)^2}{1^2-i^2}=\frac{1+i^2-2i}{1-(-1)}=\frac{1+(-1)-2i}{1+1}=\frac{-2i}{2}=-i=0-i$

Here $x=0, y=-1$

$$r=\sqrt{(0)^2+(-1)^2}=\sqrt{0+1}=\sqrt{1}=1$$

$$\theta=\tan^{-1}\left(\frac{-1}{0}\right)$$

$$\theta=\tan^{-1}(-\infty)=-\frac{\pi}{2}$$

In polar form, $z=r\left(\cos\theta+i\sin\theta\right)$

$$=1\left(\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)\right)=\cos\frac{\pi}{2}-i\sin\frac{\pi}{2}$$

vi) $\frac{\sqrt{3}+i}{1+\sqrt{3}i}$

Sol: $=\frac{\sqrt{3}+i}{1+\sqrt{3}i}\cdot\frac{1-\sqrt{3}i}{1-\sqrt{3}i}=\frac{\left(\sqrt{3}+i\right)\left(1-\sqrt{3}i\right)}{1^2-\left(\sqrt{3}i\right)^2}=\frac{\sqrt{3}-\left(\sqrt{3}\right)^2 i+i-\sqrt{3}i^2}{1-(-3)}$

$$=\frac{\sqrt{3}-3i+i-\sqrt{3}(-1)}{1+3}=\frac{\sqrt{3}-2i+\sqrt{3}}{2}=\frac{2\sqrt{3}-2i}{2}=\frac{\cancel{2}\left(\sqrt{3}-i\right)}{\cancel{2}}=\sqrt{3}-i$$

Here $x=\sqrt{3}, y=-1$

$$r=\sqrt{\left(\sqrt{3}\right)^2+(-1)^2}\quad r=\sqrt{3+1}=\sqrt{4}=2$$

$\theta=-\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=\frac{-\pi}{6}$ (θ is in quad IV because x is +ve & y is –ve)

In polar form $z=r\left(\cos\theta+i\sin\theta\right)=2\left(\cos\left(\frac{-\pi}{6}\right)+i\sin\left(\frac{-\pi}{6}\right)\right)=2\left(\cos\left(\frac{\pi}{6}\right)-i\sin\left(\frac{\pi}{6}\right)\right)$

vii) $\dfrac{3+4i}{4+3i}$

Sol: $=\dfrac{3+4i}{4+3i}\cdot\dfrac{4-3i}{4-3i}=\dfrac{(3+4i)(4-3i)}{4^2-(3i)^2}$

$$=\frac{12-9i+16i-12i^2}{16-(-9)}=\frac{12+7i-(-12)}{16+9}=\frac{12+7i+12}{25}=\frac{24+7i}{25}=\frac{24}{25}+\frac{7}{25}i$$

Here $x=\dfrac{24}{25}, y=\dfrac{7}{25}$

$$r=\sqrt{\left(\frac{24}{25}\right)^2+\left(\frac{7}{25}\right)^2}$$

$$r=\sqrt{\frac{576}{625}+\frac{49}{625}}=\sqrt{\frac{576+49}{625}}=\sqrt{\frac{625}{625}}=\sqrt{1}=1$$

$$\theta=\tan^{-1}\left(\frac{\frac{7}{25}}{\frac{24}{25}}\right)\Rightarrow\theta=\tan^{-1}\left(\frac{7}{24}\right)$$

(θ is in quad I because x is +ve & y is +ve)

In polar form, $z=r(\cos\theta+i\sin\theta)$

$$=1\left(\cos\left(\tan^{-1}\left(\frac{7}{24}\right)\right)+i\sin\left(\tan^{-1}\left(\frac{7}{24}\right)\right)\right)$$

$$=\cos\left(\tan^{-1}\left(\frac{7}{24}\right)\right)+i\sin\left(\tan^{-1}\left(\frac{7}{24}\right)\right)$$

3. Convert each of the complex number z in the rectangular form $x+iy$

i) $4\left(\cos\dfrac{5\pi}{3}+i\sin\dfrac{5\pi}{3}\right)$

Sol: $=4\left(\dfrac{1}{2}+i\left(-\dfrac{\sqrt{3}}{2}\right)\right)=4\dfrac{1}{2}-i4.\dfrac{\sqrt{3}}{2}=2-i2\sqrt{3}$

ii) $\dfrac{3}{2}\left(\cos\dfrac{7\pi}{6}+i\sin\dfrac{7\pi}{6}\right)$

Sol: $=\dfrac{3}{2}\left(-\dfrac{\sqrt{3}}{2}+i\left(-\dfrac{1}{2}\right)\right)=-\dfrac{3\sqrt{3}}{4}-\dfrac{3}{4}i$

iii) $|z|=7$, $\arg(z)=\dfrac{23\pi}{12}$

Sol: $r=7$, $\theta=\dfrac{23\pi}{12}$

$Z=r(\cos\theta+i\sin\theta)$

$$Z=7\left(\cos\frac{23\pi}{12}+i\sin\frac{23\pi}{12}\right)=7(0.9659+i(-0.2588))=6.76-1.81i$$

iv) $|z| = 11$, $\arg(z) = -\frac{11\pi}{12}$

Sol: $r = 11$, $\theta = -\frac{11\pi}{12}$

$$Z = r(\cos\theta + i\sin\theta)$$

$$Z = 11\left(\cos\left(-\frac{11\pi}{12}\right) + i\sin\left(-\frac{11\pi}{12}\right)\right)$$

$$= 11(-0.9659 + i(-0.2588)) = -10.63 - 2.85i$$

v) $|z| = \frac{10}{3}$, $\arg(z) = -\frac{17\pi}{12}$

Sol: $r = \frac{10}{3}$, $\theta = -\frac{17\pi}{12}$

$$Z = r(\cos\theta + i\sin\theta)$$

$$Z = \frac{10}{3}\left(\cos\left(-\frac{17\pi}{12}\right) + i\sin\left(-\frac{17\pi}{12}\right)\right)$$

$$= \frac{10}{3}(-0.2588 + i(0.9659)) = -0.8627 + 3.2197i$$

vi) $2\cos(-33) + i2\sin(-33)$

Sol: $= 2(0.8386) + i2(-0.5446))$

$= 1.68 - 1.09i$

4. If $z_1 = 9\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right), z_2 = 5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ then find

i) $z_1 + z_2$

Sol: $= 9\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) + 5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$= 9(-0.707 + i(-0.707)) + 5((0.5 + i(0.866))$

$= -6.36 - (6.36)i + 2.5 + i(4.33)$

$= -6.36 + 2.5 + i(4.33) - (6.36)i$

$= -3.86 - 2.03i$

ii) $z_1 - z_2$

$= 9\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) - 5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$= 9(-0.707 + i(-0.707)) - 5((0.5 + i(0.866))$

$= -6.36 - (6.36)i - 2.5 - i(4.33)$

$= -6.36 - 2.5 + i(-4.33) - (6.36)i$

$= -8.86 - 10.69i$

iii) $z_1 . z_2$

$$= 9\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right).5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 45\left(\text{cis}\frac{5\pi}{4}\right).\left(\text{cis}\frac{\pi}{3}\right)$$

$$= 45\left(\text{cis}(\frac{5\pi}{4} + \frac{\pi}{3}\right) = 45\left(\text{cis}(\frac{15\pi + 4\pi}{12}\right) = 45\left(cis\frac{19\pi}{12}\right) = 45\left(cos\frac{19\pi}{12} + isin\frac{19\pi}{12}\right)$$

iv) $\frac{z_1}{z_2}$

$$= \frac{9\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)}{5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)} = \frac{9}{5}.\frac{\text{cis}\frac{5\pi}{4}}{\text{cis}\frac{\pi}{3}} = \frac{9}{5}\text{cis}\left(\frac{5\pi}{4} - \frac{\pi}{3}\right) = \frac{9}{5}\text{cis}\left(\frac{15\pi - 4\pi}{12}\right)$$

$$= \frac{9}{5}\left(cis\frac{11\pi}{12}\right) = \frac{9}{5}\left(cos\frac{11\pi}{12} + isin\frac{11\pi}{12}\right)$$

5. **If $z_1 = 7\left(\cos\frac{23\pi}{12} + i\sin\frac{23\pi}{12}\right), z_2 = 11\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right)$ then find the following and express the result into $x + iy$ form:**

Sol: $z_1 + z_2$

i) $z_1 + z_2$

$$= 7\left(\cos\frac{23\pi}{12} + i\sin\frac{23\pi}{12}\right) + 11\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right) = 7(0.96 + i(-0.25)) + 11(-0.96 + i(0.25))$$

$$= 6.76 - (1.81)i + \{-10.62 + 2.84i\} = 6.76 - 10.62 + (2.84)i - (1.81)i = -3.86 + 1.03i$$

ii) $z_1 - z_2$

$$= 7\left(\cos\frac{23\pi}{12} + i\sin\frac{23\pi}{12}\right) - 11\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right) = 7(0.96 + i(-0.25)) - 11(-0.96 + i(0.25))$$

$$= 6.76 - (1.81)i + \{10.62 - 2.84i\} = 6.76 + 10.62 - (2.84)i - (1.81)i = 17.38 - 4.65i$$

iii) $z_1 . z_2$

$$= 7\left(\cos\frac{23\pi}{12} + i\sin\frac{23\pi}{12}\right).11\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right) = 77\left(\text{cis}\frac{23\pi}{12}\right).\left(\text{cis}\frac{11\pi}{12}\right) = 77\left(\text{cis}(\frac{23\pi}{12} + \frac{11\pi}{12}\right)$$

$$= 77\left(\text{cis}(\frac{23\pi + 111\pi}{12}\right) = 77\left(cis\frac{34\pi}{12}\right) = 77\left(cis\frac{17\pi}{6}\right) = 77\left(cos\frac{17\pi}{6} + isin\frac{17\pi}{6}\right)$$

iv) $\frac{z_1}{z_2}$

$$= \frac{7\left(\cos\frac{23\pi}{12} + i\sin\frac{23\pi}{12}\right)}{11\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right)} = \frac{7}{11}.\frac{\text{cis}\frac{23\pi}{12}}{\text{cis}\frac{11\pi}{12}} = \frac{7}{11}\text{cis}\left(\frac{23\pi}{12} - \frac{11\pi}{12}\right) = \frac{7}{11}\text{cis}\left(\frac{23\pi - 11\pi}{12}\right)$$

$$= \frac{7}{11}\left(cis\frac{12\pi}{12}\right) = \frac{7}{11}(cos\pi + isin\pi) = \frac{7}{11}(-1 + i(0)) = -\frac{7}{11} + 0i$$

6. If z_1, z_2 are two complex numbers, show that

i) $Arg(z_1 z_2) = Arg(z_1) + Arg(z_2)$ ii) $Arg\left(\frac{z_1}{z_2}\right) = Arg(z_1) - Arg(z_2)$

Sol: Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$Arg\, z_1 = \theta_1$ and $Arg\, z_2 = \theta_2$

(i) $z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1).\ r_2(\cos\theta_2 + i\sin\theta_2)$

$= r_1 r_2(\cos\theta_1 + i\sin\theta_1).\ (\cos\theta_2 + i\sin\theta_2)$

$= r_1 r_2(\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2)$

$= r_1 r_2(\cos\theta_1\cos\theta_2 + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2) + (-1)\sin\theta_1\sin\theta_2)$

$= r_1 r_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2))$

$z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$

$Arg(z_1 z_2) = \theta_1 + \theta_2$

$= Arg(z_1) + Arg(z_2)$

ii) $\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$

$= \frac{r_1}{r_2}.\frac{\cos\theta_1 + i\sin\theta_1}{\cos\theta_2 + i\sin\theta_2}$

$= \frac{r_1}{r_2}.(\cos\theta_1 + i\sin\theta_1).(\cos\theta_2 + i\sin\theta_2)^{-1}$

$= \frac{r_1}{r_2}.(\cos\theta_1 + i\sin\theta_1).(\cos(-\theta_2) + i\sin(-\theta_2))$ By De-Moivre Theorem

$= \frac{r_1}{r_2}.(\cos\theta_1 + i\sin\theta_1).(\cos\theta_2 - i\sin\theta_2)$

$= \frac{r_1}{r_2}.(\cos\theta_1\cos\theta_2 - i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 - i^2\sin\theta_1\sin\theta_2)$

$= \frac{r_1}{r_2}.(\cos\theta_1\cos\theta_2 + i\sin\theta_1\cos\theta_2 - i\cos\theta_1\sin\theta_2 - (-1)\sin\theta_1\sin\theta_2)$

$= \frac{r_1}{r_2}.(\cos\theta_1\cos\theta_2 + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2) + \sin\theta_1\sin\theta_2)$

$= \frac{r_1}{r_2}.(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2))$

$\frac{z_1}{z_2} = \frac{r_1}{r_2}.(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$

$Arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2$

7. Divide $z_1 = 6(\cos 150^0 + i\sin 150^0)$ by $z_2 = 3(\cos 30^0 + i\sin 30^0)$ and express in $x+iy$ form.

Sol: $\frac{z_1}{z_2} = \frac{6(\cos 150^0 + i\sin 150^0)}{3(\cos 30^0 + i\sin 30^0)}$

$\frac{z_1}{z_2} = 2\frac{\text{cis}150^0}{\text{cis}30^0}$

$\frac{z_1}{z_2} = 2cis(150^0 - 30^0)$

$\frac{z_1}{z_2} = 2cis(120^0)$

$= 2(cos120^0 + isin120^0)$

$= 2\left(-\frac{1}{2} + i\left(\frac{\sqrt{3}}{2}\right)\right) = -1 + \sqrt{3}i$

8. Multiply $z_1 = 2(\cos 60^0 + i\sin 60^0)$ and $z_2 = 5(\cos 90^0 + i\sin 90^0)$ and express in $x+iy$ form.

Sol: $z_1.z_2 = 2(\cos 60^0 + i\sin 60^0).5(\cos 90^0 + i\sin 90^0)$

$z_1.z_2 = 10(\text{cis}60^0\,\text{cis}90^0)$

$z_1.z_2 = 10cis(60^0 + 90^0)$

$z_1.z_2 = 10cis(150^0) = 10(cos(150^0) + isin(150^0))$

$= 10\left(-\frac{\sqrt{3}}{2} + i\left(\frac{1}{2}\right)\right) = \frac{10}{2}(-\sqrt{3} + i) = 5(-\sqrt{3} + i) = -5\sqrt{3} + 5i$

9. Find modulus and argument $z = -2 - 2i$

Sol: $z = -2 - 2i$

Here $x = -2, y = -2$

$|z| = \sqrt{x^2 + y^2} = \sqrt{(-2)^2 + (-2)^2}$

$= \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$

$\theta = \pi + tan^{-1}\left(\frac{2}{2}\right)$ (θ is in quad III because x is +ve & y is –ve)

$\theta = \pi + tan^{-1}(1) = \pi + \frac{\pi}{4} = \frac{4\pi + \pi}{4} = \frac{5\pi}{4}$

$\arg(z) = \frac{5\pi}{4} + 2n\pi \; ; n \in Z$

10. Write the equation $\arg(\bar{z} - 2 + i) = \frac{2\pi}{3}$ in Cartesian form if $z = x + iy$.

Sol: $z = x + iy$

$\bar{z} = x - iy$

$\arg(\bar{z} - 2 + i) = \frac{2\pi}{3}$

$$\arg(x-iy-2+i)=\frac{2\pi}{3}$$

$$\arg(x-2-iy+i)=\frac{2\pi}{3}$$

$$\arg(x-2-i(y-1))=\frac{2\pi}{3}$$

$$tan^{-1}\left(\frac{-(y-1)}{x-2}\right)=\frac{2\pi}{3}$$

$$\frac{-(y-1)}{x-2}=\tan\left(\frac{2\pi}{3}\right)$$

$$-\frac{y-1}{x-2}=-\sqrt{3}\Rightarrow\frac{y-1}{x-2}=\sqrt{3}$$

$$y-1=\sqrt{3}(x-2)$$

$$y-1=\sqrt{3}x-2\sqrt{3}$$

$$y=\sqrt{3}x+1-2\sqrt{3}$$

11. **If $z=x+iy$ and $\arg(\frac{\overline{z}-1+2i}{\overline{z}+1-2i})=\frac{9\pi}{4}$, Show that $x^2+y^2-4x+2y-5=0$.**

Sol: $z=x+iy$ and $\overline{z}=x-iy$

> **نوٹ:**
> اس سوال میں reverse چلنا ہے۔

$$\arg(\frac{\overline{z}-1+2i}{\overline{z}+1-2i})=\frac{9\pi}{4}$$

$$\arg(\overline{z}-1+2i)-arg(\overline{z}+1-2i)=\frac{9\pi}{4}$$

$$\arg(x-iy-1+2i)-arg(x-iy+1-2i)=\frac{9\pi}{4}$$

$$\arg(x-1-iy+2i)-arg(x+1-iy-2i)=\frac{9\pi}{4}$$

$$\arg(x-1-i(y-2))-arg(x+1-i(y+2))=\frac{9\pi}{4}$$

$$tan^{-1}\left(\frac{-(y-2)}{x-1}\right)-tan^{-1}\left(\frac{-(y+2)}{x+1}\right)=\frac{9\pi}{4}$$

$$tan^{-1}\left(\frac{2-y}{x-1}\right)+tan^{-1}\left(\frac{2+y}{x+1}\right)=\frac{9\pi}{4}$$

$$tan^{-1}\frac{\frac{2-y}{x-1}+\frac{2+y}{x+1}}{1-\left\{\frac{2-y}{x-1}.\frac{2+y}{x+1}\right\}}=\frac{9\pi}{4}$$

$$\frac{\frac{(x+1)(2-y)+(x-1)(2+y)}{x^2-1}}{1-\frac{4-y^2}{x^2-1}}=\tan\frac{9\pi}{4}$$

$$\frac{\frac{2x - xy + 2 - y + 2x + xy - 2 - y}{x^2-1}}{\frac{x^2-1-(4-y^2)}{x^2-1}} = 1$$

$$\frac{4x-2y}{x^2-1-4+y^2} = 1$$

$$4x-2y = x^2-1-4+y^2$$

$$x^2+y^2-4x+2y-5=0$$

12. **If $z = x + iy$ & $\arg(z-2-3i) - \arg(z+2+3i) = 2\pi$, show that $2y = 3x$.**

Sol: $z = x+iy$

$$\arg(z-2-3i) - \arg(z+2+3i) = 2\pi$$

$$\arg(x+iy-2-3i) - \arg(x+iy+2+3i) = 2\pi$$

$$\arg(x-2+iy-3i) - \arg(x+2+iy+3i) = 2\pi$$

$$\arg(x-2+i(y-3)) - \arg(x+2+i(y+3)) = 2\pi$$

$$\tan^{-1}\left(\frac{y-3}{x-2}\right) - \tan^{-1}\left(\frac{y+3}{x+2}\right) = 2\pi$$

$$\tan^{-1}\frac{\frac{y-3}{x-2} - \frac{y+3}{x+2}}{1+\left\{\frac{y-3}{x-2}\cdot\frac{y+3}{x+2}\right\}} = 2\pi$$

$$\frac{\frac{y-3}{x-2} - \frac{y+3}{x+2}}{1+\left\{\frac{y-3}{x-2}\cdot\frac{y+3}{x+2}\right\}} = \tan 2\pi$$

$$\frac{\frac{y-3}{x-2} - \frac{y+3}{x+2}}{1+\left\{\frac{y-3}{x-2}\cdot\frac{y+3}{x+2}\right\}} = 0$$

$$\frac{y-3}{x-2} - \frac{y+3}{x+2} = 0$$

$$\frac{(x+2)(y-3)-(x-2)(y+3)}{(x-2)(x+2)} = 0$$

$$xy-3x+2y-6-(xy+3x-2y-6)=0$$

$$xy-3x+2y-6-xy-3x+2y+6=0$$

$$-6x+4y=0$$

$$-3x+2y=0$$

$$2y=3x$$

13. **Solve the equation $|z-2i|=|\bar{z}+2|$ for $z=x+iy$.**

Sol: $z=x+iy$ and $\bar{z}=x-iy$

$|z-2i|=|\bar{z}+2|$

$|x+iy-2i|=|x-iy+2|$

$|x+i(y-2)|=|x+2-iy|$

$\sqrt{x^2+(y-2)^2}=\sqrt{(x+2)^2+(-y)^2}$

$x^2+(y-2)^2=(x+2)^2+y^2$

$x^2+y^2-4y+4=x^2+4x+4+y^2$

$-4y=4x \Rightarrow y=-x$

14. **For $z=x+iy$, solve the equation $|5z+4+i|=|5\bar{z}-3+2i|$**

Sol: $z=x+iy$ and $\bar{z}=x-iy$

$|5z+4+i|=|5\bar{z}-3+2i|$

$|5(x+iy)+4+i|=|5(x-iy)-3+2i|$

$|5x+i5y+4+i|=|5x-i5y-3+2i|$

$|5x+4+i5y+i|=|5x-3-5iy+2i|$

$|5x+4+i(5y+1)|=|5x-3-i(5y-2)|$

$\sqrt{(5x+4)^2+(5y+1)^2}=\sqrt{(5x-3)^2+(-(5y-2))^2}$

$(5x+4)^2+(5y+1)^2=(5x-3)^2+(5y-1)^2$ (Squaring both sides)

$25x^2+40x+16+25y^2+10y+1=25x^2-30x+9+25y^2-20y+4$

$40x+30x+10y+20y+16+1-9-4=0$

$70x+30y+4=0 \Rightarrow 35x+15y+2=0 \qquad (\div\ by\ 2)$

15. **Determine the set of points $z=x+iy$, that satisfy the equation $|3\bar{z}-2+i|=|3z+i|$**

Sol: $z=x+iy$ and $\bar{z}=x-iy$

$|3\bar{z}-2+i|=|3z+i|$

$|3(x-iy)-2+i|=|3(x+iy)+i|$

$|3x-i3y-2+i|=|3x+i3y+i|$

$|3x-2-i3y+i|=|3x+3iy+i|$

$|3x-2-i(3y-1)|=|3x+i(3y+1)|$

$\sqrt{(3x-2)^2+(-(3y-1))^2}=\sqrt{(3x)^2+(3y+1)^2}$

$(3x-2)^2+(3y-1)^2=(3x)^2+(3y+1)^2$ (Squaring both sides)

$9x^2-12x+4+9y^2-6y+1=9x^2+9y^2+6y+1$

$-12x-6y-6y+4+1-1=0$

$-12x-12y+4=0$

$3x+3y-1=0 \qquad (\div\ by -4)$

16. **If $z = x + iy$ and $w = \frac{1 - iz}{z - i}$, show that $|w| = 1 \Rightarrow z$ is real.**

Sol: $z = x + iy$, $w = \frac{1 - iz}{z - i}$

Suppose $|w| = 1$

$$\left|\frac{1 - iz}{z - i}\right| = 1$$

$$\frac{|1 - i(x + iy)|}{|x + iy - i|} = 1$$

$$\frac{|1 - ix - i^2 y)|}{|x + i(y - 1)|} = 1$$

$$\frac{|1 - ix - (-1)y)|}{|x + i(y - 1)|} = 1$$

$$\frac{|1 - ix + y)|}{|x + i(y - 1)|} = 1$$

$$\frac{|1 + y - ix|}{|x + i(y - 1)|} = 1$$

$$\frac{\sqrt{(1 + y)^2 + (-x)^2}}{\sqrt{x^2 + (y - 1)^2}} = 1$$

$$1 + 2y + y^2 + x^2 = x^2 + y^2 - 2y + 1$$

$$2y = -2y$$

$$4y = 0$$

$$y = 0$$

$z = x + iy = x + i(0) = x \Rightarrow z$ is real

17. **If z_1, z_2 are two different complex numbers with $|z_1| = 1$, find $\left|\frac{z_1 - z_2}{1 - z_1 z_2}\right|$.**

Note: Some mistakes in statement.

Sol: Here $|z_1| = 1$

$$|z_1|^2 = 1$$

$$z_1 \overline{z_1} = 1$$

$$\overline{z_1} = \frac{1}{z_1}$$

$$\left|\frac{z_1 - z_2}{1 - \overline{z_1} z_2}\right| = \left|\frac{z_1 - z_2}{1 - \frac{1}{z_1} z_2}\right| = \left|\frac{z_1 - z_2}{1 - \frac{z_2}{z_1}}\right| = \left|\frac{z_1 - z_2}{\frac{z_1 - z_2}{z_1}}\right| = \left|(z_1 - z_2) \cdot \frac{z_1}{(z_1 - z_2)}\right| = 1$$

18. An AC source supplies a voltage of $V = 120\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$ volts to a circuit with impedance $Z = \frac{1+i\sqrt{3}}{2}$ ohms. Calculate the current in polar form.

Sol: $Z = \frac{1+\sqrt{3}i}{2}$

$Z = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

Here $r = \sqrt{(\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = r = \sqrt{\frac{1}{4}+\frac{3}{4}} = \sqrt{\frac{1+3}{4}} = \sqrt{\frac{4}{4}} = 1$

$\theta = tan^{-1}\left(\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right) = tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$

(θ is in quad I because x is +ve & y is –ve)

In polar form, $z = r\left(\cos\theta + i\sin\theta\right)$

$= 1\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$$I = \frac{V}{Z} = \frac{120\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)}{1\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)} = \frac{120\left(cis\frac{\pi}{4}\right)}{\left(cis\frac{\pi}{3}\right)} = 120\left(cis\left(\frac{\pi}{4} - \frac{\pi}{3}\right)\right)$$

$$= 120\left(cis\left(\frac{3\pi - 4\pi}{12}\right)\right) = 120\left(cis\left(\frac{-\pi}{12}\right)\right) = 120\left(\cos\left(\frac{-\pi}{12}\right) + i\sin\left(\frac{-\pi}{12}\right)\right) = 120\left(\cos\frac{\pi}{12} - i\sin\frac{\pi}{12}\right)$$

19. An AC circuit has an impedance of $Z = 3 - 6i$ ohms and is connected to a voltage source of $V = 90 + 30i$ volts. Find the current in both rectangular and polar form.

Sol: $Z = 3 - 6i$

$I = \frac{90+30i}{3-6i}$

$$I = \frac{30}{3}.\frac{3+i}{1-2i} = 10.\frac{3+i}{1-2i}.\frac{1+2i}{1+2i} = 10.\frac{(3+i)(1+2i)}{1^2 - (2i)^2} = 10.\frac{3+6i+i+2i^2}{1-(-4)} = 10.\frac{3+7i+2(-1)}{5}$$

$Z = 2\{3+7i-2\} = 2\{1+7i\} = 2+14i$

Here $r = \sqrt{(2+(14)^2} = r = \sqrt{4+196} = \sqrt{200} = 10\sqrt{2}$

$\theta = tan^{-1}\left(\frac{14}{2}\right) = tan^{-1}(7) = 81.86$

(θ is in quad I because x is +ve & y is +ve)

In polar form, $I = r\left(\cos\theta + i\sin\theta\right)$

$= 10\sqrt{2}\left(\cos 81.86 + i\sin 81.86\right)$

20. **Encrypt the word "CODE" by multiplying the complex encryption key $k = 2 - i$. Then decrypt it back to the original word.**

Sol: $A = 1, B = 2, C = 3, \ldots Z = 26$

Here $k = 2 - i$

$C = 3, O = 15, D = 4, E = 5$

Table of Encryption and Decryption is given for 'CODE' as

Letter	Complex Number (Z)	Encryption= z' =Z × k	Decryption=z= z' / k	Letter
C	$3+0i$	$(3+0i)(2-i)=6-3i$	$\frac{6-3i}{2-i}=3+0i$	C
O	$15+0i$	$(15+0i)(2-i)=30-15i$	$\frac{30-15i}{2-i}=15+0i$	O
D	$4+0i$	$(4+0i)(2-i)=8-4i$	$\frac{8-4i}{2-i}=4+0i$	D
E	$5+0i$	$(5+0i)(2-i)=10-5i$	$\frac{10-5i}{2-i}=5+0i$	E

21. **Consider the complex encryption key $k = 3 - 3i$. Encrypt the word "QUIZ", and then recover the original word using the inverse of the key.**

Sol: $A = 1, B = 2, C = 3, \ldots Z = 26$

Here $k = 3 - 3i$

$Q = 17, U = 21, I = 9, E = 26$

Table of Encryption and Decryption is given for 'QUIZ' as

Letter	Complex Number (Z)	Encryption= z' =Z × k	Decryption=z= z' / k	Letter
Q	$17+0i$	$(17+0i)(3-3i)=51-51i$	$\frac{51-51i}{3-3i}=17+0i$	Q
U	$21+0i$	$(21+0i)(3-3i)=63-63i$	$\frac{63-63i}{3-3i}=21+0i$	U
I	$9+0i$	$(9+0i)(3-3i)=27-27i$	$\frac{27-27i}{3-3i}=9+0i$	I
Z	$26+0i$	$(26+0i)(3-3i)=78-78i$	$\frac{78-78i}{3-3i}=26+0i$	Z

22. **Encrypt the word "CLASS" by adding the complex encryption key $k=-3+4i$, Then decrypt it back to the original word.**

Sol: $A=1, B=2, C=3, \ldots Z=26$

Here $k=-3+4i$

$C=3, L=12, A=1, S=19, S=19$

Table of Encryption and Decryption is given for 'CLASS' as

Letter	Complex Number (Z)	Encryption= z' =Z × k	Decryption=z= z' / k	Letter
C	$3+0i$	$(3+0i)(-3+4i)=-9+12i$	$\frac{-9+12i}{-3+4i}=3+0i$	C
L	$12+0i$	$(12+0i)(-3+4i)=-36+48i$	$\frac{-36+48i}{-3+4i}=12+0i$	L
A	$1+0i$	$(1+0i)(-3+4i)=-3+4i$	$\frac{-3+4i}{-3+4i}=1+0i$	A
S	$19+0i$	$(19+0i)(-3+4i)=-57+76i$	$\frac{-57+76i}{-3+4i}=19+0i$	S
S	$19+0i$	$(19+0i)(-3+4i)=-57+76i$	$\frac{-57+76i}{-3+4i}=19+0i$	S