Exercise 2.1

Q1: **Given that** (a) $f(x)=x^2-1$ (b) $f(x)=\sqrt{2x+3}$

(i) **To find $f(-3)$**

Sol: Put $x=-3$ into the function

$$\Rightarrow f(-3)=(-3)^2-1=9-1=8$$

(ii) **To find $f(0)$**

Sol: Put $x=0$ into the function

$$\Rightarrow f(0)=(0)^2-1=0-1=-1$$

(iii) **To find $f(x-2)$**

Sol: Replacing x by $x-2$ into the function

$$\Rightarrow f(x-2)=(x-2)^2-1=x^2-4x+4-1=x^2-4x+3$$

(iv) **To find $f(x^2+3)$**

Sol: Replace x by x^2+3 into the function

$$\Rightarrow f(x^2+3)=(x^2+3)^2-1=x^4+6x^2+9-1=x^4+6x^2+8$$

(b) **Given that $f(x)=\sqrt{2x+3}$**

(i) **To find $f(-3)$**

Sol: Put $x=-3$ into the function

$$\Rightarrow f(-3)=\sqrt{2(-3)+3}=\sqrt{-6+3}=\sqrt{-3}=\pm\sqrt{3}i$$

(ii) **To find $f(0)$**

Sol: Put $x=0$ into the function

$$\Rightarrow f(0)=\sqrt{2(0)+3}=\sqrt{0+3}=\sqrt{3}$$

(iii) **To find $f(x-2)$**

Sol: Put replacing $f(x-2)$ into the function

$$\Rightarrow f(x-2)=\sqrt{2(x-2)+3}=\sqrt{2x-4+3}=\sqrt{2x-1}$$

(iv) **To find $f(x^2+3)$**

Sol: Put $f(x^2+3)$ into the function

$$\Rightarrow f(x^2+3)=\sqrt{2(x^2+3)+3}=\sqrt{2x^2+6+3}=\sqrt{2x^2+9}$$

Q2: **Find $\dfrac{f(a+h)-f(a)}{h}$ and simplify where,**

(i) $f(x)=4x+7$

Sol: Here we have to find $f(a)$ and $f(a+h)$.

As; $f(a)=4a+7$ and $f(a+h)=4(a+h)+7.$

$$f(a+h)=4a+4h+7.$$

$$\frac{f(a+h)-f(a)}{h}=\frac{4a+4h+7-4a-7}{h}=\frac{4h}{h}=4$$

(ii) $f(x)=\sin x$, **Calculating** $f(a)$ **and** $f(a+h)$.

Sol: As; $f(a)=\sin a$ and $f(a+h)=\sin(a+h)$

Now, $\dfrac{f(a+h)-f(a)}{h}=\dfrac{\sin(a+h)-\sin a}{h}$ $\quad\therefore \sin P-\sin Q=2\cos\dfrac{P+Q}{2}\cdot\sin\dfrac{P+Q}{2}$

$$=\frac{2\cos\frac{a+h+a}{2}\cdot\sin\frac{a+h-a}{2}}{h}=\frac{2\cos\left(\frac{2a+h}{2}\right)\cdot\sin\left(\frac{h}{2}\right)}{h}=\frac{2\cos\left(a+\frac{h}{2}\right)\cdot\sin\left(\frac{h}{2}\right)}{h}$$

(iii) $f(x)=x^3+x^2-1$, **Finding** $f(a)$ **and** $f(a+h)$

Sol: As; $f(a)=a^3+a^2-1$

$$f(a+h)=(a+h)^3+(a+h)^2-1=a^3+h^3+3a^2h+3ah^2+a^2+2ah+h^2-1$$

Now, $\dfrac{f(a+h)-f(a)}{h}=\dfrac{a^3+h^3+3a^2h+3ah^2+a^2+2ah+h^2-1-a^3-a^2+1}{h}=\dfrac{h(h^2+3a^2+3ah+2a+h)}{h}$

$$=3a^2+2a+3ah+h+h^2$$

(iv) $f(x)=\tan x$, **Calculating** $f(a)$ **and** $f(a+h)$.

Sol: As; $f(a)=\tan a$ and $f(a+h)=\tan(a+h)$

Now, $\dfrac{f(a+h)-f(a)}{h}=\dfrac{\tan(a+h)-\tan a}{h}$

$$=\frac{1}{h}\left(\frac{\sin(a+h)}{\cos(a+h)}-\frac{\sin a}{\cos a}\right)=\frac{1}{h}\left(\frac{\sin(a+h)\cos a-\cos(a+h)\sin a}{\cos(a+h)\cos a}\right)$$ $\quad\therefore \sin\alpha\cos\beta-\cos\alpha\sin\beta=\sin(\alpha-\beta)$

$$=\frac{1}{h}\left(\frac{\sin(a+h-a)}{\cos(a+h)\cos a}\right)=\frac{1}{h}\left(\frac{\sin h}{\cos(a+h)\cos a}\right)=\frac{\sin h}{h\cos(a+h)\cos a}$$

Q3: Express the following:

a) **The area A of a square as a function of its perimeter P.**

b) **The circumference C of a circle as a function of its area A.**

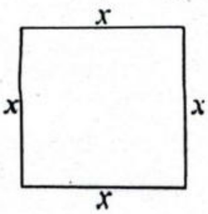
c) **The surface area S of a cube as a function of its volume V.**

(a) Let x be the side of square.

Sol: Then area A will be: $A=x^2\longrightarrow$ (i)

The perimeter (boundary wall distance) of square is given as:

$P=4x\longrightarrow$ (ii)



From equation (ii)

$x=\dfrac{P}{4}\longrightarrow$ (iii)

Putting in equation (i)

$A=\left(\dfrac{P}{4}\right)^2\Rightarrow A=\dfrac{P^2}{16}$.

(b) Let r be the radius of circle.

Sol: The circumference C (Perimeter of circle) is given as

$C=2\pi r\longrightarrow$ (i)

The area of circle is given as:

$A=\pi r^2\longrightarrow$ (ii)

From equation (ii)

$r^2=\dfrac{A}{\pi}\Rightarrow r=\sqrt{\dfrac{A}{\pi}}\longrightarrow$ (iii)

Putting in equation (iii) in equation (i)

$C=2\pi\sqrt{\dfrac{A}{\pi}}=2\sqrt{\pi A}$

(c) Let the length of the side of the cube is l.

Sol: Surface area of the cube is given as.

$S=6(l\times l)=6l^2\longrightarrow$ (i)

The volume of the cube is given as:

$V=l^3\longrightarrow$ (ii)

> **Note:**
> Cube is a diagram whose length, width, height are equal.

from equation (ii)

$$l=(V)^{\frac{1}{3}}\longrightarrow \text{(iii)}$$

Putting in equation (i)

$$S=6\left(V^{\frac{1}{3}}\right)^2=6V^{\frac{2}{3}}=6\sqrt[3]{V^2}$$

Q4: Find the domain and the range of the function g defined below:

(i) $g(x)=5-x$

Sol: The function is real valued so, its domain and range are:

Domain of $g(x)=\mathbb{R}=(-\infty,\infty)=$ (Set of all real numbers)

Range of $g(x)=\mathbb{R}=(-\infty,\infty)=$ (Set of all real numbers)

(ii) $g(x)=\sqrt{x+2}$

Sol: The given function is radical function in which radicand must be positive.

For domain, using the condition as: $x+2\geq 0$

$\Rightarrow \quad x\geq -2 \quad \text{or} \quad x\in[-2,\infty)$

Domain of $g(x)=[-2,\infty)$

For range, $\sqrt{x+2}\geq 0$ as $x+2\geq 0$

Range of $g(x)=[0,\infty)$

(iii) $g(x)=\begin{cases}6x+7, & x\leq -2\\ 4-3x, & x>-2\end{cases}$

Sol: It is a piecewise function and working for two conditions.

When input is $x\leq -2$ or $x\in(\infty,-2]$

Then $g(x)=6x+7$ is working in this case output of the functions are:

$g(-2)=6(-2)+7=-12+7=-5$

$g(-3)=6(-3)+7=-18+7=-11$

Order in output values on real line is less than so, the range for this piece is $(-\infty,-5]$.

When input is $x>-2$ or $x\in(-2,\infty)$

Then $g(x)=4-3x$ is working in this case output of the functions are: $g(-2)=4-3(-2)=4+6=10$

$g(-1)=4-3(-1)=4+3=7$

Order in output values on real line is less than so, the range for this piece is $(-\infty,10)$.

Domain of $g(x)=(-\infty,-2]\cup(-2,\infty)=(-\infty,\infty)=\mathbb{R}$

Range of $g(x)=(-\infty,-5]\cup(-\infty,10)$

Range of $g(x)=(-\infty,10)$

(iv) $g(x)=|x-5|$,

Sol: It is an absolute value function (Whose Range must be the positive real number)

For domain $|x-5|\geq 0 \Rightarrow \pm(x-5)\geq 0$

$x-5\geq 0 \Rightarrow x\geq 5$

$x-5\leq 0 \Rightarrow x\leq 5$

Domain of $g(x)=(-\infty,5]\cup(5,\infty)=(-\infty,\infty)=\mathbb{R}$

Range of $g(x)=(0,\infty)=$ set of all positive real numbers

(v) $g(x)=\dfrac{x+2}{3-x}$

Sol: $g(x)=\dfrac{x+2}{3-x}$

Domain of $g(x)=\mathbb{R}-\{3\}=(-\infty,3)\cup(3,\infty)$

Range of $g(x)=\mathbb{R}-\{-1\}=(-\infty,-1)\cup(-1,\infty)$

Q5: Given $f(x)=x^3-ax^2+bx+1$. If $f(2)=-3$ and $f(-1)=0$ Find the values of a & b

Sol: Given function is $f(x)=x^3-ax^2+bx+1\longrightarrow(i)$

Given $f(2)=-3$

$\Rightarrow f(2)=(2)^3-a(2)^2+b(2)+1=-3$

$=8-4a+2b+1=-3$

$=9-4a+2b=-3$

$\Rightarrow 4a-2b=12 \quad \text{or} \quad 2a-b=6\longrightarrow(ii)$

Also given $f(-1)=0$

$\Rightarrow f(-1)=(-1)^3-a(-1)^2+b(-1)+1=0$

$=-1-a-b+1=0$

$=-a-b=0\Rightarrow a+b=0$

$\Rightarrow a=-b\longrightarrow(iii)$

Putting equation (iii) into equation (ii)

$2(-b)-b=6 \Rightarrow -2b-b=6$

$-3b=6 \Rightarrow b=-2$

From equation (iii)

$a=-(-2) \Rightarrow a=2$

Q6: A stone falls from a height of 60m on the ground, the height h after x seconds is approximately given by $h(x)=40-10x^2$.

i. ***What is the height of the stone when:***

a. $x=1\,\text{sec}$? b. $x=1.5\,\text{sec}$?

c. $x=1.7\,\text{sec}$?

ii. **When does stone strike the ground?**

Sol: The height h after x sec. is given as:

$h(x)=40-10x^2 \longrightarrow$ (i)

$\therefore$ (Height is the function of time)

(i) When $x=1\text{sec}$

Then $h(1)=40-10(1)^2=40-10=30m$

(b) When $x=1.5\text{sec}$

Then

$h(1.5)=40-10(1.5)^2=40-10(2.25)$

$=40-22.5=17.5m$

(c) When $x=1.7\text{sec}$

Then $h(1.7)=40-10(1.7)^2=40-10(2.89)$

$=40-28.9=11.1m$

(ii) When stone strikes the ground its height is zero.

Then

$h(x)=0 \Rightarrow 40-10x^2=0 \Rightarrow 10x^2=40$

$\Rightarrow x^2=4 \Rightarrow x=2\text{sec}$

Q7: Consider the function $f(x)=3x-5$.

i. Determine the domain and range of $f(x)$.

ii. Is the function f one-to-one? Justify your answer.

iii. Is the function f onto if the co-domain is all real numbers? Explain.

Sol: Given function $f(x)=3x-5 \longrightarrow$ (i) linear algebraic function.

(i) For domain it is clear that any of the real number can be put into their function

Domain of $f(x)=\mathbb{R}=(-\infty,\infty)$

Range of $f(x)=\mathbb{R}=(-\infty,\infty)$

(ii) *The* function $f(x)=3x-5 \longrightarrow$ (i)

Will be one-to-one if there is unique output for each and every input.

When $f(x_1)=f(x_2) \Rightarrow x_1=x_2$

$\Rightarrow 3x_1-5=3x_2-5 \Rightarrow x_1=x_2$

So, $f(x)$ is one-to-one function for $x\in\mathbb{R}$

(iii) The function $f(x)=3x-5 \longrightarrow$ *(i) will be onto if*

$f(x)=y \longrightarrow$ (ii) $y=3x-5 \longrightarrow$ (iii)

From (iii) $3x-5=y$

Or $3x=y+5$

Or $x=\dfrac{y+5}{3} \longrightarrow$ (iv)

Putting into the function

$$f\left(\frac{y+5}{3}\right)=3\left(\frac{y+5}{3}\right)-5$$

$$f\left(\frac{y+5}{3}\right)=y+\not{5}-\not{5}$$

$$f\left(\frac{y+5}{3}\right)=y=f(x)$$

$\Rightarrow$ $f(x)$ is a function and $f^{-1}(x)$ is also a function

$\therefore$ $f\left(f^{-1}(x)\right)=f^{-1}\left(f(x)\right)=x$

So, $f(x)$ is onto function $\forall x\in\mathbb{R}$

Q8: Left $f:R\rightarrow R$ be defined by $f(x)=\dfrac{2x-3}{x+1}$

i. Find the domain and range of $f(x)$.

ii. Determine whether $f(x)$ is onto.

iii. Prove that $f(x)$ is one-to-one.

Sol:

(i) As $f(x)=\dfrac{2x-3}{x+1} \longrightarrow$ (i) be the rational function defined by $f:\mathbb{R}\rightarrow\mathbb{R}$.

Which shows $f(x)$ is real valued function.

$f(x)=\dfrac{2x-3}{x+1}$ can exist if $x\neq-1$

Domain of $f(x)=\mathbb{R}-\{-1\}=(-\infty,-1)\cup(-1,\infty)$

(all real number except -1)

For range: To obtain $f(x)=0$ (as output)

$\Rightarrow \dfrac{2x-3}{x+1}=0 \Rightarrow 2x-3=0$

$\Rightarrow x=\dfrac{3}{2}\in Df$

To obtain $f(x)=1$, as output, taking

$\Rightarrow \dfrac{2x-3}{x+1}=1 \Rightarrow 2x-3=x+1$

$\Rightarrow 2x-x=4$ or $x=4\in Df$

As in domain -1 is central value between 0 and 1

Similarly, in Range 2 is central value between $\dfrac{2}{3}$ and 4.

Hence when in domain -1 is not possible then in range 2 in not possible

Range of $f(x)=\mathbb{R}-\{2\}=$ all real numbers except 2

(ii) The given function $f(x)=\frac{2x-3}{x+1} \longrightarrow$ (i)

Will be onto if $f(x)=y \longrightarrow$ (ii)

Where $y=\frac{2x-3}{x+1} \longrightarrow$ (iii)

From (iii) $xy+y=2x-3$

$2x-xy=y+3$

or $x(2-y)=y+3$

or $x=\frac{y+3}{2-y} \longrightarrow$ (iv)

Putting in equation (i)

$$f(x)=\frac{2\left(\frac{y+3}{2-y}\right)-3}{\frac{y+3}{2-y}+1}=\frac{\frac{2y+6-6+3y}{2-y}}{\frac{y+3+2-y}{2-y}}=\frac{5y}{5}=y$$

Justified that $f(x)$ is onto function (when $y=f(x)$ and $f\left(f^{-1}(x)\right)=f^{-1}\left(f(x)\right)=x$)

(iii) The given function $f(x)=\frac{2x-3}{x+1} \longrightarrow$ (i) is called one-to-one if it assigns a unique value for each and every input.

When $f(x_1)=f(x_2) \Rightarrow x_1=x_2 \longrightarrow$ (ii)

So, $f(x_1)=\frac{2x_1-3}{x_1+1}$, $f(x_2)=\frac{2x_2-3}{x_2+1}$

Using the condition $f(x_1)=f(x_2)$

$$\frac{2x_1-3}{x_1+1}=\frac{2x_2-3}{x_2+1}$$

$\Rightarrow (2x_1-3)(x_2+1)=(x_1+1)(2x_2-3)$

$2x_1x_2+2x_1-3x_2-3=2x_1x_2-3x_1+2x_2-3$

$2x_1+3x_1=2x_2+3x_2$

$5x_1=5x_2$

$\Rightarrow x_1=x_2$

Prove that $f(x)$ is one-to-one function

Q9: Consider the function $f:\mathbb{R}^+\to\mathbb{R}^+$ defined by $f(x)=e^{-x}$. Show that $f(x)$ is a bijective

Sol: This function will be bijective if it is one-to-one and onto.

When $f(x_1)=f(x_2) \Rightarrow x_1=x_2$

Then $e^{-x_1}=e^{-x_2}$

$-x_1=-x_2 \Rightarrow x_1=x_2$

This shows that $f(x)=e^{-x}\to$ (i) is one-to-one

$f(x)=y\to$ (ii) when $y=e^{-x}\to$ (iii)

From (iii) $\ln y=-x$ or $x=-\ln y$

$f(x)=f(-\ln y)=e^{-(\ln y)}=e^{\ln y}=y$

$f(x)=y$, This shows that $f(x)=e^{-x}$ is onto function.

So, $f(x)=e^{-x}$ is bijective functions

Q10: Let $g:\mathbb{R}\to\mathbb{R}$ be given by $f(x)=x^3-3x$. Determine if $g(x)$ is injective and/ or surjective.

Sol: $f(x_1)=f(x_2)$ where $x_1=x_2$

$\Rightarrow (x_1)^3-3(x_1)=x_2^3-3x_2$

$x_1^3-x_2^3=3x_1-3x_2$

$(x_1-x_2)(x_1^2+x_1x_2+x_2^2)=3(x_1-x_2)$

$x_1^2+x_1x_2+x_2^2=3 \Rightarrow x_1\neq x_2$

For example, $f(1)=(1)^3-3(1)=-2$

When $x_1=1$, then $y=f(1)=-2$

When $x_2=-2$, then $y=f(-2)=-2$

As $x_1\neq x_2$, while $f(x_1)=f(x_2)$

$f(x)$ is not injective function.

When $y=g(x)$

Then $y=x^3-3x$

As x cannot be expresses in term of $y \Rightarrow g(x)$ cannot be onto function $\Rightarrow$ ($f^{-1}(y)$ does not exist). So $g(x)$ is surjective.