Q.1. Evaluate the following determinants.

i. $\begin{vmatrix} 1 & -2 & -4 \\ 3 & -1 & -3 \\ -2 & 3 & 2 \end{vmatrix}$

Sol: $\begin{vmatrix} 1 & -2 & -4 \\ 3 & -1 & -3 \\ -2 & 3 & 2 \end{vmatrix} = 1\begin{vmatrix} -1 & -3 \\ 3 & 2 \end{vmatrix} - (-2)\begin{vmatrix} 3 & -3 \\ -2 & 2 \end{vmatrix} + (-4)\begin{vmatrix} 3 & -1 \\ -2 & 3 \end{vmatrix}$ Expanding by 1st row

$= 1(-2+9)+2(6-6)-4(9-2)$

$= 1(7)+2(0)-4(7)$

$= 7+0-28 = -21$

ii. $\begin{vmatrix} a+b & a-b & a \\ a & a+b & a-b \\ a-b & a & a+b \end{vmatrix}$

Sol: $\begin{vmatrix} a+b & a-b & a \\ a & a+b & a-b \\ a-b & a & a+b \end{vmatrix} = (a+b)\begin{vmatrix} a+b & a-b \\ a & a+b \end{vmatrix} - (a-b)\begin{vmatrix} a & a-b \\ a-b & a+b \end{vmatrix} + a\begin{vmatrix} a & a+b \\ a-b & a \end{vmatrix}$

$= (a+b)\left[(a+b)^2 - a(a-b)\right] - (a-b)\left[a(a+b)-(a-b)^2\right] + a\left[a^2-(a-b)(a+b)\right]$

$= (a+b)\left[\cancel{a^2}+b^2+2ab-\cancel{a^2}+ab\right] - (a-b)\left[\cancel{a^2}+ab-\cancel{a^2}-b^2+2ab\right] + a\left[\cancel{a^2}-\cancel{a^2}+b^2\right]$

$= (a+b)(b^2+3ab)-(a-b)(3ab-b^2)+a(b^2)$

$= ab^2+\cancel{3a^2b}+\cancel{b^3}+3ab^2-\cancel{3a^2b}+ab^2+3ab^2-\cancel{b^3}+ab^2 = 9ab^2$

iii. $\begin{vmatrix} 2x & x & x \\ y & 2y & y \\ z & z & 2z \end{vmatrix}$

Sol: $\begin{vmatrix} 2x & x & x \\ y & 2y & y \\ z & z & 2z \end{vmatrix}$

$= 2x\begin{vmatrix} 2y & y \\ z & 2z \end{vmatrix} - x\begin{vmatrix} y & y \\ z & 2z \end{vmatrix} + x\begin{vmatrix} y & 2y \\ z & z \end{vmatrix}$ Expending by 1st row

$= 2x(4yz-yz)-x(2yz-yz)+x(yz-2yz)$

$= 2x(3yz)-x(yz)+x(-yz)$

$= 6xyz-xyz-xyz = 4xyz$

Q.3. Using properties of determinants, show that:

> **نوٹ:**
> اگر R.H.S پہ زیرو یا determinant یا factor ہوں تو operation سے حل کریں ورنہ صرف expand کر دیں۔

i. $\begin{vmatrix} 3 & 5 & 0 \\ 5 & 25 & 10 \\ 7 & 25 & 1 \end{vmatrix} = 25\begin{vmatrix} 3 & 1 & 0 \\ 1 & 1 & 2 \\ 7 & 5 & 1 \end{vmatrix}$

Sol: $\text{L.H.S} = \begin{vmatrix} 3 & 5 & 0 \\ 5 & 25 & 10 \\ 7 & 25 & 1 \end{vmatrix} = 5\begin{vmatrix} 3 & 5 & 0 \\ 1 & 5 & 2 \\ 7 & 25 & 1 \end{vmatrix}$ taking 5 common from R_2

$= 5.5\begin{vmatrix} 3 & 1 & 0 \\ 1 & 1 & 2 \\ 7 & 5 & 1 \end{vmatrix}$ Taking 5 common from C_2

$= 25\begin{vmatrix} 3 & 1 & 0 \\ 1 & 1 & 2 \\ 7 & 5 & 1 \end{vmatrix}$

= R.H.S

ii. $\begin{vmatrix} a+x & a & a \\ a & a+x & a \\ a & a & a+x \end{vmatrix} = x^2(3a+x)$

Sol: $\text{L.H.S} = \begin{vmatrix} a+x & a & a \\ a & a+x & a \\ a & a & a+x \end{vmatrix}$ Add R_2, R_3 in R_1

$= \begin{vmatrix} 3a+x & 3a+x & 3a+x \\ a & a+x & a \\ a & a & a+x \end{vmatrix}$

> **نوٹ:**
> R.H.S میں $3a+x$ سے اندازہ ہو جاتا ہے کہ R_2 اور R_3 کو R_1 میں جمع کر کے common لینا ہے۔

$= (3a+x)\begin{vmatrix} 1 & 1 & 1 \\ a & a+x & a \\ a & a & a+x \end{vmatrix}$ Take common $(3a+x)$ from R_1

$= (3a+x)\begin{vmatrix} 1 & 0 & 0 \\ a & x & 0 \\ a & 0 & x \end{vmatrix}$ $C_2 - C_1, C_3 - C_1$

$= (3a+x)\left[1\begin{vmatrix} x & 0 \\ 0 & x \end{vmatrix} - 0\begin{vmatrix} a & 0 \\ a & x \end{vmatrix} + 0\begin{vmatrix} a & x \\ a & 0 \end{vmatrix}\right]$

$= (3a+x)1(x^2-0)$

$= (3a+x)(x^2) = x^2(3a+x)$

= R.H.S

iii. $\begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$

Sol: $\text{L.H.S} = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$

$$= \frac{1}{xyz}\begin{vmatrix} x & x^2 & xyz \\ y & y^2 & xyz \\ z & z & xyz \end{vmatrix} \quad x\times R_1, y\times R_2, z\times R_3$$

$$= \frac{\cancel{xyz}}{\cancel{xyz}}\begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z & 1 \end{vmatrix} \quad \text{Take common } xyz \text{ from } C_3$$

$$= \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} = -\begin{vmatrix} x & 1 & x^2 \\ y & 1 & y^2 \\ z & 1 & z^2 \end{vmatrix} \quad \text{Interchange } C_2 \text{ and } C_3$$

$$= (-)(-)\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \quad \text{Again interchange } C_2 \text{ and } C_1$$

$$= \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = \text{R.H.S}$$

iv. $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)$

Sol: $\text{L.H.S} = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$

$$= \begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{vmatrix} \quad R_2 - R_1,\ R_3 - R_1$$

$$= \begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & (y-x)(y+x) \\ 0 & z-x & (z-x)(z+x) \end{vmatrix}$$

Take common $(y-x)$ from R_2 and $(z-x)$ from R_3

نوٹ:

اگر R.H.S پر factors بنے ہوں تو پہلے جس Row یا Column میں 1 ہے اسے باقی میں سے minus کر کے zero بنائیں۔ پھر جو دو factor ان میں سے Common آرہے ہوں common لیں پھر جس Row یا Column میں zero بنائیں اسی سے open کریں۔

$= a^3 + \cancel{ab^2} + \cancel{ac^2} - \cancel{a^2b} - abc - \cancel{ca^2} + \cancel{ba^2} + b^3 + \cancel{bc^2} - \cancel{ab^2} - \cancel{b^2c} - abc$

$\cancel{a^2c} + \cancel{cb^2} + c^3 - abc - \cancel{bc^2} - \cancel{c^2a} = a^3 + b^3 + c^3 - 3abc = R.H.S$

viii. $\begin{vmatrix} a+t & a & a \\ b & b+t & b \\ c & c & c+t \end{vmatrix} = t^2(a+b+c+t)$

Sol: L.H.S $= \begin{vmatrix} a+t & a & a \\ b & b+t & b \\ c & c & c+t \end{vmatrix}$

$= \begin{vmatrix} a+b+c+t & a+b+c+t & a+b+c+t \\ b & b+t & b \\ c & c & c+t \end{vmatrix}$ Add R_2, R_3 in R_1

$= (a+b+c+t)\begin{vmatrix} 1 & 1 & 1 \\ b & b+t & b \\ c & c & c+t \end{vmatrix}$ Take common $(a+b+c+t)$ from R_1

$= (a+b+c+t)\begin{vmatrix} 1 & 0 & 0 \\ b & t & 0 \\ c & 0 & t \end{vmatrix}$ $C_2 - C_1, C_3 - C_1$

$(a+b+c+t)\left|1\begin{vmatrix} t & 0 \\ 0 & t \end{vmatrix} - 0 + 0\right|$ Expand by R_1

$= (a+b+c+t)(t^2 - 0) = t^2(a+b+c+t)$

$= R.H.S$

ix. $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

Sol: Expand the determinant by R_1

$= (a-b-c)\begin{vmatrix} b-c-a & 2b \\ 2c & c-a-b \end{vmatrix} - 2a\begin{vmatrix} 2b & 2b \\ 2c & c-a-b \end{vmatrix} + 2a\begin{vmatrix} 2b & b-c-a \\ 2c & 2c \end{vmatrix}$

$= (a-b-c)[(c-a-b)(b-c-a) - 4bc] - 2a[2b(c-a-b) - 4bc] + 2a[4bc - 2c(b-c-a)]$

$= (a-b-c)[cb - c^2 - \cancel{ac} - \cancel{ab} + \cancel{ac} + a^2 - b^2 + bc + \cancel{ab} - 4bc] - 2a[2bc - 2ba - 2b^2 - 4bc] + 2a[4bc - (2cb - 2c^2 - 2ca)]$

$= (a-b-c)(a^2 - b^2 - c^2 - 2bc) - 2a(-2b^2 - 2ab - 2bc) + 2a(2c^2 + 2bc + 2ca)$

$= a^3 - ab^2 - ac^2 - 2abc - ba^2 + b^3 + bc^2 + 2b^2c - ca^2 + cb^2 + c^3 + 2bc^2 + 4ab^2 + 4a^2b + 4abc + 4ac^2 + 4abc + 4a^2c$

$= a^3 + b^3 + c^3 + 3a^2b + 3a^2c + 3ab^2 + 3b^2c + 3ac^2 + 3bc^2 + 6abc = (a+b+c)^3$

$= R.H.S$

x. $\begin{vmatrix} y+z & z+x & x+y \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = (x+y+z)(x-y)(y-z)(z-x)$

Sol: L.H.S $= \begin{vmatrix} y+z & z+x & x+y \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$

$= \begin{vmatrix} x+y+z & x+y+z & x+y \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$ Add R_2 in R_1

$= (x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$ Take common $(x+y+z)$ from R_1

$= (x+y+z)\begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ x^2 & y^2-x^2 & z^2-x^2 \end{vmatrix}$ $C_2 - C_1$ *and* $C_3 - C_1$

$= (x+y+z)\left|1\begin{vmatrix} y-x & z-x \\ y^2-x^2 & z^2-x^2 \end{vmatrix} - 0 + 0\right|$ Expand by R_1

$= (x+y+z)\begin{vmatrix} y-x & z-x \\ (y-x)(y+x) & (z-x)(z+x) \end{vmatrix}$

$= (x+y+z)(y-x)(z-x)\begin{vmatrix} 1 & 1 \\ (y+x) & z+x \end{vmatrix}$ Take common $(y-x)$ from C_1 and $(z-x)$ from C_2

$= (x+y+z)(y-x)(z-x)(z+\cancel{x}-y-\cancel{x}) = (x+y+z)(y-x)(z-x)(z-y)$

$= (x+y+z)((-1)(x-y))(z-x)(-1)(y-z) = (x+y+z)(x-y)(y-z)(z-x) =$ R.H.S

xi. $\begin{vmatrix} 1 & 1 & 1 \\ a^2+1 & b^2+1 & c^2+1 \\ a^3+a & b^3+b & c^3+c \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca-1)$

Sol: L.H.S $= \begin{vmatrix} 1 & 0 & 0 \\ a^2+1 & b^2+\cancel{1}-a^2-\cancel{1} & c^2+\cancel{1}-a^2-\cancel{1} \\ a^3+a & b^3+b-a^3-a & c^3+c-a^3-a \end{vmatrix}$ by $C_2 - C_1$ and $C_3 - C_1$

$= \begin{vmatrix} 1 & 0 & 0 \\ a^2+1 & b^2-a^2 & c^2-a^2 \\ a^3+a & b^3-a^3+b-a & c^3-a^3+c-a \end{vmatrix}$

$= \begin{vmatrix} 1 & 0 & 0 \\ a^2+1 & (b-a)(b+a) & (b-a)(b+a) \\ a^3+a & (b-a)(b^2+ab+a^2)+(b-a) & (c-a)(c^2+ac+a^2)+(c-a) \end{vmatrix}$

$$=(b-a)(c-a)\begin{vmatrix}1 & 0 & 0\\ a^2+1 & (b+a) & c+a\\ a^3+a & b^2+ab+a^2+1 & c^2+ac+a^2+1\end{vmatrix}$$ Take $(b-a)$ common from C_2 and $(c-a)$ from C_3

$$=(b-a)(c-a)\left[1\begin{vmatrix}b+a & c+a\\ b^2+ab+a^2+1 & c^2+ac+a^2+1\end{vmatrix}-0+0\right]$$ Expand by R_1

$$=(b-a)(c-a)\begin{vmatrix}b+a & c+a\\ b^2+ab+a^2+1 & c^2+ac+a^2+1\end{vmatrix}$$

$$=(b-a)(c-a)\begin{vmatrix}b+a & c+\cancel{a}-b-\cancel{a}\\ b^2+ab+a^2+1 & c^2+ac+\cancel{a^2}+\cancel{1}-b^2-ab-\cancel{a^2}-\cancel{1}\end{vmatrix}$$ C_2-C_1

$$=(b-a)(c-a)\begin{vmatrix}b+a & c-b\\ b^2+ab+a^2+1 & c^2-b^2+ac-ab\end{vmatrix}$$

$$=(b-a)(c-a)\begin{vmatrix}b+a & c-b\\ b^2+ab+a^2+1 & (c-b)(c+b)+a(c-b)\end{vmatrix}$$

$$=(b-a)(c-a)(c-b)\begin{vmatrix}b+a & 1\\ b^2+ab+a^2+1 & c+b+a\end{vmatrix}$$ Take $(c-b)$ common from C_2

$$=(-)(a-b)(-)(b-c)(c-a)\left[(b+a)(c+b+a)-b^2-ab-a^2-1\right]$$

$$=(a-b)(b-c)(c-a)\left[bc+\cancel{b^2}+\cancel{ab}+ac+ab+\cancel{a^2}-b^2-\cancel{ab}-\cancel{a^2}-1\right]$$

$$=(a-b)(b-c)(c-a)[ab+bc+ca-1]=R.H.S$$

Hence Proved.

xii. $$\begin{vmatrix}1+a & 1 & 1\\ 1 & 1+b & 1\\ 1 & 1 & 1+c\end{vmatrix}=abc+ab+bc+ca$$

Sol: L.H.S $$\begin{vmatrix}1+a & 1 & 1\\ 1 & 1+b & 1\\ 1 & 1 & 1+c\end{vmatrix}$$ expanding by 1st row

$$=(1+a)[(1+b)(1+c)-(1)(1)]-1[(1+c)-1]+1[1-(1+b)]=(1+a)[1+c+b+bc-1]-1[c]+1[-b]$$

$$=(1+a)(c+b+bc)-c-b$$

$$=\cancel{c}+\cancel{b}+bc+ac+ab+abc-\cancel{c}-\cancel{b}$$

$$=abc+ab+bc+ca$$ =R.H.S

xiii. $$\begin{vmatrix}1 & a & a^2-bc\\ 1 & b & b^2-ca\\ 1 & c & c^2-ab\end{vmatrix}=0$$

Sol: L.H.S $$=\begin{vmatrix}1 & a & a^2-bc\\ 1 & b & b^2-ca\\ 1 & c & c^2-ab\end{vmatrix}$$

$$=\begin{vmatrix}1 & a & a^2-bc\\ 0 & b-a & b^2-a^2+bc-ca\\ 0 & c-a & c^2-a^2+bc-ab\end{vmatrix}$$ By R_2-R_1 and R_3-R_1

$$=\begin{vmatrix}1 & a & a^2-bc\\ 0 & (b-a) & (b-a)(b+a)+c(b-a)\\ 0 & (c-a) & (c-a)(c+a)+b(c-a)\end{vmatrix}$$

$$=(b-a)(c-a)\begin{vmatrix}1 & a & a^2-bc\\ 0 & 1 & b+a+c\\ 0 & 1 & c+a+b\end{vmatrix}$$ Take $(b-a)$ common from R_2 and $(c-a)$ from R_3

$=(b-a)(c-a)1\left[(c+a+b)(1)-(b+a+c)1\right]$ Expand by C_1

$=(b-a)(c-a)\left[\not c+\not a+\not b-\not b-\not a-\not c\right]$

$(b-a)(c-a)(0)=0=R.H.S$

xiv. $\begin{vmatrix}2x+3 & x+2 & x+a\\ 2x+5 & x+3 & x+b\\ 2x+7 & x+4 & x+c\end{vmatrix}=0$ **where** $2b=a+c$

Sol: L.H.S $=\begin{vmatrix}2x+3 & x+2 & x+a\\ 2x+5 & x+3 & x+b\\ 2x+7 & x+4 & x+c\end{vmatrix}$

$$=\begin{vmatrix}2x+3 & x+2 & x+a\\ 2 & 1 & b-a\\ 4 & 2 & c-a\end{vmatrix}$$ by R_2-R_1 and R_3-R_1

Given that $2b=a+c$ its mean a,b,c are in A.P so,

$d=\not b-a$

$d=c-\not b$

$2d=c-a$

$$=\begin{vmatrix}2x+3 & x+2 & x+a\\ 2 & 1 & d\\ 4 & 2 & 2d\end{vmatrix}$$

$$=\begin{vmatrix}2x+3 & x+2 & x+a\\ 2 & 1 & d\\ 2 & 1 & d\end{vmatrix}$$ 2 common from R_3

$=2(0)$ (R_2 and R_3 are identical)

$=0=R.H.S$

xv. $\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = (a+b+c)(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$ where ω is an imaginary cube root of unity.

Sol: we have, let $\Delta = \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$ ______ (I)

$$\Delta = \begin{vmatrix} a+c\omega+b\omega^2 & b+a\omega+c\omega^2 & c+b\omega+c\omega^2 \\ c & a & b \\ b & c & a \end{vmatrix} \text{ By } R_1+\omega R_2+\omega^2 R_3$$

$$\Delta = \begin{vmatrix} a+b\omega^2+c\omega & \omega(a+b\omega^2+c\omega) & \omega^2(a+b\omega^2+c\omega) \\ c & a & b \\ b & c & a \end{vmatrix} \text{ Take } b = bw^3 \text{ and } c = cw^3$$

$$\Delta = (a+b\omega^2+c\omega)\begin{vmatrix} 1 & \omega & \omega^2 \\ c & a & b \\ b & c & a \end{vmatrix} \text{ Take } (a+b\omega^2+c\omega) \text{ common from } R_1 \text{ ______ } (II)$$

Again from (I) $\Delta = \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$

Now $R_1+\omega^2 R_2+\omega R_3$ we get

$$= \begin{vmatrix} a+c\omega^2+b\omega & b+a\omega^2+c\omega & c+b\omega^2+a\omega \\ c & a & b \\ b & c & a \end{vmatrix}$$

$$\Delta = \begin{vmatrix} a+b\omega+c\omega^2 & \omega^2(a+b\omega+c\omega^2) & \omega(a+b\omega+c\omega^2) \\ c & a & b \\ b & c & a \end{vmatrix}$$

$$= (a+b\omega+c\omega^2)\begin{vmatrix} 1 & \omega^2 & \omega \\ c & a & b \\ b & c & a \end{vmatrix} \text{ Take } (a+b\omega+c\omega^2) \text{ common from } R_1 \text{ ___ } (III)$$

Again from I

$$\Delta = \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} \qquad \text{Add } R_2, R_3 \text{ in } R_1$$

$$\Delta = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ c & a & b \\ b & c & a \end{vmatrix}$$

$$\Delta=(a+b+c)\begin{vmatrix}1&1&1\\c&a&b\\b&c&a\end{vmatrix}$$ Take common $a+b+c$ from R_1 ______ (IV)

From II, III, IV

$\Delta=K(a+b+c)(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$ Take $K=1$ (K in any constant)

Hence

$$\begin{vmatrix}a&b&c\\c&a&b\\b&c&a\end{vmatrix}=(a+b+c)(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$$

Q.4. **If** $A=\begin{bmatrix}1&2&-3\\0&-5&0\\-2&-2&7\end{bmatrix}$ *and* $B=\begin{bmatrix}-5&-2&5\\-3&-1&4\\-2&-1&2\end{bmatrix}$ **then find**

(i) A_{13}, A_{23}, A_{33} *and* $|A|$ **(ii)** B_{31}, B_{32}, B_{33} *and* $|B|$

Sol (i): Let $A=\begin{bmatrix}1&2&-3\\0&-5&0\\-2&-2&7\end{bmatrix}$

$$|A|=\begin{vmatrix}1&2&-3\\0&-5&0\\-2&-2&7\end{vmatrix}=1\begin{vmatrix}-5&0\\-2&7\end{vmatrix}-2\begin{vmatrix}0&0\\-2&7\end{vmatrix}+(-3)\begin{vmatrix}0&-5\\-2&-2\end{vmatrix}=1(-35-0)-2(0-0)-3(0-10)$$

$|A|=-35+30\Rightarrow|A|=-5$

$$A_{13}=(-1)^{1+3}\begin{vmatrix}0&-5\\-2&-2\end{vmatrix}=(-1)^4(0-10)=-10$$

$$A_{23}=(-1)^{2+3}\begin{vmatrix}1&2\\-2&-2\end{vmatrix}=(-1)^5(-2+4)=-(2)=-2$$

$$A_{33}=(-1)^{3+3}\begin{vmatrix}1&2\\0&-5\end{vmatrix}=(-1)^6(-5-0)=-5$$

ii. B_{31}, B_{32}, B_{33} *and* $|B|$

Sol: Let $B=\begin{bmatrix}-5&-2&5\\-3&-1&4\\-2&-1&2\end{bmatrix}$

$$|B|=\begin{vmatrix}-5&-2&5\\-3&-1&4\\-2&-1&2\end{vmatrix}=-5\begin{vmatrix}-1&4\\-1&2\end{vmatrix}-(-2)\begin{vmatrix}-3&4\\-2&2\end{vmatrix}+5\begin{vmatrix}-3&-1\\-2&-1\end{vmatrix}=-5(-2+4)+2(-6+8)+5(3-2)$$

$=-5(2)+2(2)+5(1)=-10+4+5=-10+9=-1\Rightarrow|B|=-1$

Now,

B_{31}, B_{32}, B_{33}

$$B_{31}=(-1)^{3+1}\begin{vmatrix}-2 & 5\\ -1 & 4\end{vmatrix}=(-1)^4(-8+5)=1(-3)=-3$$

$$B_{32}=(-1)^{3+2}\begin{vmatrix}-5 & 5\\ -3 & 4\end{vmatrix}=(-1)^5(-20+15)=(-1)(-5)=5$$

$$B_{33}=(-1)^{3+3}\begin{vmatrix}-5 & -2\\ -3 & -1\end{vmatrix}=(-1)^6(5-6)=1(-1)=-1$$

Q.5. Find the values of x if

i. $\begin{vmatrix}2 & 1 & x\\ -1 & -4 & -3\\ x & 1 & 0\end{vmatrix}=5$ ii. $\begin{vmatrix}1 & x-1 & 3\\ -1 & x+1 & 2\\ 2 & -3 & x\end{vmatrix}=9$ iii. $\begin{vmatrix}1 & 1 & 1\\ 2 & x & 2\\ 3 & 6 & x\end{vmatrix}=0$

i. $\begin{vmatrix}2 & 1 & x\\ -1 & -4 & -3\\ x & 1 & 0\end{vmatrix}=5$

Sol: Expand the determinant from R_1

$$2\begin{vmatrix}-4 & -3\\ 1 & 0\end{vmatrix}-1\begin{vmatrix}-1 & -3\\ x & 0\end{vmatrix}+x\begin{vmatrix}-1 & -4\\ x & 1\end{vmatrix}=5$$

$$2(0+3)-1(0+3x)+x(-1+4x)=5$$

$$6-3x-x+4x^2-5=0$$

$$4x^2-4x+1=0$$

$$\Rightarrow(2x-1)^2=0$$

Taking square root on the both sides

$$(2x-1)=0 \quad \Rightarrow 2x=1 \Rightarrow x=\frac{1}{2}$$

ii. $\begin{vmatrix}1 & x-1 & 3\\ -1 & x+1 & 2\\ 2 & -3 & x\end{vmatrix}=9$

Sol: $1\begin{vmatrix}x+1 & 2\\ -3 & x\end{vmatrix}-(x-1)\begin{vmatrix}-1 & 2\\ 2 & x\end{vmatrix}+3\begin{vmatrix}-1 & x+1\\ 2 & -3\end{vmatrix}=9$ Expand by R_1

$$(x(x+1)+6)-(x-1)(-x-4)+3(3-2(x+1))=9 \quad (x^2+x+6)-(-x^2-4x+x+4)+3(3-2x-2)=9$$

$$(x^2+x+6)-(-x^2-3x+4)+3(-2x+1)=9$$

$$x^2+x+6+x^2+3x-4-6x+3=9$$

$$2x^2-2x+5=9$$

$$\Rightarrow 2x^2-2x-4=0\Rightarrow x^2-x-2=0$$

$$x^2-2x+x-2=0\Rightarrow x(x-2)+1(x-2)=0$$

$$(x-2)(x+1)=0$$

$\Rightarrow x-2=0$ *or* $x+1=0$

$x=2$ *or* $x=-1$

iii. $\begin{vmatrix}1 & 1 & 1\\2 & x & 2\\3 & 6 & x\end{vmatrix}=0$

Sol: $1\begin{vmatrix}x & 2\\6 & x\end{vmatrix}-1\begin{vmatrix}2 & 2\\3 & x\end{vmatrix}+1\begin{vmatrix}2 & x\\3 & 6\end{vmatrix}=0$

$1\left(x^2-12\right)-1(2x-6)+1(12-3x)=0$

$x^2-\cancel{12}-2x+6+\cancel{12}-3x=0$

$x^2-5x+6=0$

$x^2-3x-2x+6=0$

$x(x-3)-2(x-3)=0$

$(x-2)(x-3)=0$

$x-2=0 \text{ or } x-3=0$

$x=2 \text{ or } x=3$

Q.6. **Find $|AA'|$ and $|A'A|$ if (*i*)** $A=\begin{bmatrix}-3 & 2 & -1\\2 & 1 & 3\end{bmatrix}$ (*ii*) $A=\begin{bmatrix}3 & 1\\2 & 2\\1 & 3\end{bmatrix}$

i. $A=\begin{bmatrix}-3 & 2 & -1\\2 & 1 & 3\end{bmatrix}$

Sol: $A=\begin{bmatrix}-3 & 2 & -1\\2 & 1 & 3\end{bmatrix}$ then $A'=\begin{bmatrix}-3 & 2\\2 & 1\\-1 & 3\end{bmatrix}$

$$AA'=\begin{bmatrix}-3 & 2 & -1\\2 & 1 & 3\end{bmatrix}\begin{bmatrix}-3 & 2\\2 & 1\\-1 & 3\end{bmatrix}=\begin{bmatrix}9+4+1 & -6+2-3\\-6+2-3 & 4+1+9\end{bmatrix}=\begin{bmatrix}14 & -7\\-7 & 14\end{bmatrix}$$

$|AA'|=\begin{vmatrix}14 & -7\\-7 & 14\end{vmatrix}=196-49=147$

Now, For $|A'A|$

$$A'A=\begin{bmatrix}-3 & 2\\2 & 1\\-1 & 3\end{bmatrix}\begin{bmatrix}-3 & 2 & -1\\2 & 1 & 3\end{bmatrix}=\begin{bmatrix}9+4 & -6+2 & 3+6\\-6+2 & 4+1 & -2+3\\3+6 & -2+3 & 1+9\end{bmatrix}=\begin{bmatrix}13 & -4 & 9\\-4 & 5 & 1\\9 & 1 & 10\end{bmatrix}$$

$|A'A|=\begin{vmatrix}13 & -4 & 9\\-4 & 5 & 1\\9 & 1 & 10\end{vmatrix}$

$|A'A|=13\begin{vmatrix}5 & 1\\1 & 10\end{vmatrix}-(-4)\begin{vmatrix}-4 & 1\\9 & 10\end{vmatrix}+9\begin{vmatrix}-4 & 5\\9 & 1\end{vmatrix}=13(50-1)+4(-40-9)+9(-4-45)$

$|A'A|=13(49)+4(-49)+9(-49)=637-196-441=0=0$

ii. $A=\begin{bmatrix}3 & 1\\ 2 & 2\\ 1 & 3\end{bmatrix}$

Sol: $A=\begin{bmatrix}3 & 1\\ 2 & 2\\ 1 & 3\end{bmatrix}$ then $A'=\begin{bmatrix}3 & 2 & 1\\ 1 & 2 & 3\end{bmatrix}$

$$AA'=\begin{bmatrix}3 & 1\\ 2 & 2\\ 1 & 3\end{bmatrix}\begin{bmatrix}3 & 2 & 1\\ 1 & 2 & 3\end{bmatrix}=\begin{bmatrix}9+1 & 6+2 & 3+3\\ 6+2 & 4+4 & 2+6\\ 3+3 & 2+6 & 1+9\end{bmatrix}$$

$$AA'=\begin{vmatrix}10 & 8 & 6\\ 8 & 8 & 8\\ 6 & 8 & 10\end{vmatrix} \text{ then } |AA'|=\begin{vmatrix}10 & 8 & 6\\ 8 & 8 & 8\\ 6 & 8 & 10\end{vmatrix}$$

$$|AA'|=10\begin{vmatrix}8 & 8\\ 8 & 10\end{vmatrix}-8\begin{vmatrix}8 & 8\\ 6 & 10\end{vmatrix}+6\begin{vmatrix}8 & 8\\ 6 & 8\end{vmatrix}$$

$$|AA'|=10(80-64)-8(80-48)+6(64-48)=10(16)-8(32)+6(16)=160-256+96$$

$$|AA'|=256-256=0$$

Now $A'A$

$$A'A=\begin{bmatrix}3 & 2 & 1\\ 1 & 2 & 3\end{bmatrix}\begin{bmatrix}3 & 1\\ 2 & 2\\ 1 & 3\end{bmatrix}=\begin{bmatrix}9+4+1 & 3+4+3\\ 3+4+3 & 1+4+9\end{bmatrix}=\begin{bmatrix}14 & 10\\ 10 & 14\end{bmatrix} \text{then}$$

$$|A'A|=\begin{vmatrix}14 & 10\\ 10 & 14\end{vmatrix}=14\times14-10\times10=196-100=96$$

Q.7. If A is a square matrix of order 3, then show that $|KA|=K^3|A|$

Sol: let $A=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}$ then $KA=K\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}=\begin{bmatrix}ka_{11} & ka_{12} & ka_{13}\\ ka_{21} & ka_{22} & ka_{23}\\ ka_{31} & ka_{32} & ka_{33}\end{bmatrix}$

$|KA|=\begin{vmatrix}ka_{11} & ka_{12} & ka_{13}\\ ka_{21} & ka_{22} & ka_{23}\\ ka_{31} & ka_{32} & ka_{33}\end{vmatrix}$ Take k common from R_1, R_2, R_3

$$k.k.k=\begin{vmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}$$

$=k^3|A|=$ R.H.S

Q.8. Find the values of λ if A and B are singular & $A=\begin{bmatrix}4 & 2 & 3\\ 7 & \lambda & 6\\ 2 & 3 & 1\end{bmatrix}$ $B=\begin{bmatrix}-2 & 4 & 5\\ 1 & -2 & 1\\ 2 & \lambda & 0\end{bmatrix}$

i. $A=\begin{bmatrix}4 & 2 & 3\\ 7 & \lambda & 6\\ 2 & 3 & 1\end{bmatrix}$

Sol: Given matrix is singular so $|A|=\begin{vmatrix}4 & 2 & 3\\ 7 & \lambda & 6\\ 2 & 3 & 1\end{vmatrix}=0$

$$4\begin{vmatrix}\lambda & 6\\ 3 & 1\end{vmatrix}-2\begin{vmatrix}7 & 6\\ 2 & 1\end{vmatrix}+3\begin{vmatrix}7 & \lambda\\ 2 & 3\end{vmatrix}=0$$

$$4(\lambda-18)-2(7-12)+3(21-2\lambda)=0$$

$$4\lambda-72-14+24+63-6\lambda=0$$

$$-2\lambda+1=0$$

$$\not-2\lambda=\not-1\Rightarrow\lambda=\frac{1}{2}$$

ii: $B=\begin{bmatrix}-2 & 4 & 5\\ 1 & -2 & 1\\ 2 & \lambda & 0\end{bmatrix}$

Sol: Given matrix is singular so $|B|=0$

$$|B|=\begin{vmatrix}-2 & 4 & 5\\ 1 & -2 & 1\\ 2 & \lambda & 0\end{vmatrix}=0$$

$$|B|=-2\begin{vmatrix}-2 & 1\\ \lambda & 0\end{vmatrix}-4\begin{vmatrix}1 & 1\\ 2 & 0\end{vmatrix}+5\begin{vmatrix}1 & -2\\ 2 & \lambda\end{vmatrix}=0$$

$$|B|=-2(0-\lambda)-4(0-2)+5(\lambda+4)=0$$

$$2\lambda+8+5\lambda+20=0$$

$$7\lambda+28=0\Rightarrow7\lambda=-28$$

$$\lambda=\frac{-28^{4}}{7}=-4\Rightarrow\lambda=-4$$

Q.9: Find the inverse of $A=\begin{bmatrix}1 & 2 & 1\\ -5 & 0 & 4\\ 5 & 4 & 0\end{bmatrix}$ and show that $A^{-1}A=I_3$

Sol: $A=\begin{bmatrix}1 & 2 & 1\\ -5 & 0 & 4\\ 5 & 4 & 0\end{bmatrix}$

Then $|A| = \begin{vmatrix} 1 & 2 & 1 \\ -5 & 0 & 4 \\ 5 & 4 & 0 \end{vmatrix} = 1\begin{vmatrix} 0 & 4 \\ 4 & 0 \end{vmatrix} - 2\begin{vmatrix} -5 & 4 \\ 5 & 0 \end{vmatrix} + 1\begin{vmatrix} -5 & 0 \\ 4 & 4 \end{vmatrix}$

$= 1(0-16) - 2(0-20) + 1(-20-0) = -16 + 40 - 20 = 4$

$A_{11} = (-1)^{1+1}\begin{vmatrix} 0 & 4 \\ 4 & 0 \end{vmatrix} = (-1)^2(0-16) = -16$

$A_{12} = (-1)^{1+2}\begin{vmatrix} -5 & 4 \\ 5 & 0 \end{vmatrix} = (-1)^3(0-20) = -(-20) = 20$ $A_{13} = (-1)^{1+3}\begin{vmatrix} -5 & 0 \\ 5 & 4 \end{vmatrix} = (-1)^4(-20-0) = -20$

$A_{21} = (-1)^{2+1}\begin{vmatrix} 2 & 1 \\ 4 & 0 \end{vmatrix} = (-1)^3(0-4) = -(-4) = 4$

$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 1 \\ 5 & 0 \end{vmatrix} = (-1)^4(0-5) = -5$

$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 2 \\ 5 & 4 \end{vmatrix} = (-1)^5(4-10) = -(-6) = 6$

$A_{31} = (-1)^{3+1}\begin{vmatrix} 2 & 1 \\ 0 & 4 \end{vmatrix} = (-1)^4(8-0) = 8$

$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 1 \\ -5 & 4 \end{vmatrix} = (-1)^5(4+5) = -(9) = -9$

$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 2 \\ -5 & 0 \end{vmatrix} = (-1)^6(0+10) = (+1)(10) = +10$ ***cofactor of***

$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} -16 & 20 & -20 \\ 4 & -5 & 6 \\ 8 & -9 & +10 \end{bmatrix}$

$\mathbf{AdjA} = (\mathbf{cofactorA})' = \begin{bmatrix} -16 & 4 & 8 \\ 20 & -5 & -9 \\ -20 & 6 & +10 \end{bmatrix}$

$A^{-1} = \frac{Adj}{|A|} = \frac{1}{4}\begin{bmatrix} -16 & 4 & 8 \\ 20 & -5 & -9 \\ -20 & 6 & +10 \end{bmatrix}$

$A^{-1}A = \frac{1}{4}\begin{bmatrix} -16 & 4 & 8 \\ 20 & -5 & -9 \\ -20 & 6 & +10 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ -5 & 0 & 4 \\ 5 & 4 & 0 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -16-20+40 & -32+0+32 & -16+16+0 \\ 20+25-45 & 40+0-36 & 20-20+0 \\ -20-30+50 & -40+0+40 & -20+24+0 \end{bmatrix}$

$A^{-1}A = \frac{1}{4}\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \frac{\cancel{4}}{\cancel{4}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$

Hence proved $A^{-1}A = I_3$

Q.10: Verify that $(AB)^t = B^tA^t$ If

(i) $A=\begin{bmatrix}1 & -1 & 2\\ 0 & -3 & 1\end{bmatrix}$ and $B=\begin{bmatrix}1 & 1\\ -3 & -2\\ 0 & 1\end{bmatrix}$ (ii) $A=\begin{bmatrix}1 & 2\\ 1 & 4\\ 2 & 1\end{bmatrix}$ *and* $B=\begin{bmatrix}1 & -3\\ -2 & 1\end{bmatrix}$

Sol: $AB=\begin{bmatrix}1 & -1 & 2\\ 0 & -3 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ -3 & -2\\ 0 & 1\end{bmatrix}$

$AB=\begin{bmatrix}1+3+0 & 1+2+2\\ 0+9+0 & 0+6+1\end{bmatrix}=\begin{bmatrix}4 & 5\\ 9 & 7\end{bmatrix}$

L.H.S $=(AB)^t=\begin{bmatrix}4 & 9\\ 5 & 7\end{bmatrix}$

Now $B^t=\begin{bmatrix}1 & -3 & 0\\ 1 & -2 & 1\end{bmatrix}, A^t=\begin{bmatrix}1 & 0\\ -1 & -3\\ 2 & 1\end{bmatrix}$

R.H.S $B^tA^t=\begin{bmatrix}1 & -3 & 0\\ 1 & -2 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ -1 & -3\\ 2 & 1\end{bmatrix}$

$B^tA^t=\begin{bmatrix}1+3+0 & 0+9+0\\ 1+2+2 & 0+6+1\end{bmatrix}=\begin{bmatrix}4 & 9\\ 5 & 7\end{bmatrix}$

Hence prove that $(AB)^t = B^tA^t$

L.H.S = R.H.S

ii: $A=\begin{bmatrix}1 & 2\\ 1 & 4\\ 2 & 1\end{bmatrix}$ *and* $B=\begin{bmatrix}1 & -3\\ -2 & 1\end{bmatrix}$

Sol: $AB=\begin{bmatrix}1 & 2\\ 1 & 4\\ 2 & 1\end{bmatrix}\begin{bmatrix}1 & -3\\ -2 & 1\end{bmatrix}$

$AB=\begin{bmatrix}1-4 & -3+2\\ 1-8 & -3+4\\ 2-2 & -6+1\end{bmatrix}=\begin{bmatrix}-3 & -1\\ -7 & 1\\ 0 & -5\end{bmatrix}$

$AB=\begin{bmatrix}-3 & -1\\ -7 & 1\\ 0 & -5\end{bmatrix}$

L.H.S = $(AB)^t=\begin{bmatrix}-3 & -7 & 0\\ -1 & 1 & -5\end{bmatrix}$

Now

$$B' = \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}, \quad A' = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 1 \end{bmatrix}$$

R.H.S = $B'A'$

$$B'A' = \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 1 \end{bmatrix}$$

$$B'A' = \begin{bmatrix} 1-4 & 1-8 & 2-2 \\ -3+2 & -3+4 & -6+1 \end{bmatrix}$$

$$B'A' = \begin{bmatrix} -3 & -7 & 0 \\ -1 & 1 & -5 \end{bmatrix}$$

Hence Prove that $(AB)' = B'A'$ L.H.S = R.H.S