Q.1. Find the inverse of the following matrices by using row operations.

(i) $\begin{bmatrix} 2 & 6 & -3 \\ 0 & -2 & 0 \\ -2 & 5 & 6 \end{bmatrix}$

نوٹ:

Row option کے ذریعے inverse معلوم کرنے کے لیے دیے گئے matrix کے سامنے اُسی order کا identity matrix لیتا ہے پھر جو matrix دیا گیا ہے اسے Row operation سے identity بناتا ہے لیکن ہر operation دونوں Matrices پر لگاتا ہے جب دیا گیا matrix مکمل طور پر identity بن جائے گا تو ساتھ والا inverse بن جائے گا۔

Sol: Let $A = \begin{bmatrix} 2 & 6 & -3 \\ 0 & -2 & 0 \\ -2 & 5 & 6 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & 6 & -3 \\ 0 & -2 & 0 \\ -2 & 5 & 6 \end{vmatrix}$$

$$= 2\begin{vmatrix} -2 & 0 \\ 5 & 6 \end{vmatrix} - 6\begin{vmatrix} 0 & 0 \\ -2 & 6 \end{vmatrix} + (-3)\begin{vmatrix} 0 & -2 \\ -2 & 5 \end{vmatrix} = 2(-12-0) - 6(0-0) - 3(0-4) = -24+12$$

$$|A| = -12 \neq 0$$

As $|A| \neq 0$ so A is non-singular.

Appending I_3 on the right of the matrix A, we have

$$\left[\begin{array}{ccc:ccc} 2 & 6 & -3 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 & 0 \\ -2 & 5 & 6 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:ccc} 2 & 6 & -3 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 & 0 \\ 0 & 11 & 3 & 1 & 0 & 1 \end{array}\right] \text{ by } R_3 + R_1$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:ccc} 1 & 3 & -3/2 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/2 & 0 \\ 0 & 11 & 3 & 1 & 0 & 1 \end{array}\right] \text{ by } \frac{R_1}{2}, \frac{R_2}{-2}$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:ccc} 1 & 0 & -3/2 & 1/2 & 3/2 & 0 \\ 0 & 1 & 0 & 0 & -1/2 & 0 \\ 0 & 0 & 3 & 1 & 11/2 & 1 \end{array}\right] \text{ by } \begin{matrix} R_1 - 3R_2 \\ R_3 - 11R_2 \end{matrix}$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:ccc} 1 & 0 & -3/2 & 1/2 & 3/2 & 0 \\ 0 & 1 & 0 & 0 & -1/2 & 0 \\ 0 & 0 & 1 & 1/3 & 11/6 & 1/3 \end{array}\right] \text{ by } \frac{R_3}{3}$$

$$\underset{\sim}{R}\begin{bmatrix}1&0&0&\vdots&1&17/4&1/2\\0&1&0&\vdots&0&-1/2&0\\0&0&1&\vdots&1/3&11/6&1/3\end{bmatrix} \text{ by } R_1+\frac{3}{2}R_3$$

Hence

$$A^{-1}=\begin{bmatrix}1&17/4&1/2\\0&-1/2&0\\1/3&11/6&1/3\end{bmatrix}$$

ii. $\begin{bmatrix}1&2&-1\\0&-2&8\\1&0&2\end{bmatrix}$

Sol: let $A=\begin{bmatrix}1&2&-1\\0&-2&8\\1&0&2\end{bmatrix}$ then $|A|=\begin{vmatrix}1&2&-1\\0&-2&8\\1&0&2\end{vmatrix}$

$$=1\begin{vmatrix}-2&8\\0&2\end{vmatrix}-2\begin{vmatrix}0&8\\1&2\end{vmatrix}+(-1)\begin{vmatrix}0&-2\\1&0\end{vmatrix}=1(-4-0)-2(0-8)-1(0+2)=1(-4)-2(-8)-1(2)$$

$$=-4+16-2=10$$

As $|A|\neq 0$ So A is non-singular

For Row operation

$$\begin{bmatrix}1&2&-1&:&1&0&0\\0&-2&8&:&0&1&0\\1&0&2&:&0&0&1\end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix}1&2&-1&\vdots&1&0&0\\0&-2&8&\vdots&0&1&0\\0&-2&3&\vdots&-1&0&1\end{bmatrix} \text{ by } R_3-R_1$$

$$\underset{\sim}{R}\begin{bmatrix}1&2&-1&\vdots&1&0&0\\0&1&-4&\vdots&0&\frac{-1}{2}&0\\0&-2&3&\vdots&-1&0&1\end{bmatrix} \text{ by } \left(\frac{-1}{2}R_2\right)$$

$$\underset{\sim}{R}\begin{bmatrix}1&0&2&:&0&0&1\\0&1&-4&:&0&\frac{-1}{2}&0\\0&-2&3&:&-1&0&1\end{bmatrix} \text{ by } R_1+R_3$$

$$\underset{\sim}{R}\begin{bmatrix}1&0&2&\vdots&0&0&1\\0&1&-4&\vdots&0&\frac{-1}{2}&0\\0&0&-5&\vdots&-1&-1&1\end{bmatrix} \text{ by } R_3+2R_2$$

$$\Rightarrow \underset{\sim}{R}\left[\begin{array}{ccc:ccc} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & -4 & 0 & \frac{-1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{5} & \frac{1}{5} & \frac{-1}{5} \end{array}\right] by \left(\frac{-1}{5}\right)R_3$$

$$\Rightarrow \underset{\sim}{R}\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & \frac{-2}{5} & \frac{-2}{5} & \frac{7}{5} \\ 0 & 1 & 0 & \frac{4}{5} & \frac{3}{10} & \frac{-4}{5} \\ 0 & 0 & 1 & \frac{1}{5} & \frac{1}{5} & \frac{-1}{5} \end{array}\right] by \begin{array}{l} R_1 - 2R_3 \\ R_2 + 4R_3 \end{array}$$

Hence $A^{-1} = \begin{bmatrix} \frac{-2}{5} & \frac{-2}{5} & \frac{7}{5} \\ \frac{4}{5} & \frac{3}{10} & \frac{-4}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{-1}{5} \end{bmatrix}$

iii. $\begin{bmatrix} \mathbf{1} & \mathbf{6} & \mathbf{2} \\ \mathbf{2} & \mathbf{13} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} & \mathbf{1} \end{bmatrix}$

Sol: let $A = \begin{bmatrix} 1 & 6 & 2 \\ 2 & 13 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ then $|A| = \begin{vmatrix} 1 & 6 & 2 \\ 2 & 13 & 0 \\ 0 & -1 & 1 \end{vmatrix}$

$$|A| = 1\begin{vmatrix} 13 & 0 \\ -1 & 1 \end{vmatrix} - 6\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 13 \\ 0 & -1 \end{vmatrix} = 1(13-0) - 6(2-0) + 2(-2-0) = 13 - 12 - 4 = -3$$

For Row operation.

$$\Rightarrow \underset{\sim}{R}\left[\begin{array}{ccc:ccc} 1 & 6 & 2 & 1 & 0 & 0 \\ 2 & 13 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \Rightarrow \underset{\sim}{R}\left[\begin{array}{ccc:ccc} 1 & 6 & 2 & 1 & 0 & 0 \\ 0 & 1 & -4 & -2 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] by\ R_2 - 2R_1$$

$$\Rightarrow \underset{\sim}{R}\left[\begin{array}{ccc:ccc} 1 & 0 & 26 & 13 & -6 & 0 \\ 0 & 1 & -4 & -2 & 1 & 0 \\ 0 & 0 & -3 & -2 & 1 & 1 \end{array}\right] \begin{array}{l} R_1 - 6R_2 \\ R_3 + R_2 \end{array}$$

$$\Rightarrow \underset{\sim}{R}\left[\begin{array}{ccc:ccc} 1 & 0 & 26 & 13 & -6 & 0 \\ 0 & 1 & -4 & -2 & 1 & 0 \\ 0 & 0 & 1 & 2/3 & -1/3 & -1/3 \end{array}\right] by \left(\frac{-1}{3}\right)R_3$$

$$\Rightarrow \underset{\sim}{R}\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & \frac{-13}{3} & \frac{8}{3} & \frac{26}{3} \\ 0 & 1 & 0 & \frac{2}{3} & \frac{-1}{3} & \frac{-4}{3} \\ 0 & 0 & 1 & \frac{2}{3} & \frac{-1}{3} & \frac{-1}{3} \end{array}\right] \text{by } \begin{array}{l} R_1-26R_3 \\ R_2+4R_3 \end{array}$$

$$\text{Hence } A^{-1}=\begin{bmatrix} \frac{-13}{3} & \frac{8}{3} & \frac{26}{3} \\ \frac{2}{3} & \frac{-1}{3} & \frac{-4}{3} \\ \frac{2}{3} & \frac{-1}{3} & \frac{-1}{3} \end{bmatrix}$$

Q.2. Find the rank of the following matrices

i. $\begin{bmatrix} 1 & -1 & 3 & 1 \\ -2 & -6 & 1 & -1 \\ 3 & 1 & 4 & -2 \end{bmatrix}$

> Note: Rank = No. of non-zero rows in echelon form or reduced echelon form

> نوٹ:
> Rank معلوم کرنے کے لیے پہلی Row کے پہلے Element کو 1 بنائیں
> پھر اس سے نیچے تمام Elements کو زیرو بنائیں اس طرح پھر دوسری Row
> کے دوسرے Element کو 1 بنائیں پھر اس سے نیچے سے زیرو بنائیں۔

Sol: $\underset{\sim}{R}\begin{bmatrix} 1 & -1 & 3 & 1 \\ 0 & -8 & 7 & 1 \\ 0 & 4 & -5 & -5 \end{bmatrix} \text{by } \begin{array}{l} R_2+2R_1 \\ R_3-3R_1 \end{array}$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -1 & 3 & 1 \\ 0 & 1 & \frac{-7}{8} & \frac{-1}{8} \\ 0 & 4 & -5 & -5 \end{bmatrix} \text{by } \left(\frac{-1}{8}\right)R_2$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & \frac{17}{8} & \frac{1}{8} \\ 0 & 1 & \frac{-7}{8} & \frac{-1}{8} \\ 0 & 0 & \frac{-3}{2} & \frac{-9}{2} \end{bmatrix} \text{by } \begin{array}{l} R_1+R_2 \\ R_3-4R_2 \end{array}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & \frac{17}{8} & \frac{1}{8} \\ 0 & 1 & \frac{-7}{8} & \frac{-1}{8} \\ 0 & 0 & \frac{-3}{2} & \frac{-9}{2} \end{bmatrix} \text{by } \begin{array}{l} R_1+R_2 \\ R_3-4R_2 \end{array}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & \frac{17}{8} & \frac{1}{8} \\ 0 & 1 & \frac{-7}{8} & \frac{-1}{8} \\ 0 & 0 & 1 & 3 \end{bmatrix} \text{by } \left(\frac{-2}{3}\right)R_3$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 0 & \frac{-25}{4} \\ 0 & 1 & 0 & \frac{5}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix} \text{ by } \begin{matrix} R_1 - \frac{17}{8}R_3 \\ R_2 + \frac{7}{8}R_3 \end{matrix}$$

Note: Here All three rows are non zero , so Rank = 3

Hence Rank = 3

ii. $\begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ -1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$

Sol: $\underset{\sim}{R}\begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \\ 0 & -2 & 5 \\ 0 & 1 & -1 \end{bmatrix} \text{ by } \begin{matrix} R_2 - 2R_1 \\ R_3 + R_1 \end{matrix}$

Note: Here All three rows are non zero , so Rank = 3

Hence Rank = 3

iii. $\begin{bmatrix} 3 & -1 & 3 & 0 & 1 \\ 1 & 2 & -1 & -3 & -2 \\ 2 & 3 & 4 & 2 & 1 \\ 2 & 5 & -2 & -3 & 3 \end{bmatrix}$

Sol: $\underset{\sim}{R}\begin{bmatrix} 1 & -4 & -1 & -2 & 0 \\ 1 & 2 & -1 & -3 & -2 \\ 2 & 3 & 4 & 2 & 1 \\ 2 & 5 & -2 & -3 & 3 \end{bmatrix} \text{ by } R_1 - R_3$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -4 & -1 & -2 & 0 \\ 0 & 6 & 0 & -1 & -2 \\ 0 & 11 & 6 & 6 & 1 \\ 0 & 13 & 0 & 1 & 3 \end{bmatrix} \text{ by } \begin{matrix} R_2 - R_1 \\ R_3 - 2R_1 \\ R_4 - 2R_1 \end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -4 & -1 & -2 & 0 \\ 0 & 1 & 0 & \frac{-1}{6} & \frac{-1}{3} \\ 0 & 11 & 6 & 6 & 1 \\ 0 & 13 & 0 & 1 & 3 \end{bmatrix} \text{ by } \left(\frac{R_2}{6}\right)$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & -1 & \frac{-8}{3} & \frac{-4}{3} \\ 0 & 1 & 0 & \frac{-1}{6} & \frac{-1}{3} \\ 0 & 0 & 6 & \frac{47}{6} & \frac{14}{3} \\ 0 & 0 & 0 & \frac{19}{6} & \frac{22}{3} \end{bmatrix} \text{ by } \begin{matrix} R_1 + 4R_2 \\ R_3 - 11R_2 \\ R_4 - 13R_2 \end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 0 & -1 & \frac{-8}{3} & \frac{-4}{3}\\ 0 & 1 & 0 & \frac{-1}{6} & \frac{-1}{3}\\ 0 & 0 & 1 & \frac{47}{36} & \frac{14}{78}\\ 0 & 0 & 0 & \frac{19}{6} & \frac{22}{3}\end{bmatrix} by\left(\frac{R_3}{6}\right)$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 0 & 0 & \frac{49}{36} & \frac{-5}{9}\\ 0 & 1 & 0 & \frac{-1}{6} & \frac{-1}{3}\\ 0 & 0 & 1 & \frac{47}{36} & \frac{7}{9}\\ 0 & 0 & 0 & 1 & \frac{44}{19}\end{bmatrix} \begin{matrix}R_1+R_3 \\ by\ \frac{6}{19}R_4\end{matrix}\ and$$

Here non zero are four, so Rank = 4

Hence Rank = 4

Q.3. Solve the following systems of linear equations by Cramer's rule.

i.
$$2x+y-z=1$$
$$x-y+2z=3$$
$$3x+2y+z=4$$

Sol: $A=\begin{bmatrix}2 & 1 & -1\\ 1 & -1 & 2\\ 3 & 2 & 1\end{bmatrix}$

$$|A|=\begin{vmatrix}2 & 1 & -1\\ 1 & -1 & 2\\ 3 & 2 & 1\end{vmatrix}=2\begin{vmatrix}-1 & 2\\ 2 & 1\end{vmatrix}-1\begin{vmatrix}1 & 2\\ 3 & 1\end{vmatrix}+(-1)\begin{vmatrix}1 & -1\\ 3 & 2\end{vmatrix}=2(-1-4)-1(1-6)-1(2+3)$$

$$=2(-5)-1(-5)-1(5)=-10+5-5=-10\Rightarrow |A|=-10\neq 0$$

$$x=\frac{|A_x|}{|A|}=\frac{\begin{vmatrix}1 & 1 & -1\\ 3 & -1 & 2\\ 4 & 2 & 1\end{vmatrix}}{-10}=\frac{1(-1-4)-1(3-8)-1(6+4)}{-10}=\frac{-5+5-10}{-10}=1\Rightarrow x=1$$

$$y=\frac{|A_y|}{|A|}=\frac{\begin{vmatrix}2 & 1 & -1\\ 1 & 3 & 2\\ 3 & 4 & 1\end{vmatrix}}{-10}=\frac{2(3-8)-1(1-6)-1(4-9)}{-10}=\frac{2(-5)-(-5)-(-5)}{-10}=\frac{-10+5+5}{-10}=\frac{0}{-10}=0$$

$$z=\frac{|A_z|}{|A|}=\frac{\begin{vmatrix}2 & 1 & 1\\ 1 & -1 & 3\\ 3 & 2 & 4\end{vmatrix}}{-10}=\frac{2(-4-6)-1(4-9)+1(2+3)}{-10}=\frac{2(-10)-1(-5)+1(5)}{-10}$$

$$z=\frac{-20+5+5}{-10}=\frac{-20+10}{-10}=\frac{-10}{-10}=1$$

$$S.S=\{(1,0,1)\}$$

$x_1+2x_2-3x_3=0$

ii. $4x_1-x_2+x_3=5$

$-2x_1+3x_2+2x_3=3$

Sol: $A=\begin{bmatrix}1 & 2 & -3\\ 4 & -1 & 1\\ -2 & 3 & 2\end{bmatrix}$

$$|A|=\begin{vmatrix}1 & 2 & -3\\ 4 & -1 & 1\\ -2 & 3 & 2\end{vmatrix}=1\begin{vmatrix}-1 & 1\\ 3 & 2\end{vmatrix}-2\begin{vmatrix}4 & 1\\ -2 & 2\end{vmatrix}+(-3)\begin{vmatrix}4 & -1\\ -2 & 3\end{vmatrix}=1(-2-3)-2(8+2)-3(12-2)$$

$$=1(-5)-2(10)-3(10)$$

$$|A|=-5-20-30=-55$$

$$x_1=\frac{|A_{x_1}|}{|A|}=\frac{\begin{vmatrix}0 & 2 & -3\\ 5 & -1 & 1\\ 3 & 3 & 2\end{vmatrix}}{-55}=\frac{0(-2-3)-2(10-3)+(-3)(15+3)}{-55}=\frac{0-2(7)-3(18)}{-55}=\frac{-14-54}{-55}=\frac{68}{55}$$

$$x_2=\frac{|A_{x_2}|}{|A|}=\frac{\begin{vmatrix}1 & 0 & -3\\ 4 & 5 & 1\\ -2 & 3 & 2\end{vmatrix}}{-55}=\frac{1(10-3)-0(8+2)+(-3)(12+10)}{-55}=\frac{7-0-3(22)}{-55}=\frac{7-66}{-55}=\frac{59}{55}$$

$$x_3=\frac{|A_{x_3}|}{|A|}=\frac{\begin{vmatrix}1 & 2 & 0\\ 4 & -1 & 5\\ -2 & 3 & 3\end{vmatrix}}{-55}=\frac{1(-3-15)-2(12+10)+0(12-2)}{-55}=\frac{-18-2(22)}{-55}=\frac{-18-44}{-55}=\frac{62}{55}$$

$$S.S=\left\{\left(\frac{68}{55},\frac{59}{55},\frac{62}{55}\right)\right\}$$

$2x_1-x_2+x_3=1$

iii. $x_1+2x_2+2x_3=2$

$x_1-2x_2-x_3=1$

Sol: Here $A=\begin{bmatrix}2 & -1 & 1\\ 1 & 2 & 2\\ 1 & -2 & -1\end{bmatrix}$

$$|A|=\begin{vmatrix}2 & -1 & 1\\ 1 & 2 & 2\\ 1 & -2 & -1\end{vmatrix}\ |A|=2\begin{vmatrix}2 & 2\\ -2 & -1\end{vmatrix}-(-1)\begin{vmatrix}1 & 2\\ 1 & -1\end{vmatrix}+1\begin{vmatrix}1 & 2\\ 1 & -2\end{vmatrix}=2(-2+4)+1(-1-2)+1(-2-2)$$

$$|A|=2(2)+1(-3)+1(-4)=\cancel{4}-3-\cancel{4}=-3$$

$$|A|=-3\neq 0$$

$$x_1=\frac{|A_{x_1}|}{|A|}=\frac{\begin{vmatrix}1 & -1 & 1\\2 & 2 & 2\\1 & -2 & -1\end{vmatrix}}{-3}=\frac{1(-2+4)+1(-2-2)+1(-4-2)}{-3}=\frac{2-4-6}{-3}=\frac{-8}{-3}=\frac{8}{3}\Rightarrow x_1=\frac{8}{3}$$

$$x_2=\frac{|A_{x_2}|}{|A|}=\frac{\begin{vmatrix}2 & 1 & 1\\1 & 2 & 2\\1 & 1 & -1\end{vmatrix}}{-3}=\frac{2(-2-2)-1(-1-2)+1(1-2)}{-3}=\frac{2(-4)-1(-3)+1(-1)}{-3}$$

$$x_2=\frac{-8+3-1}{-3}=2\Rightarrow x_2=2$$

$$x_3=\frac{|A_{x_3}|}{|A|}=\frac{\begin{vmatrix}2 & -1 & 1\\1 & 2 & 2\\1 & -2 & 1\end{vmatrix}}{-3}=\frac{2(2+4)+1(1-2)+1(-2-2)}{-3}=\frac{2(6)+1(-1)+1(-4)}{-3}=\frac{12-1-4}{-3}=\frac{7}{-3}$$

$$\Rightarrow x_3=\frac{-7}{3}$$

$$S.S=\left\{\left(\frac{8}{3},2,\frac{-7}{3}\right)\right\}$$

Q.4. Solve the following systems of linear equations by matrix inversion method

i.
$$\begin{aligned}x-2y+z&=-1\\3x+y-2z&=4\\y-z&=1\end{aligned}$$

Sol: In matrix forms

$$\begin{bmatrix}1 & -2 & 1\\3 & 1 & -2\\0 & 1 & -1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-1\\4\\1\end{bmatrix}$$

$$A\quad X=B\Rightarrow X=A^{-1}B\_\_\_\_ I$$

Where $A=\begin{bmatrix}1 & -2 & 1\\3 & 1 & -2\\0 & 1 & -1\end{bmatrix}, X=\begin{bmatrix}x\\y\\z\end{bmatrix} B=\begin{bmatrix}-1\\4\\1\end{bmatrix}$

$$|A|=\begin{vmatrix}1 & -2 & 1\\3 & 1 & -2\\0 & 1 & -1\end{vmatrix}=1(-1+2)-(-2)(-3+0)+1(3-0)=1+2(-3)+3=1-6+3=-2\neq 0$$

$$A_{11}=(-1)^{1+1}\begin{vmatrix}1 & -2\\1 & -1\end{vmatrix}=(-1+2)=1$$

$$A_{12}=(-1)^{1+2}\begin{vmatrix}3 & -2\\0 & -1\end{vmatrix}=-(-3+0)=3$$

$$A_{13}=(-1)^{1+3}\begin{vmatrix}3 & 1\\0 & 1\end{vmatrix}=(3-0)=3$$

$$A_{21}=(-1)^{2+1}\begin{vmatrix}-2 & 1\\1 & -1\end{vmatrix}=-(2-1)=-1$$

$$A_{22}=(-1)^{2+2}\begin{vmatrix}1 & 1\\0 & -1\end{vmatrix}=(-1-0)=-1$$

$$A_{23}=(-1)^{2+3}\begin{vmatrix}1 & -2\\0 & 1\end{vmatrix}=-(1+0)=-1$$

$$A_{31}=(-1)^{3+1}\begin{vmatrix}-2 & 1\\1 & -2\end{vmatrix}=(4-1)=3$$

$$A_{32}=(-1)^{3+2}\begin{vmatrix}1 & 1\\3 & -2\end{vmatrix}=-(-2-3)=5$$

$$A_{33}=(-1)^{3+3}\begin{vmatrix}1 & -2\\3 & 1\end{vmatrix}=(1+6)=7$$

$$\text{Cofactor of } A=\begin{bmatrix}A_{11} & A_{12} & A_{13}\\A_{21} & A_{22} & A_{23}\\A_{31} & A_{32} & A_{33}\end{bmatrix}=\begin{bmatrix}1 & 3 & 3\\-1 & -1 & -1\\3 & 5 & 7\end{bmatrix}$$

$$adjA=(co-factor\ of\ A)'=\begin{bmatrix}1 & -1 & 3\\3 & -1 & 5\\3 & -1 & 7\end{bmatrix}$$

$$A^{-1}=\frac{AdjA}{|A|}=\frac{1}{-2}\begin{bmatrix}1 & -1 & 3\\3 & -1 & 5\\3 & -1 & 7\end{bmatrix}$$

$$I \Rightarrow X=A^{-1}B$$

$$X=\frac{-1}{2}\begin{bmatrix}1 & -1 & 3\\3 & -1 & 5\\3 & -1 & 7\end{bmatrix}\begin{bmatrix}-1\\4\\1\end{bmatrix}=\frac{-1}{2}\begin{bmatrix}-1-4+3\\-3-4+5\\-3-4+7\end{bmatrix}$$

$$X=\frac{-1}{2}\begin{bmatrix}-2\\-2\\0\end{bmatrix}=\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\1\\0\end{bmatrix}\Rightarrow x=1, y=1, z=0$$

$$S.S=\{(1,1,0)\}$$

ii.
$$2x_1 + x_2 + 3x_3 = 3$$
$$x_1 + 3x_2 - 2x_3 = 0$$
$$-3x_1 - x_2 + 2x_3 = 4$$

Sol: In matrix forms

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 3 & -2 \\ -3 & -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}$$

$$A \quad X = B \Rightarrow X = A^{-1}B \text{------}(I)$$

$$|A| = \begin{vmatrix} 2 & 1 & 3 \\ 1 & 3 & -2 \\ -3 & -1 & 2 \end{vmatrix}$$

$$= 2(6-2) - 1(2-6) + 3(-1+9)$$
$$= 2(4) - 1(-4) + 3(8) = 8 + 4 + 24 = 36$$

Now,

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 3 & -2 \\ -3 & -1 & 2 \end{bmatrix}$$

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 3 & -2 \\ -1 & 2 \end{vmatrix} = (-1)^2(6-2) = 4$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 1 & -2 \\ -3 & 2 \end{vmatrix} = (-1)^3(2-6) = -1(-4) = 4$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 1 & 3 \\ -3 & -1 \end{vmatrix} = (-1)^4(-1+9) = 8$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} 1 & 3 \\ -1 & 2 \end{vmatrix} = (-1)^3(2+3) = -(+5) = -5$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 2 & 3 \\ -3 & 2 \end{vmatrix} = (-1)^4(4+9) = 13$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 2 & 1 \\ -3 & -1 \end{vmatrix} = (-1)^5(-2+3) = -(1) = -1$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} 1 & 3 \\ 3 & -2 \end{vmatrix} = (-1)^4(-2-9) = -11$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = (-1)^5(-4-3) = -(-7) = 7$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = (-1)^6(6-1) = 5$$

$$\text{cofactor of } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} 4 & 4 & 8 \\ -5 & 13 & -1 \\ -11 & 7 & 5 \end{bmatrix}$$

$$AdjA = (\text{cofactor of } A)^t = \begin{bmatrix} 4 & -5 & -11 \\ 4 & 13 & 7 \\ 8 & -1 & 5 \end{bmatrix}$$

$$A^{-1} = \frac{AdjA}{|A|} = \frac{1}{36}\begin{bmatrix} 4 & -5 & -11 \\ 4 & 13 & 7 \\ 8 & -1 & 5 \end{bmatrix}$$

$$I \Rightarrow X = A^{-1}B$$

$$X = \frac{1}{36}\begin{bmatrix} 4 & -5 & -11 \\ 4 & 13 & 7 \\ 8 & -1 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}$$

$$X = \frac{1}{36}\begin{bmatrix} 12+0-44 \\ 12+0+28 \\ 24-0+20 \end{bmatrix} = \frac{1}{36}\begin{bmatrix} -32 \\ 40 \\ 44 \end{bmatrix} = \begin{bmatrix} \frac{-8}{9} \\ \frac{10}{9} \\ \frac{11}{9} \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{-8}{9} \\ \frac{10}{9} \\ \frac{11}{9} \end{bmatrix} \text{ so } x_1 = \frac{-8}{9},\ x_2 = \frac{10}{9},\ x_3 = \frac{11}{9}$$

$$S.S = \left\{\left(\frac{-8}{9}, \frac{10}{9}, \frac{11}{9}\right)\right\}$$

iii.
$$x + y = 2$$
$$2x - z = 1$$
$$2y - 3z = -1$$

Sol: In matrix forms

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & -1 \\ 0 & 2 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$$A \quad X = B \Rightarrow X = A^{-1}B \underline{\qquad} I$$

Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & -1 \\ 0 & 2 & -3 \end{bmatrix}, X\begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 1 & 0 \\ 2 & 0 & -1 \\ 0 & 2 & -3 \end{vmatrix} = 1(0+2) - 1(-6+0) + 0(4-0)$$

$$= 2 + 6 + 0 = 8 \neq 0$$

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 0 & -1 \\ 2 & -3 \end{vmatrix} = (0+2) = 2$$

$$A_{12} = (-1)^{1+2} \begin{vmatrix} 2 & -1 \\ 0 & -3 \end{vmatrix} = -(-6+0) = 6$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = (4-0) = 4$$

$$A_{21} = (-1)^{2+1} \begin{vmatrix} 1 & 0 \\ 2 & -3 \end{vmatrix} = (-3-0) = 3$$

$$A_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 0 \\ 0 & -3 \end{vmatrix} = (-3-0) = -3$$

$$A_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = -(2-0) = -2$$

$$A_{31} = (-1)^{3+1} \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = (-1-0) = -1$$

$$A_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} = (-1-0) = 1$$

$$A_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = (0-2) = -2$$

$$cofactor\ of\ A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} 2 & 6 & 4 \\ 3 & -3 & -2 \\ -1 & 1 & -2 \end{bmatrix}$$

$$AdjA = (co-factor\ of\ A)' = \begin{bmatrix} 2 & 3 & -1 \\ 6 & -3 & 1 \\ 4 & -2 & -2 \end{bmatrix}$$

$$A^{-1} = \frac{adjA}{|A|} = \frac{1}{8}\begin{bmatrix} 2 & 3 & -1 \\ 6 & -3 & 1 \\ 4 & -2 & -2 \end{bmatrix}$$

$$I \Rightarrow X = A^{-1}B = \frac{1}{8}\begin{bmatrix} 2 & 3 & -1 \\ 6 & -3 & 1 \\ 4 & -2 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$$X = \frac{1}{8}\begin{bmatrix} 4+3+1 \\ 12-3-1 \\ 8-2+2 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 8 \\ 8 \\ 8 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$ Hence $x = 1,\ y = 1,\ z = 1$

$S.S\{(1,1,1)\}$

Q.5. Solve the following systems by reducing their augmented matrices to the echelon form and the reduced echelon forms.

i.
$$\mathbf{x_1 + 2x_2 - 2x_3 = -1}$$
$$\mathbf{2x_1 + 3x_2 + x_3 = 1}$$
$$\mathbf{5x_1 + 4x_2 - 3x_3 = 1}$$

Sol: The Augmented matrix is

$$\begin{bmatrix} 1 & 2 & -2 & : & -1 \\ 2 & 3 & 1 & : & 1 \\ 5 & 4 & -3 & : & 1 \end{bmatrix}$$

For Echelon form

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -2 & : & -1 \\ 0 & -1 & 5 & : & 3 \\ 0 & -6 & 7 & : & 6 \end{bmatrix} by \begin{matrix} R_2 - 2R_1 \\ R_3 - 5R_1 \end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -2 & : & -1 \\ 0 & 1 & -5 & : & -3 \\ 0 & -6 & 7 & : & 6 \end{bmatrix} by\ (-1)R_2$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -2 & : & -1 \\ 0 & 1 & -5 & : & -3 \\ 0 & 0 & -23 & : & -12 \end{bmatrix} by\ R_3 + 6R_2$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -2 & : & -1 \\ 0 & 1 & -5 & : & -3 \\ 0 & 0 & 1 & : & \frac{12}{23} \end{bmatrix} by \left(\frac{-1}{23}\right)6R_2 \quad \text{\_\_\_\_\_\_ (I)}$$

Is required Echelon from where

$x_1 + 2x_2 - 2x_3 = -1$ ________(i)

$x_2 - 5x_3 = -3$ ________(ii)

$x_3 = \frac{12}{23}$ ________(iii)

Put (iii) in (ii)

$$x_2 - 5\left(\frac{12}{23}\right) = -3$$

$$x_2-\frac{60}{23}=-3\Rightarrow x_2=-3+\frac{60}{23}$$

$$x_2=\frac{-69+60}{23}=\frac{-9}{23}\Rightarrow x_2=\frac{-9}{23}$$

Put in x_2 and x_3 in (i)

$$x_1+2\left(\frac{-9}{23}\right)-2\left(\frac{12}{23}\right)=-1$$

$$x_1-\frac{18}{23}-\frac{24}{23}=-1$$

$$x_1-\frac{18}{23}-\frac{24}{23}+1=0$$

$$x_1+\frac{-18-24+23}{23}=0$$

$$x_1-\frac{19}{23}=0\Rightarrow x_1=\frac{19}{23}$$

For Reduced Echelon form continue (1)

$$\underset{\sim}{R}\begin{bmatrix}1 & 2 & -2 & : & -1\\ 0 & 1 & -2 & : & -3\\ 0 & 0 & 1 & : & \frac{12}{23}\end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 0 & 8 & : & 5\\ 0 & 1 & -5 & : & -3\\ 0 & 0 & 1 & : & \frac{12}{23}\end{bmatrix} \text{ by } R_1-2R_2$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 0 & 0 & : & \frac{19}{23}\\ 0 & 1 & 0 & : & \frac{-9}{23}\\ 0 & 0 & 1 & : & \frac{12}{23}\end{bmatrix} \text{ by } \begin{matrix}R_1-8R_3\\ R_2+5R_3\end{matrix}$$

So,

$$x_1=\frac{19}{23},\ x_2=\frac{-9}{23}, x_3=\frac{12}{23}$$

Answer is $\left\{\left(\frac{19}{23},\frac{-9}{23},\frac{12}{23}\right)\right\}$

ii.

$$x+2y+z=2$$
$$2x+y+2z=3$$
$$2x+3y-z=7$$

Sol. The augmented matrix is

$$\begin{bmatrix}1 & 2 & 1 & : & 2\\ 2 & 1 & 2 & : & 3\\ 2 & 3 & -1 & : & 7\end{bmatrix}$$

For Echelon form

$$\underset{\sim}{R}\begin{bmatrix}1 & 2 & 1 & : & 2\\ 0 & -3 & 0 & : & -1\\ 0 & -1 & -3 & : & 3\end{bmatrix}\begin{matrix}R_3-2R_1\\ R_2-2R_1\end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 2 & 1 & : & 2\\ 0 & 1 & 12 & : & -13\\ 0 & -1 & -3 & : & 3\end{bmatrix} R_2-4R_3$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 2 & 1 & : & 2\\ 0 & 1 & 12 & : & -13\\ 0 & 0 & 9 & : & -10\end{bmatrix} R_3+R_2$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 2 & 1 & : & 2\\ 0 & 1 & 12 & : & -13\\ 0 & 0 & 1 & : & \frac{-10}{9}\end{bmatrix}\frac{1}{9}R_3 \quad \text{\_\_\_\_\_ (A)}$$

Which is required echelon form

$$x+2y+z=2 \quad \text{\_\_\_\_\_ (i)}$$
$$y+12z=-13 \quad \text{\_\_\_\_\_ (ii)}$$
$$z=\frac{-10}{9} \quad \text{\_\_\_\_\_ (iii)}$$

Put (iii) in (ii)

$$y+\cancel{12}^{4}\left(\frac{-10}{\cancel{9}}\right)=-13$$

$$y-\frac{40}{3}=-13\Rightarrow y=-13+\frac{40}{3}$$

$$y=\frac{-39+40}{3}=\frac{1}{3}$$

$$\Rightarrow y=\frac{1}{3}$$

Put in y and z in (i)

$$x+2\left(\frac{1}{3}\right)-\frac{10}{9}=2$$

$$x=2+\frac{10}{9}-\frac{2}{3}=\frac{18+10-6}{9}=\frac{22}{9}$$

$$\Rightarrow x=\frac{22}{9}$$

For Reduced Echelon form continued (A)

$$R\begin{bmatrix} 1 & 2 & 1 & : & 2 \\ 0 & 1 & 12 & : & -13 \\ 0 & 0 & 1 & : & \frac{-10}{9} \end{bmatrix}$$

$$R\begin{bmatrix} 1 & 0 & -23 & : & 28 \\ 0 & 1 & 12 & : & -13 \\ 0 & 0 & 1 & : & \frac{-10}{9} \end{bmatrix} R_1 - 2R_2$$

$$R\begin{bmatrix} 1 & 0 & 0 & : & \frac{22}{9} \\ 0 & 1 & 0 & : & \frac{1}{3} \\ 0 & 0 & 1 & : & \frac{-10}{9} \end{bmatrix} by \begin{matrix} R_2 - 12R_3 \\ R_1 + 23R_3 \end{matrix}$$

So, $x = \frac{22}{9}$, $y = \frac{1}{3}$, $z = \frac{-10}{9}$

Answer is $= \left\{\left(\frac{22}{9}, \frac{1}{3}, \frac{-10}{9}\right)\right\}$

iii.
$$\begin{aligned} x_1 + 4x_2 + x_3 &= 2 \\ 2x_1 + x_2 - 2x_3 &= 9 \\ 3x_1 + x_2 - x_3 &= 12 \end{aligned}$$

Sol. The augmented matrix is

$$\begin{bmatrix} 1 & 4 & 1 & : & 2 \\ 2 & 1 & -2 & : & 9 \\ 3 & 1 & -1 & : & 12 \end{bmatrix}$$

For Echelon form

$$R\begin{bmatrix} 1 & 4 & 1 & : & 2 \\ 0 & -7 & -4 & : & 5 \\ 0 & -11 & -4 & : & 6 \end{bmatrix} by \begin{matrix} R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix}$$

$$R\begin{bmatrix} 1 & 4 & 1 & : & 2 \\ 0 & 1 & \frac{4}{7} & : & \frac{-5}{7} \\ 0 & -11 & -4 & : & 6 \end{bmatrix} \frac{-1}{7}R_2$$

$$R\begin{bmatrix} 1 & 4 & 1 & : & 2 \\ 0 & 1 & \frac{4}{7} & : & \frac{-5}{7} \\ 0 & 0 & \frac{16}{7} & : & \frac{-13}{7} \end{bmatrix} R_3 + 11R_2$$

$$R\begin{bmatrix} 1 & 4 & 1 & : & 2 \\ 0 & 1 & \frac{4}{7} & : & \frac{-5}{7} \\ 0 & 0 & 1 & : & \frac{-13}{16} \end{bmatrix} \frac{7}{16}R_3 \quad \text{\_\_\_\_\_ (A)}$$

$x_1 + 4x_2 + x_3 = 2$ _____ **(i)**

$x_2 + \frac{4}{7}x_3 = \frac{-5}{7}$ _____ **(ii)**

$x_3 = \frac{-13}{16}$ _____ **(iii)**

put (iii) in (ii)

$$x_2 + \frac{4}{7}\left(\frac{-13}{16}\right) = \frac{-5}{7}$$

$$x_2 - \frac{52}{112} = \frac{-5}{7} \Rightarrow x_2 = \frac{-5}{7} + \frac{52}{112}$$

$$x_2 = \frac{-1}{4}$$

Put in x_2, x_3 in (i)

$$x_1 + 4\left(\frac{-1}{4}\right) - \frac{13}{16} = 2$$

$$x_1 = \frac{61}{16}$$

For Reduced Echelon form continued (A)

$$R\begin{bmatrix} 1 & 0 & \frac{-9}{7} & : & \frac{34}{7} \\ 0 & 1 & \frac{4}{7} & : & \frac{-5}{7} \\ 0 & 0 & 1 & : & \frac{-13}{16} \end{bmatrix} by\ R_1 - 4R_2$$

$$R\begin{bmatrix} 1 & 0 & 0 & : & \frac{61}{16} \\ 0 & 1 & 0 & : & \frac{-1}{4} \\ 0 & 0 & 1 & : & \frac{-13}{16} \end{bmatrix} by \begin{matrix} R_1 + \frac{9}{7}R_3 \\ R_2 - \frac{4}{7}R_3 \end{matrix}$$

So, $x_1 = \frac{61}{16}$, $x_2 = \frac{-1}{4}$, $x_3 = \frac{-13}{16}$

Answer is $= \left\{\left(\frac{61}{16}, \frac{-1}{4}, \frac{-13}{16}\right)\right\}$

Q.6. Solve the following system of Homogenous linear equation by using Gaussian elimination method.

i.
$$x+4y-2z=0$$
$$2x+y+5z=0$$
$$5x+2y+8z=0$$

Sol. The augmented matrix is

$$\underset{\sim}{R}\begin{bmatrix}1 & 4 & -2 & : & 0\\ 2 & 1 & 5 & : & 0\\ 5 & 2 & 8 & : & 0\end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 4 & -2 & : & 0\\ 0 & -7 & 9 & : & 0\\ 0 & -18 & 18 & : & 0\end{bmatrix}\begin{matrix}R_2-2R_1\\ R_3-5R_1\end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 4 & -2 & : & 0\\ 0 & 1 & -9/7 & : & 0\\ 0 & -18 & 18 & : & 0\end{bmatrix}\frac{R_2}{-7}$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 4 & -2 & : & 0\\ 0 & 1 & -9/7 & : & 0\\ 0 & 0 & -36/7 & : & 0\end{bmatrix}R_3+18R_2$$

Solve for x, y and z

$$-\frac{36}{7}z=0 \Rightarrow z=0$$

$$y-\frac{9}{7}z=0 \Rightarrow y-\frac{9}{7}(0)=0 \Rightarrow y=0$$

$$x+4y-2z=0$$
$$x+4(0)-(0)=0$$
$$x=0$$

The solution is $x=y=z=0$

which is the trivial solution

So Answers is $\{(0,0,0)\}$

ii.
$$x_1+4x_2+2x_3=0$$
$$2x_1+x_2-3x_3=0$$
$$3x_1+2x_2-4x_3=0$$

Sol. The augmented matrix is

$$\underset{\sim}{R}\begin{bmatrix}1 & 4 & 2 & : & 0\\ 2 & 1 & -3 & : & 0\\ 3 & 2 & -4 & : & 0\end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 4 & 2 & : & 0\\ 2 & 1 & -3 & : & 0\\ 3 & 2 & -4 & : & 0\end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 4 & 2 & : & 0\\ 0 & -7 & -7 & : & 0\\ 0 & -10 & -10 & : & 0\end{bmatrix}\begin{matrix}R_2-2R_1\\ R_3-3R_1\end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 4 & 2 & : & 0\\ 0 & -7 & -7 & : & 0\\ 0 & 0 & 0 & : & 0\end{bmatrix}R_3-\frac{10}{7}R_2$$

Solve for x_1, x_2 and x_3

$$-7x_2-7x_3=0 \Rightarrow x_2=-x_3$$

$$x_1+4x_2+2x_3=0 \Rightarrow x_1+4(-x_3)+2x_3=0$$

$$x_1-4x_2+2x_3=0 \Rightarrow x_1=2x_3$$

Let $x_3=t$ (any constant)

Then $x_1=2t, x_2=-t, x_3=t$ in the solution

Where t is any real number. This system has infinitely many solutions. ,

iii.
$$x_1+2x_2-x_3=0$$
$$x_1-x_2+5x_3=0$$
$$2x_1+x_2+4x_3=0$$

Sol. The augmented matrix is

$$\underset{\sim}{R}\begin{bmatrix}1 & 2 & -1 & : & 0\\ 1 & -1 & 5 & : & 0\\ 2 & 1 & 4 & : & 0\end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 2 & -1 & : & 0\\ 0 & -3 & 6 & : & 0\\ 0 & -3 & 6 & : & 0\end{bmatrix}\begin{matrix}R_2-R_1\\ R_3-2R_1\end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 2 & -1 & : & 0\\ 0 & -3 & 6 & : & 0\\ 0 & 0 & 0 & : & 0\end{bmatrix}R_3-R_2$$

Solve for x_1 and x_2 in terms of x_3

$$-3x_2+6x_3=0 \Rightarrow x_2=2x_3$$

$$x_1+2x_2-x_3=0 \Rightarrow x_1+2(2x_3)-x_3=0$$

$$x_1=-3x_3$$

Let $x_3=t$ (any constant)

Then $x_1=-3t, x_2=2t, x_3=t$

The solution is $x_1=-3t$, $x_2=2t$, $x_3=t$

Where t is any real number. This system has infinitely many solutions.

Q.7. A triangle has vertices at $A(4,1), B(-2,5)$ and $C(0,-3)$. Find the vertices of the reflected triangle over the y-axis using a transformation matrix.

Sol. To reflect a point across a certain axis or line, we have multiply the point as a column vector by the corresponding transformation matrix. Here, to reflect the given points over the y-axis we use the transformation matrix. $\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$

Write the points as column matrices

$$A=\begin{bmatrix}4\\1\end{bmatrix}, B=\begin{bmatrix}-2\\5\end{bmatrix}, C=\begin{bmatrix}0\\-3\end{bmatrix}$$

The vertex A′ of the reflected triangle

$$=\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}4\\1\end{bmatrix}=\begin{bmatrix}-4+0\\0+1\end{bmatrix}=\begin{bmatrix}4\\1\end{bmatrix}=(-4,1)$$

The vertex B′ of the reflected Triangle

$$=\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}-2\\5\end{bmatrix}=\begin{bmatrix}2+0\\0+5\end{bmatrix}=\begin{bmatrix}2\\5\end{bmatrix}=(2,5)$$

The vertex C′ of the reflected image

$$=\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}0\\-3\end{bmatrix}=\begin{bmatrix}0+0\\0-3\end{bmatrix}=\begin{bmatrix}0\\-3\end{bmatrix}=(0,-3)$$

Thus the vertices of the reflected triangle are

$$A'(-4,1)\ ,\ B'(2,5)\ and\ C'(0,-3)$$

Coding is the process of converting a message into a specific format using a code. A code is a system of symbols, words or signals used to represent other words meanings. It's often used to hide the actual meaning of a message.

Q.8. The point A is mapped to $(30,20,-5)$ by the scaling matrix $p=\begin{bmatrix}-5 & 0 & 0\\0 & -5 & 0\\0 & 0 & -5\end{bmatrix}$

Find the coordinates of A.

Sol. Given $P=\begin{bmatrix}-5 & 0 & 0\\0 & -5 & 0\\0 & 0 & -5\end{bmatrix}$

Let $A=\begin{bmatrix}x\\y\\z\end{bmatrix}$ and $A'=\begin{bmatrix}30\\30\\-5\end{bmatrix}$

If A is mapped to A′ by scaling Matrix P, then $AP=A'$

$P.A=A'$

$A=P^{-1}A'$ __________(i)

Now,

$$P^{-1}=\begin{bmatrix}\frac{-1}{5} & 0 & 0\\0 & \frac{-1}{5} & 0\\0 & 0 & \frac{-1}{5}\end{bmatrix}$$

Put values A and P^{-1} in (i)

$A=P^{-1}A'$

$$A=\begin{bmatrix}\frac{-1}{5} & 0 & 0\\0 & \frac{-1}{5} & 0\\0 & 0 & \frac{-1}{5}\end{bmatrix}\begin{bmatrix}30\\20\\-5\end{bmatrix}$$

$$A=\begin{bmatrix}\frac{-1}{5}\times 30\\\frac{-1}{5}\times 20\\\frac{-1}{5}\times -5\end{bmatrix}=\begin{bmatrix}-6\\-4\\1\end{bmatrix}$$

Hence the coordinate of A is $A=\begin{bmatrix}-6\\-4\\1\end{bmatrix}$

Q.9. Find the equation of the image of the curve with equation $y=x^2$ under the transformation with associated matrix. $\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$

Sol. This transformation matrix represents linear transformation that maps the original curve to its image. To find the equation of the image, we need to apply the transformation to the original curve equation. Let's denote the original coordinates as (x,y) and the transformation coordinates as (x',y'). The transformation matrix maps the original coordinates to the transformed coordinates as follows.

$x'=x+2y$ __________(i)

$y'=3x+4y$ __________(ii)

Now, substitute the original curve equation $y = x^2$ into the transformation equation.

$y' = 3x + 4x^2$

$y' = 3x' - 2y' + 4\left(x'^2 - 2x'y' + y'^2\right)$

Put $x = y' - 2x'$ by solving (i) & (ii)

Simplify the equation to get the image curve equation.

$y' = 3x' - 2y' + 4x'^2 - 8x'y' + 4y'^2$

This is the equation of the image curve under the given transformation.

Q.10. Use the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ to encode the message: KEEP IT UP, where letters A to Z are corresponding to the numbers 1 to 26 .

Sol. Here

A^1 B^2 C^3 D^4 E^5 F^6 G^7 H^8 I^9
J^{10} K^{11} L^{12} M^{13} N^{14} O^{15} P^{16} Q^{17} R^{18}
S^{19} T^{20} U^{21} V^{22} W^{23} X^{24} Y^{25} Z^{26}

Divide the letters of the message into groups of three,

KEE , PIT , UP

Assign the numbers to these letters and convert each pair of numbers into 3×1 matrices.

$$\begin{bmatrix} K \\ E \\ E \end{bmatrix} = \begin{bmatrix} 11 \\ 5 \\ 5 \end{bmatrix}, \begin{bmatrix} P \\ I \\ T \end{bmatrix} = \begin{bmatrix} 16 \\ 9 \\ 20 \end{bmatrix}, \begin{bmatrix} U \\ P \\ O \end{bmatrix} = \begin{bmatrix} 21 \\ 16 \\ 0 \end{bmatrix}$$

So the message in 3×1 matrices is

$$\begin{bmatrix} 11 \\ 5 \\ 5 \end{bmatrix}\begin{bmatrix} 16 \\ 9 \\ 20 \end{bmatrix}\begin{bmatrix} 21 \\ 16 \\ 0 \end{bmatrix}$$

Now to encode, we multiply on the left each matrix of our message by the matrix A i.e

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 11 \\ 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 11+0+5 \\ 22-5+5 \\ 0+5+10 \end{bmatrix} = \begin{bmatrix} 16 \\ 22 \\ 15 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 16 \\ 9 \\ 20 \end{bmatrix} = \begin{bmatrix} 16+0+20 \\ 32-9+20 \\ 0+9+40 \end{bmatrix} = \begin{bmatrix} 36 \\ 43 \\ 49 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 21 \\ 16 \\ 0 \end{bmatrix} = \begin{bmatrix} 21+0+0 \\ 42-16+0 \\ 0+16+0 \end{bmatrix} = \begin{bmatrix} 21 \\ 26 \\ 16 \end{bmatrix}$$

So, the derived coded message is,

$$\begin{bmatrix} 16 \\ 22 \\ 15 \end{bmatrix}\begin{bmatrix} 36 \\ 43 \\ 49 \end{bmatrix}\begin{bmatrix} 21 \\ 26 \\ 16 \end{bmatrix}$$

Q.11. Decode the message $\begin{bmatrix} 11 \\ 20 \\ 43 \end{bmatrix}\begin{bmatrix} 25 \\ 10 \\ 41 \end{bmatrix}\begin{bmatrix} 22 \\ 14 \\ 41 \end{bmatrix}$ that was encoded using matrix $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$ where the number 1 to 26 are corresponding to the letters A to Z, and 27 is representing space or "-"

Sol: The encoding matrix A is

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

Each number in the decoded vectors corresponds to a letter from $A(1)$ to $Z(26)$

First we need to find A^{-1}.

$$|A| = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{vmatrix} = 1(0-1) - 1(1-2) + (-1)(1-0)$$

$$|A| = -1 + 1 - 1 = -1 \Rightarrow |A| = -1$$

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = (1)^2(0-1) = -1$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = (-1)^3(1-2) = 1$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = (-1)^4(1-0) = 1$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = (-1)^3(1+1) = -2$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = (-1)^4(1+2) = 3$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = (-1)^5(1-2) = 1$$

$$A_{31}=(-1)^{3+1}\begin{vmatrix}1 & -1\\ 0 & 1\end{vmatrix}=(-1)^4(1-2)=1$$

$$A_{32}=(-1)^{3+2}\begin{vmatrix}1 & -1\\ 1 & 1\end{vmatrix}=(-1)^5(1+1)=-2$$

$$A_{33}=(-1)^{3+3}\begin{vmatrix}1 & 1\\ 1 & 0\end{vmatrix}=(-1)^6(0-1)=-1$$

Co-factor of A $A=\begin{bmatrix}A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{bmatrix}$

$$=\begin{bmatrix}-1 & 1 & 1\\ -2 & 3 & 1\\ 1 & -2 & -1\end{bmatrix}$$

$$Adj\,A=(Co-factor\ of\ A)'=\begin{bmatrix}-1 & -2 & 1\\ 1 & 3 & -2\\ 1 & 1 & -1\end{bmatrix}$$

$$A^{-1}=\frac{AdjA}{|A|}=\frac{1}{-1}\begin{bmatrix}-1 & -2 & 1\\ 1 & 3 & -2\\ 1 & 1 & -1\end{bmatrix}$$

$$A^{-1}=\begin{bmatrix}1 & 2 & -1\\ -1 & -3 & 2\\ -1 & -1 & 1\end{bmatrix}$$

Now, we multiply each encoded vector by A^{-1} to get original message vector.

First Encoded vector $=\begin{bmatrix}11\\ 20\\ 43\end{bmatrix}$

Original $A^{-1}\begin{bmatrix}11\\ 20\\ 43\end{bmatrix}=\begin{bmatrix}1 & 2 & -1\\ -1 & -3 & 2\\ -1 & -1 & 1\end{bmatrix}\begin{bmatrix}11\\ 20\\ 43\end{bmatrix}$

$$=\begin{bmatrix}11+40-43\\ -11-60+86\\ -11-20+43\end{bmatrix}=\begin{bmatrix}8\\ 15\\ 12\end{bmatrix}$$

So the first original vector is $\begin{bmatrix}8\\ 15\\ 12\end{bmatrix}$

Second encoded vector: $\begin{bmatrix}25\\ 10\\ 41\end{bmatrix}$

Original $=A^{-1}\begin{bmatrix}25\\ 10\\ 41\end{bmatrix}=\begin{bmatrix}1 & 2 & -1\\ -1 & -3 & 2\\ -1 & -1 & 1\end{bmatrix}\begin{bmatrix}25\\ 10\\ 41\end{bmatrix}$

Original $=\begin{bmatrix}25+20-41\\ -25-30+82\\ -25-10+41\end{bmatrix}=\begin{bmatrix}4\\ 27\\ 6\end{bmatrix}$

We notice that 27 is outside our (1 – 26) range for letters. Indeed it's 27. We need to consider module 26: 27 mod 26 = 1

Third Encoded vector: $\begin{bmatrix}22\\ 14\\ 41\end{bmatrix}$

Original $=A^{-1}\begin{bmatrix}22\\ 14\\ 41\end{bmatrix}$

Original $=\begin{bmatrix}1 & 2 & -1\\ -1 & -3 & 2\\ -1 & -1 & 1\end{bmatrix}\begin{bmatrix}22\\ 14\\ 41\end{bmatrix}$

$$=\begin{bmatrix}22+28-41\\ -22-42+82\\ -22-14+41\end{bmatrix}=\begin{bmatrix}9\\ 18\\ 5\end{bmatrix}$$

So, the third original vector is $\begin{bmatrix}9\\ 18\\ 5\end{bmatrix}$

So, adjusted original vectors:

1. $\begin{bmatrix}8\\ 15\\ 12\end{bmatrix}$ 2. $\begin{bmatrix}4\\ 27\\ 6\end{bmatrix}$ 3. $\begin{bmatrix}9\\ 18\\ 5\end{bmatrix}$

Now, map each number to its corresponding letter $(A=1,\ B=2,\ .....z=26)$

First vector: $H=8,\ O=15,\ L=12$

Second vector: $D=4,\ A=1,\ F=6$

Third vector: $I=9,\ R=18,\ E=5$

Combining the letters. Putting them together in order HOLDFIRE

Final Answer: Thus, decoded message is: **HOLDFIRE**