Exercise 6.4

Q1: **Sum the series:**

i. $3+6+9+\ldots+a_{20}$

Sol: $3+6+9+\ldots+a_{20}$

$a_1=3, d=3, n=20$

$$S_n=\frac{n}{2}\left[2a_1+(n-1)d\right]$$

$$S_{20}=\frac{20}{2}\left[2(3)+(20-1)3\right]=10(6+57)=10(63)=630$$

ii. $\frac{4}{\sqrt{5}}+\sqrt{5}+\frac{6}{\sqrt{5}}+\ldots+a_n$

Sol: $\frac{4}{\sqrt{5}}+\sqrt{5}+\frac{6}{\sqrt{5}}+\ldots+a_n$

$$a_1=\frac{4}{\sqrt{5}}, d=\sqrt{5}-\frac{4}{\sqrt{5}}=\frac{5-4}{\sqrt{5}}=\frac{1}{\sqrt{5}}, n=n$$

$$S_n=\frac{n}{2}\left[2a_1+(n-1)d\right]$$

$$S_n=\frac{n}{2}\left[2\left(\frac{4}{\sqrt{5}}\right)+(n-1)\frac{1}{\sqrt{5}}\right]=\frac{n}{2}\left(\frac{8}{\sqrt{5}}+\frac{n-1}{\sqrt{5}}\right)=\frac{n}{2}\left(\frac{8+n-1}{\sqrt{5}}\right)=\frac{n(n+7)}{2\sqrt{5}}$$

Q2: **Find S_n for each arithmetic series:**

i. $a_1=4, n=25, a_n=100$

Sol: $a_1=4, n=25, a_n=100$

$$S_n=\frac{n}{2}(a_1+a_n)=\frac{25}{2}(4+100)=\frac{25}{2}(104)=25(52)=1300$$

ii. $a_1=40, n=20, d=-3$

Sol: $a_1=40, n=20, d=-3$

$$S_n=\frac{n}{2}(2a_1+(n-1)d)=\frac{20}{2}(2(40)+(20-1)(-3))=10(80-57)=10(23)=230$$

iii. $a_n=52, n=21, d=-4$

Sol: $a_n=52, n=21, d=-4$

$a_n=a_1+(n-1)d$

$52=a_1+(21-1)(-4)$

$52=a_1-80 \Rightarrow a_1=132$

$$S_n=\frac{n}{2}(a_1+a_n)=\frac{21}{2}(132+52)=\frac{21}{2}(184)=21(92)=1932$$

Q3: **Find a_1 for the arithmetic series: $d=8, n=19, S_n=1786$.**

Sol: $d=8, n=19, S_n=1786$

$$S_n=\frac{n}{2}(2a_1+(n-1)d)$$

$$1786=\frac{19}{2}(2a_1+(19-1)(8))$$

$$1786=\frac{19}{2}(2a_1+144)$$
$$3572=19(2a_1+144)$$
$$188=2a_1+144$$
$$44=2a_1 \Rightarrow a_1=22$$

Q4: **How many terms of the series: $96+93+90+\ldots$ amount to 1071.**

Sol: $96+93+90+\ldots+1071$

$a_1=96, d=93-96=-3, S_n=1071$

$$S_n=\frac{n}{2}(2a_1+(n-1)d)$$
$$1071=\frac{n}{2}(2(96)+(n-1)(-3))$$
$$2142=n(192-3n+3)$$
$$2142=n(195-3n)$$
$$2142=195n-3n^2$$
$$3n^2-195n+2142=0$$
$n^2-65n+714=0$ (Divided by 3)
$$\Rightarrow n^2-51n-14n+714=0$$
$$n(n-51)-14(n-51)=0$$
$(n-51)(n-14)=0 \Rightarrow n-51=0$ or $n-14=0 \quad \Rightarrow n=51$ or $n=14$

Q5: **If the three sides of a right-angled triangle having perimeter $36cm$ are in A.P., find them.**

Sol: $a, a+d, a+2d$ are the sides of a right-angle triangle

$$a+(a+d)+(a+2d)=36$$
$$3a+3d=36$$
$a+d=12 \longrightarrow$ (i)

By Pythagoras theorem

$$(a+2d)^2=a^2+(a+d)^2$$
$$a^2+4ad+4d^2=a^2+a^2+2ad+d^2 \Rightarrow \cancel{a^2}+a^2+2ad+d^2-\cancel{a^2}-4ad-4d^2=0$$
$$a^2-2ad-3d^2=0 \Rightarrow a^2-3ad+ad-3d^2=0$$
$$a(a-3d)+d(a-3d)=0 \Rightarrow (a-3d)(a+d)=0$$
$a-3d=0$ or $a+d=0$

$a=3d$ or $a=-d$ (*cm is always positive so ignore it*)

When (i) $\Rightarrow 3d+d=12 \Rightarrow 4d=12 \Rightarrow d=3$ cm

$\Rightarrow a=3(3)=9$ cm

Sides are $a, a+d, a+2d=9, 9+3, 9+2(3)=9cm, 12cm, 15cm$

نوٹ:
جب تین تین کو جمع کر کے ایک ایک بنالیں گے تو ان کی
تعداد 3n کے بجائے n رہ جائے گی۔

Q6: **Sum the series**

i. $3+5-7+9+11-13+15+17-19+\ldots to\ 3n.$ terms

Sol: $3+5-7+9+11-13+15+17-19+\ldots to\ 3n$ terms:

$=(3+5-7)+(9+11-13)+(15+17-19)+\ldots+3n$ terms

$=1+7+13+\ldots$ to n terms

$a=1, d=6, n=n$

$$S_n=\frac{n}{2}(2a+(n-1)d)=\frac{n}{2}(2(1)+(n-1)(6))=\frac{n}{2}(2+6n-6)=\frac{n}{2}(6n-4)=n(3n-2)$$

ii. $1+4-7+10+13-16+19+22-25+\ldots$ to $3n$. terms.

Sol: $1+4-7+10+13-16+19+22-25+\ldots$ to $3n$ terms:

$=(1+4-7)+(10+13-16)+(19+22-25)+\ldots+3n\ terms$

$=-2+7+16+\ldots$ to n terms

$a=-2, d=9, n=n$

$$S_n=\frac{n}{2}(2a+(n-1)d)=\frac{n}{2}(2(-2)+(n-1)(9))=\frac{n}{2}(-4+9n-9)=\frac{n}{2}(9n-13)$$

Q7: Find the sum of 20 terms of the series whose r^{th} term is $3r+1$.

Sol: Here $a_1=3(1)+1=4$

$a_2=3(2)+1=7$

$a_3=3(3)+1=10$

We have $4+7+10+\ldots.+a_{20}$

$a_1=4, d=3, n=20$

$$S_n=\frac{n}{2}(2a_1+(n-1)d)=\frac{20}{2}(2(4)+(20-1)3)$$

$=10(8+57)=10(65)=650$

Q8: The 5^{th} and 9^{th} term of an A.P. are 11 and 17 respectively. Find the sum of 20 terms.

Sol: We have

$a_5=a+4d=11$ ---------(I)

$a_9=a+8d=17$ ---------(II)

By Subtracting the above two equations: $4d=6 \Rightarrow d=\frac{3}{2}$

$(I) \Rightarrow a+4\left(\frac{3}{2}\right)=11$

$a+6=11 \Rightarrow a=5$

We know that $S_n=\frac{n}{2}(2a_2+(n-1)d)$ put values

$$S_{20}=\frac{20}{2}\left(2(5)+(20-1)\frac{3}{2}\right)$$

$$S_{20}=10\left(10+\frac{57}{2}\right)=10\left(\frac{20+57}{2}\right)$$

$S_{20}=5(77)=385$

Q9: Obtain the sum of all integers in the first 1000 positive integers which are neither divisible by 5 nor by 2.

Sol: Sum of first 1000 integers which are neither divisible by 5 nor by 2 are

$1+3+7+9+11+13+17+19+21+23+27+29+991+993+997+999$

$=20+60+100+\ldots\ldots+3980$

$a_1=20, d=40, a_n=3980, n=100$

$a_n=a_1+(n-1)d$

$3980=20+(n-1)d$

$3960=(n-1)40$

$n-1=99 \Rightarrow n=100$

Now $S_n=\frac{n}{2}(a_1+a_n)$

$$S_{100}=\frac{100}{2}(20+3980)=50(4000)=200000$$

Q10: The sum of 9 terms of an A.P. is 171 and its eighth term is 31. Find the series.

Sol: Given

$S_9 = \frac{9}{2}(2a+8d) = 171$ and $a_8 = a_1 + 7d = 31 - - - - (I)$

$2a_1 + 8d = 38$

$a_1 + 4d = 19 - - - - - (II)$

Subtracting the two equations: $3d = 12$

$\Rightarrow \boxed{d = 4}$

$(II) \Rightarrow a_1 + 4(4) = 19$

$a = 3$

$a_1 = 3$

$a_2 = a_1 + d = 3 + 4 = 7$

$a_3 = a_1 + 2d = 3 + 2(4) = 11$

Series is $3 + 7 + 11 + ...$

Q11: The 5^{th} term of an arithmetic progression is 21 and the sum of first six terms is 90. Find the 18^{th} term.

Sol: Given

$a_5 = a_1 + 4d = 21 - - - - - - - (I)$

And $S_6 = \frac{6}{2}(2a_1 + 5d) = 90$

$2a + 5d = 30 - - - - - (II)$

$II - 2 \times I$

$2a_1 + 5d = 30$

$2a_1 \pm 8d = \pm 42$

$-3d = -12 \Rightarrow d = 4$

$I \Rightarrow a_1 + 4(4) = 21 \Rightarrow \boxed{a_1 = 5}$

$a_n = a_1 + (n-1)d$

$a_{18} = 5 + (18-1)(4)$

$a_{18} = 5 + (17)4 = 5 + 68 = 73$

Q12: The sum of three numbers in an A.P. is 24 and their product is 440. Find the numbers.

Sol: Suppose numbers are $a-d, a, a+d$ in A.P.

Condition I $\Rightarrow a - d + a + a + d = 24$

$3a = 24 \Rightarrow a = 8$

Condition II $\Rightarrow (a-d)(a)(a+d) = 440$

$(8-d)(8)(8+d) = 440$

$(64 - d^2)(8) = 440$

$512 - 8d^2 = 440$

$8d^2 = 72 \Rightarrow d^2 = 9 \Rightarrow d = \pm 3$

When $a = 8, d = 3$

$a - d = 8 - 3 = 5$

$a = 8$

$a + d = 8 + 3 = 11$

When $a=8, d=-3$

$a-d=8-(-3)=11$

$a=8$

$a+d=8+(-3)=5$

Hence numbers are 5,8,11 or 11,8,5

Q13: The first four terms of an A.P. are 2,6,10 and 14. Find the least number of terms needed so that the sum of the terms is greater than 2000.

Sol: Given 2,6,10,14

Here $a_1=2, d=4$

$a_n=a_1+(n-1)d=2+(n-1)4=2+4n-4=4n-2$

$S_n=\frac{n}{2}(a_1+a_n)=\frac{n}{2}(\cancel{2}+4n-\cancel{2})=2n^2$

According to the given condition

$2n^2>2000 \Rightarrow n^2>1000 \Rightarrow n> 1000\approx 31.6$

$n=32$

Q14: Find four numbers in A.P. whose sum is 32 and the sum of whose squares is 276.

Sol: Suppose the four numbers in A.P are

Condition I $\Rightarrow a-3d, a-d, a+d, a+3d$

$(a-3d)+(a-d)+(a+d)+(a+3d)=32$

$4a=32 \Rightarrow a=8$

Condition II $\Rightarrow (a-3d)^2+(a-d)^2+(a+d)^2+(a+3d)^2=276$

$a^2-6ad+9d^2+a^2-2ad+d^2+a^2+2ad+d^2+a^2+6ad+9d^2=276$

$4a^2+20d^2=276 \Rightarrow a^2+5d^2=69$

$8^2+5d^2=69 \Rightarrow 5d^2=69-64=5 \Rightarrow d^2=1 \Rightarrow d=\pm 1$

When $a=8, d=-1$

$a-3d=8-3(-1)=8+3=11$

$a-d=8-(-1)=8+1=9$

$a+d=8+(-1)=8-1=7$

$a+3d=8+3(-1)=8-3=5$

When $a=8, d=1$

$a-3d=8-3(1)=5$

$a-d=8-1=7$

$a+d=8+1=9$

$a+3d=8+3(1)=11$

Hence four numbers are 5,7,9,11 or 11,9,7,5

Q15: Find the five numbers in A.P. whose sum is 25 and the sum of whose squares is 135.

Sol: Suppose five numbers in A.P are

$a-2d, a-d, a, a+d, a+2d$

Condition I $\Rightarrow (a-2\cancel{d})+(a-\cancel{d})+a+(a+\cancel{d})+(a+2\cancel{d})=25$

$5a=25 \Rightarrow a=5$

Condition II $\Rightarrow (a-2d)^2+(a-d)^2+a^2+(a+d)^2+(a+2d)^2=135$

$a^2-4ad+4d^2+a^2-2ad+d^2+a^2+a^2+2ad+d^2+a^2+4ad+4d^2=135$

$5a^2+10d^2=135 \Rightarrow a^2+2d^2=27 \Rightarrow (5)^2+2d^2=27$ Put values of a

$2d^2=27-25 \Rightarrow 2d^2=2 \Rightarrow d^2=1 \Rightarrow d=\pm 1$

When $a=5, d=-1$

$a-2d=5-2(-1)=5+2=7$

$a-d=5-(-1)=5+1=6$

$a=5$
$a+d=5+(-1)=5-1=4$
$a+2d=5+2(-1)=5-2=3$
When $a=5, d=1$
$a-2d=5-2(1)=5-2=3$
$a-d=5-1=4$
$a=5$
$a+d=5+1=6$
$a+2d=5+2(1)=7$
Required numbers are
7,6,5,4,3 or 3,4,5,6,7

Q16: if $\frac{1}{a+b},\frac{1}{c+a},\frac{1}{b+c}$ **are in A.P. Then show that** a^2,b^2,c^2 **are in A.P**

Sol: Given $\frac{1}{a+b},\frac{1}{c+a},\frac{1}{b+c}$ are in A.P. then $\frac{1}{b+c}-\frac{1}{c+a}=\frac{1}{c+a}-\frac{1}{a+b}$.

$$\frac{\not{c}+a-b-\not{c}}{(b+c)(c+\not{a})}=\frac{\not{a}+b-c-\not{a}}{(a+b)(c+\not{a})}$$

$$\frac{(a-b)}{b+c}=\frac{b-c}{a+b}\Rightarrow a^2-b^2=b^2-c^2 \;---(I)$$

We want to prove that a^2,b^2,c^2 are in A.P. for this
$c^2-b^2=b^2-a^2$
Both sides by (-1)
$b^2-c^2=a^2-b^2$
$a^2-b^2=a^2-b^2\,(Use\,I)$
Common difference is same hence a^2,b^2,c^2 are in A.P.

Q17: **The sum of the first four terms of an A.P. is 56. The sum of the last four terms is 112. If its first term is 11, then find number of terms.**

Sol: Suppose first four terms are $a, a+d, a+2d, a+3d$
Then $a+a+d+a+2d+a+3d=56$
$\Rightarrow\ 4a+6d=56$
$\Rightarrow\ 44+6d=56\ \Rightarrow\ d=2$ (Put $a=11$)
Last 4 terms are : $a+(n-4)d, a+(n-3)d, a+(n-2)d, a+(n-1)d$
$a+(n-4)d+a+(n-3)d+a+(n-2)d+a+(n-1)d=112$
$\Rightarrow 4a+(4n-10)d=112$
$4(11)+(4n-10)\cdot 2=112\ \Rightarrow\ 44+8n-20=112$
$8n=88\ \Rightarrow\ n=11$

Q18: **The first, second and last terms of an A.P. are *a*, *b* and *c* respectively. Show that the sum of A.P. is** $\frac{(b+c-2a)(c+a)}{2(b-a)}$.

Sol: Given terms are: $a,b,c\ \Rightarrow$ here $a_1=a, a_n=c, a_2=b$

$a_2=a+d=b,\Rightarrow\ d=b-a$

$a_n=a+(n-1)d=c,\Rightarrow\ c=a+(n-1)(b-a)\Rightarrow(c-a)=(n-1)(b-a)$

$$\Rightarrow n-1=\frac{c-a}{b-a}\ \Rightarrow n=\frac{c-a}{b-a}+1=\frac{c-a+b-a}{b-a}=\frac{b+c-2a}{b-a}$$

Sum: $S=\frac{n}{2}(a+a_n)=\frac{n}{2}(a+c)=\frac{b+c-2a}{2(b-a)}(a+c)\Rightarrow S=\frac{(b+c-2a)(c+a)}{2(b-a)}$ Hence Proved.

Q19: Show that the sum of n A.Ms. between a and b is n times the single A.M. between them.

Sol. Let $A_1, A_2, A_3 ...., A_n$ be n A.Ms between a & b.

Then $a, A_1, A_2, A_3, ..., A_n, b$ are in A.P.

Here $a_1 = a$ & $n = n+2, a_{n+2} = b, d = ?$ $\boxed{a_{n+2} = b}$

$a_n = a_1 + (n-1)d$ put $n = n+2$

$a_{n+2} = a_1 + (n+2-1)d = a_1 + (n+1)d$

$\Rightarrow a_{n+2} = a_1 + (n+1)d \Rightarrow b = a_1 + (n+1)d \Rightarrow d = \frac{b-a_1}{n+1}$

Now $A_1 + A_2 + A_3 ....A_n = \frac{n}{2}[A_1 + A_n]$

$= \frac{n}{2}[a_1 + d + a_1 + nd] = \frac{n}{2}[2a_1 + (n+1)d]$

$= \frac{n}{2}\left[2a_1 + (n+1).\frac{(b-a_1)}{(n+1)}\right] = \frac{n}{2}[2a_1 + b - a_1]$

$= \frac{n}{2}[a_1 + b] \Rightarrow n\left(\frac{a_1+b}{2}\right) = n\left(\frac{a+b}{2}\right) \therefore \quad a_1 = a$

$= n(A.M \text{ between } a \text{ and } b)$

Hence $A_1 + A_2 ....A_n = n$ (A.M)