



Exercise 6.6



Q.1. Find G.M. between:

i. -2 and 8

Sol: Here $a = -2$ and $b = 8$

$$G = \pm\sqrt{ab} = \pm\sqrt{(-2)(8)} = \pm\sqrt{-16} = \pm 4i$$

ii. $-2i$ and $8i$

Sol: $G = \pm\sqrt{ab} = \pm\sqrt{(-2i)(8i)} = \pm\sqrt{-16i^2} = \pm\sqrt{-16(-1)} = \pm\sqrt{16} = \pm 4$

iii. 6 and 9

Sol: $a = 6$, $b = 9$

$$G = \pm\sqrt{ab} = \pm\sqrt{(6)(9)} = \pm\sqrt{54} = \pm 3\sqrt{6}$$

Q.2. Insert four real geometric means between 3 and 96 .

Sol: Let G_1, G_2, G_3, G_4 are four G.M.s between 3 & 96

$3, G_1, G_2, G_3, G_4, 96$ are in G.P

$$a_1 = 3 \text{ \& } a^6 = a_1 r^5 = 96 \Rightarrow 3r^5 = 96 \Rightarrow r^5 = \frac{96}{3} = 32 = 2^5 \Rightarrow r = 2$$

$$\text{Then } G_1 = a_1 r = 3(2) = 6$$

$$G_2 = a_1 r^2 = 3(2)^2 = 3(4) = 12$$

$$G_3 = a_1 r^3 = 3(2)^3 = 3(8) = 24$$

$$G_4 = a_1 r^4 = 3(2)^4 = 3(16) = 48$$

So, four G.M's are $b = 6, 12, 24, 48$

Q.3. If both x and y are positive distinct real numbers, show that the geometric mean between x and y is less than their arithmetic mean.

Sol: Given $x > 0$ & $y > 0$ then

Here $a = x$, & $b = y$

$$G.M = \sqrt{xy} \text{ \& } A.M = \frac{a+b}{2} = \frac{x+y}{2}$$

$$A.M - G.M = \frac{x+y}{2} - \sqrt{xy} = \frac{x+y-2\sqrt{xy}}{2} = \frac{(\sqrt{x})^2 + (\sqrt{y})^2 - 2\sqrt{xy}}{2} = \frac{(\sqrt{x}-\sqrt{y})^2}{2} = \frac{(\sqrt{x}-\sqrt{y})^2}{2} > 0$$

$$\Rightarrow A.M - G.M > 0 \Rightarrow A.M > G.M \Rightarrow \text{or } G.M < A.M$$

Q.4. For what value of n , $\frac{a^n + b^n}{a^{n-1} + b^{n-1}}$ is the positive geometric mean between a and b ?

Sol: If $\frac{a^n + b^n}{a^{n-1} + b^{n-1}}$ be G.M between a & b

$$\text{Then } \frac{a^n + b^n}{a^{n-1} + b^{n-1}} = \sqrt{ab} = (ab)^{1/2} \Rightarrow a^n + b^n = (a^{n-1} + b^{n-1}) a^{1/2} b^{1/2}$$

$$a^n + b^n = a^{n-1+1/2} b^{1/2} + a^{1/2} b^{n-1+1/2} \Rightarrow a^n + b^n = a^{n-1/2} b^{1/2} + a^{1/2} b^{n-1/2}$$

$$a^n - a^{n-1/2} b^{1/2} = a^{1/2} b^{n-1/2} - b^n \Rightarrow a^{n-1/2} (a^{1/2} - b^{1/2}) = b^{n-1/2} (a^{1/2} - b^{1/2}) \Rightarrow \frac{a^{n-1/2}}{b^{n-1/2}} = \frac{a^{1/2} - b^{1/2}}{a^{1/2} - b^{1/2}}$$

$$\Rightarrow \frac{a^{n-1/2}}{b^{n-1/2}} = 1 \Rightarrow \left(\frac{a}{b}\right)^{n-1/2} = 1 = \left(\frac{a}{b}\right)^0 \Rightarrow n - \frac{1}{2} = 0 \Rightarrow n = \frac{1}{2}$$



Q.5. The A.M. of two positive integral numbers exceeds their (positive) G.M. by 2 and their sum is 20, find the numbers.

Sol: Condition I $\Rightarrow \frac{a+b}{2} = \sqrt{ab} + 2$

'x' by 2 $\Rightarrow a+b = 2\sqrt{ab} + 4$ I

Condition II $a+b = 20 \Rightarrow a = 20 - b$ II

$20 - b + b = 2\sqrt{(20-b)b} + 4$ (Put II in I)

$\Rightarrow 20 - 4 = 2\sqrt{20b - b^2} \Rightarrow 16 = 2\sqrt{20b - b^2} \Rightarrow 8 = \sqrt{20b - b^2}$

$64 = 20b - b^2 \Rightarrow b^2 - 20b + 64 = 0$

$\Rightarrow b^2 - 16b - 4b + 64 = 0$

$b(b-16) - 4(b-16) = 0$

$\Rightarrow (b-16)(b-4) = 0 \Rightarrow b-16 = 0$ or $b-4 = 0 \Rightarrow b = 16$ or $b = 4$

When $b = 16$ then $a = 20 - 16 = 4$

When $b = 4$ then $a = 20 - 4 = 16$

Hence two number are 4, 16 or 16, 4

Q.6. The A.M. between two numbers is 5 and their (positive) G.M. is 4. Find the numbers.

Sol: Suppose two number are a & b then

Condition I $\Rightarrow \frac{a+b}{2} = 5$

$\Rightarrow a+b = 10$ I $\Rightarrow a = 10 - b$ II

Condition II $\sqrt{ab} = 4 \Rightarrow ab = 16$ III

Put II in III $\Rightarrow (10-b)b = 16 \Rightarrow 10b - b^2 = 16$

$\Rightarrow b^2 - 10b + 16 = 0 \Rightarrow b^2 - 8b - 2b + 16 = 0 \Rightarrow b(b-8) - 2(b-8) = 0$

$(b-8)(b-2) = 0 \Rightarrow b-8 = 0$ or $b-2 = 0 \Rightarrow b = 8$ or $b = 2$

When $b = 8$ then $a = 10 - 8 = 2$

When $b = 2$ then $a = 10 - 2 = 8$

Hence two number are 2, 8 or 8, 2

Q.7. The arithmetic mean between two positive numbers a and b is double their geometric mean. Prove that $a : b = 2 + \sqrt{3} : 2 - \sqrt{3}$.

Sol: $AM = \frac{a+b}{2}$ and $GM = \sqrt{ab}$

Given that $AM = 2 \times GM$

$\frac{a+b}{2} = 2\sqrt{ab} \Rightarrow \frac{(a+b)^2}{4} = 4ab$ (square both sides)

$(a+b)^2 = 16ab$

$a^2 + 2ab + b^2 = 16ab$

$a^2 - 14ab + b^2 = 0$

$\left(\frac{a}{b}\right)^2 - 14\left(\frac{a}{b}\right) + 1 = 0$

Let $x = \frac{a}{b}$

$x^2 - 14x + 1 = 0$

$$a = 1, b = -14, c = 1$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-14) \pm \sqrt{(-14)^2 - 4(1)(1)}}{2(1)} = \frac{14 \pm \sqrt{196 - 4}}{2} = \frac{14 \pm \sqrt{192}}{2} = \frac{14 \pm 8\sqrt{3}}{2} = 7 \pm 4\sqrt{3}$$

$$\Rightarrow \frac{a}{b} = 7 + 4\sqrt{3} \text{ or } \frac{a}{b} = 7 - 4\sqrt{3}$$

Now let's simplify the target ratio

$$2 + \sqrt{3} : 2 - \sqrt{3}$$

$$\frac{2 + \sqrt{3}}{2 - \sqrt{3}} = \frac{(2 + \sqrt{3})(2 + \sqrt{3})}{(2 - \sqrt{3})(2 + \sqrt{3})} = \frac{4 + 2\sqrt{3} + 2\sqrt{3} + 3}{4 - 3} = 7 + 4\sqrt{3} = \frac{a}{b}$$

Hence proved that

$$a : b = 2 + \sqrt{3} : 2 - \sqrt{3}$$

Q.8, If one geometric mean G and two arithmetic means p and q are inserted between two positive numbers, show that $G^2 = (2p - q)(2q - p)$.

Sol: Let G be a GM between a and b then a, G, b forms a GP.

Therefore, common ratio is

$$\frac{G}{a} = \frac{b}{G} \Rightarrow G^2 = ab$$

Also, given that p and q are the two AM's between a and b .

Then

a, p, q, b are in AP.

Let's denote the common difference as d , then

$$p = a + d$$

$$q = a + 2d$$

$$b = a + 3d$$

From these equations, we can express d in terms a, p, q , and b :

$$d = p - a$$

$$d = \frac{q - a}{2}$$

$$d = \frac{b - a}{3}$$

Now, let's express b and a in terms of p and q .

From $p = a + d$ and $q = a + 2d$, we can eliminate a :

$$q - p = (a + 2d) - (a + d) = d$$

So, the common difference is $d = q - p$.

Now we can express a and b in terms of p and q :

$$a = p - d = p - (q - p) = 2p - q$$

$$b = a + 3d = (2p - q) + 3(q - p) = 2p - q + 3q - 3p = 2q - p$$

Now, recall from equation 1 that $G^2 = ab$.

Substituting the expressions for a and b in terms of p and q :

$$G^2 = (2p - q)(2q - p)$$

This is exactly what we needed to show.

Final Answer: The final answer is $G^2 = (2p - q)(2q - p)$