



Exercise 6.9



Q.1. Find the 9th term of the following harmonic sequences:

i. $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \dots$

Sol: Given $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \dots$ are in H.P, $a_9 = ?$

Then $\frac{3}{1}, \frac{5}{1}, \frac{7}{1}, \dots$ are in A.P

Or 3, 5, 7, ... are in A.P

$$a_1 = 3, d = 5 - 3 = 2, n = 9$$

$$a_n = a_1 + (n-1)d$$

$$a_9 = 3 + (9-1)2$$

$$= 3 + (8)2 = 3 + 16 = 19 \text{ in A.P} \Rightarrow a_9 = \frac{1}{19} \text{ in H.P.}$$

ii. $\frac{-1}{5}, \frac{-1}{3}, -1, \dots$

Sol: Given $\frac{-1}{5}, \frac{-1}{3}, -1$ are in H.P or $-5, -3, -1, \dots$ are in A.P

$$a_1 = 5, d = -3 - (-5) = -3 + 5 = 2, n = 9$$

$$a_n = a_1 + (n-1)d \Rightarrow a_9 = 5 + (9-1)2$$

$$a_9 = -5 + (8)(2) = -5 + 16 = 11 \text{ in A.P then } a_9 = \frac{1}{11} \text{ in H.P}$$

Q.2. Insert five harmonic means between the following given numbers:

i) $\frac{-2}{5}$ & $\frac{2}{13}$

Sol: Let H_1, H_2, H_3, H_4, H_5 , be five

H.M between $\frac{-2}{5}$ & $\frac{2}{13}$ then $\frac{-2}{5}, H_1, H_2, H_3, H_4, H_5, \frac{2}{13}$ are in H.P

$\Rightarrow \frac{-5}{2}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3}, \frac{1}{H_4}, \frac{1}{H_5}, \frac{13}{2}$ are in A.P.

$$a_1 = \frac{-5}{2} \text{ \& } a_7 = a_1 + 6d = \frac{13}{2} \Rightarrow \frac{-5}{2} + 6d = \frac{13}{2} \Rightarrow 6d = \frac{13}{2} + \frac{5}{2} = \frac{18}{2} \Rightarrow d = \frac{18}{2} \times \frac{1}{6} \Rightarrow d = \frac{3}{2}$$

$$\frac{1}{H_1} = a_2 = a_1 + d = \frac{-5}{2} + \frac{3}{2} = \frac{-5+3}{2} = \frac{-2}{2} = -1$$

$$\frac{1}{H_2} = a_3 = a_1 + 2d = \frac{-5}{2} + 2\left(\frac{3}{2}\right) = \frac{-5}{2} + \frac{6}{2} = \frac{-5+6}{2} = \frac{1}{2}$$

$$\frac{1}{H_3} = a_4 = a_1 + 3d = \frac{-5}{2} + 3\left(\frac{3}{2}\right) = \frac{-5}{2} + \frac{9}{2} = \frac{-5+9}{2} = \frac{4}{2} = 2$$

$$\frac{1}{H_4} = a_5 = a_1 + 4d = \frac{-5}{2} + 4\left(\frac{3}{2}\right) = \frac{-5}{2} + \frac{12}{2} = \frac{-5+12}{2} = \frac{7}{2}$$

$$\frac{1}{H_5} = a_6 = a_1 + 5d = \frac{-5}{2} + 5\left(\frac{3}{2}\right) = \frac{-5}{2} + \frac{15}{2} = \frac{-5+15}{2} = \frac{10}{2} = 5$$

$$\text{Hence } H_1 = -1, H_2 = \frac{1}{2}, H_3 = \frac{1}{2}, H_4 = \frac{2}{7}, H_5 = \frac{1}{5}$$



ii) $\frac{1}{4}$ & $\frac{1}{24}$

Sol: Let H_1, H_2, H_3, H_4, H_5 , H.M.s between $\frac{1}{4}$ & $\frac{1}{24}$ then

$$\frac{1}{4}, H_1, H_2, H_3, H_4, H_5, \frac{1}{24} \text{ are in H.P}$$

$$\Rightarrow 4, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3}, \frac{1}{H_4}, \frac{1}{H_5}, 24 \text{ are in A.P.}$$

$$a_1 = 4 \text{ \& } a_7 = a_1 + 6d = 24$$

$$4 + 6d = 24 \Rightarrow 6d = 24 - 4$$

$$\Rightarrow 6d = 24 - 4 = 20 \Rightarrow d = \frac{20}{6} = \frac{10}{3}$$

Now

$$\frac{1}{H_1} = a_2 = a_1 + d = 4 + \frac{10}{3} = \frac{12+10}{3} = \frac{22}{3}$$

$$\frac{1}{H_2} = a_3 = a_1 + 2d = 4 + 2\left(\frac{10}{3}\right) = \frac{12+20}{3} = \frac{32}{3}$$

$$\frac{1}{H_3} = a_4 = a_1 + 3d = 4 + 3\left(\frac{10}{3}\right) = \frac{12+30}{3} = \frac{42}{3}$$

$$\frac{1}{H_4} = a_5 = a_1 + 4d = 4 + 4\left(\frac{10}{3}\right) = \frac{12+40}{3} = \frac{52}{3}$$

$$\frac{1}{H_5} = a_6 = a_1 + 5d = 4 + 5\left(\frac{10}{3}\right) = \frac{12+50}{3} = \frac{62}{3}$$

$$\text{Hence } H_1 = \frac{3}{22}, H_2 = \frac{3}{32}, H_3 = \frac{3}{42}, H_4 = \frac{3}{52}, H_5 = \frac{3}{62}$$

Q.3. The first term of an H.P. is $-\frac{1}{3}$ and the fifth term is $\frac{1}{5}$. Find its 9th term.

Sol: $a_1 = -\frac{1}{3}, a_5 = \frac{1}{5}, a_9 = ?$ in H.P

$$a_1 = -3 \quad (i) \quad , \quad a_5 = 5 \text{ in A.P}$$

$$a_5 = a_1 + 4d = 5 \Rightarrow -3 + 4d = 5 \Rightarrow 4d = 8 \Rightarrow d = 2 \quad \text{use (i)}$$

$$a_n = a_1 + (n-1)d$$

$$a_9 = -3 + (9-1)(2)$$

$$a_9 = -3 + (8)(2) = -3 + 16 = 13$$

$$a_9 = 13 \text{ in A.P} \Rightarrow a_9 = \frac{1}{13} \text{ in H.P}$$

Q.4. If 5 is the harmonic mean between 2 & b . Find b .

Sol: Given $a = 2, b = b, H.M = 5$

$$H.M = \frac{2ab}{a+b}$$

Put values

$$5 = \frac{2(2)b}{2+b} \Rightarrow 5 = \frac{4b}{2+b}$$

$$\Rightarrow 5(2+b) = 4b \Rightarrow 10+5b = 4b$$

$$\Rightarrow 10+5b-4b=0 \Rightarrow 10+b=0 \Rightarrow b=-10$$

Q.5. If the numbers $\frac{1}{k}, \frac{1}{2k+1}$ & $\frac{1}{4k-1}$ are in harmonic sequence, find k.

Sol: $\frac{1}{k}, \frac{1}{2k+1}$ & $\frac{1}{4k-1}$ are in H.P

$k, 2k+1, 4k-1$ are in A.P

$$\Rightarrow 4k-1-(2k+1) = 2k+1-k$$

$$4k-1-2k-1 = k+1 \Rightarrow 2k-2 = k+1$$

$$\Rightarrow k-3=0 \Rightarrow k=3$$

Q.6. Find n so that $\frac{a^{n+1} + b^{n+1}}{a^n + b^n}$ may be H.M. between a & b.

Sol: If $\frac{a^{n+1} + b^{n+1}}{a^n + b^n}$ be H.M between a & b then

$$\frac{a^{n+1} + b^{n+1}}{a^n + b^n} = \frac{2ab}{a+b}$$

$$(a^{n+1} + b^{n+1})(a+b) = (a^n + b^n)(2ab)$$

$$a^{n+2} + a^{n+1}b + ab^{n+1} + b^{n+2} = 2a^{n+1}b + 2ab^{n+1}$$

$$a^{n+2} + b^{n+2} = 2ab^{n+1} + 2ab^{n+1} - a^{n+1}b - ab^{n+1}$$

$$a^{n+2} + b^{n+2} = a^{n+1}b + ab^{n+1}$$

$$a^{n+2} - a^{n+1}b = ab^{n+1} - b^{n+2}$$

$$a^{n+1}(a-b) = b^{n+1}(a-b)$$

$$\frac{a^{n+1}}{b^{n+1}} = \frac{\cancel{a-b}}{\cancel{a-b}} = 1 \Rightarrow \left(\frac{a}{b}\right)^{n+1} = \left(\frac{a}{b}\right)^0 \Rightarrow n+1=0 \Rightarrow n=-1$$

$$\text{Note } 1 = \left(\frac{a}{b}\right)^0$$

Q.7. If a^2, b^2 & c^2 are in A.P., show that $a+b, c+a$ & $b+c$ are in H.P.

Sol: a^2, b^2 & c^2 are in A.P

$$\text{Then } b^2 - a^2 = c^2 - b^2 \quad (i)$$

Now $a+b, c+a, b+c$ are in H.P

$$\text{If } \frac{1}{a+b}, \frac{1}{c+a}, \frac{1}{b+c} \text{ are in A.P} \Rightarrow \frac{1}{b+c} - \frac{1}{c+a} = \frac{1}{c+a} - \frac{1}{a+b}$$

$$\Rightarrow \frac{(c+a)-(b+c)}{(b+c)(c+a)} = \frac{a+b-c-a}{(b+c)(c+a)} \Rightarrow \frac{\cancel{c}+a-b-\cancel{c}}{(b+c)(c+a)} = \frac{\cancel{a}+b-c-\cancel{a}}{(a+b)(c+a)}$$

'x' both sides by $(c+a)$

$$\frac{\cancel{c+a} (a-b)}{(b+c)\cancel{c+a}} = \frac{\cancel{c+a} (b-c)}{(a+b)\cancel{c+a}} \Rightarrow \frac{a-b}{b+c} = \frac{b-c}{a+b}$$

By cross multiplication

$$(a+b)(a-b) = (b+c)(b-c)$$

$$a^2 - b^2 = b^2 - c^2$$

'x' both sides by (-1)

$$b^2 - a^2 = c^2 - b^2 \Rightarrow b^2 - a^2 = b^2 - a^2 \quad (\text{use i})$$

Hence proved.

Q.8. If the H.M. and A.M. between two numbers are 4 and $\frac{9}{2}$ respectively, find the numbers.

Sol: $H.M = 4, A.M = \frac{9}{2}, a, b = ?$

$$H.M = \frac{2ab}{a+b} \Rightarrow 4 = \frac{2ab}{a+b} \quad \text{_____ (i)}$$

$$A.M = \frac{a+b}{2} \Rightarrow \frac{9}{2} = \frac{a+b}{2}$$

$$\Rightarrow a+b = 9 \quad \text{_____ (ii)}$$

$$\text{Put equation ii in i} \Rightarrow 4 = \frac{2ab}{9} \Rightarrow 2ab = 36$$

$$\Rightarrow ab = \frac{36}{2} = 18 \Rightarrow ab = 18 \quad \text{_____ (iii) from ii} \Rightarrow a+b = 9 \Rightarrow a = 9-b$$

Put value in iii

$$(9-b)b = 18 \Rightarrow 9b - b^2 - 18 = 0$$

$$\Rightarrow b^2 - 9b + 18 = 0$$

$$\Rightarrow b^2 - 3b - 6b + 18 = 0$$

$$\Rightarrow b(b-3) - 6(b-3) = 0$$

$$\Rightarrow b-3 = 0 \text{ or } b-6 = 0 \Rightarrow b = 3 \text{ or } b = 6$$

$$\text{When } b = 3 \text{ then } a = 9 - 3 = 6$$

$$\text{When } b = 6 \text{ then } a = 9 - 6 = 3$$

Numbers are 6, 3 or 3, 6

Q.9. If the (positive) G.M. and H.M. between two numbers are 4 and $\frac{16}{5}$ find the numbers.

Sol: $G.M = 4, H.M = \frac{16}{5}, a, b = ?$

$$G.M = \sqrt{ab} \Rightarrow 4 = \sqrt{ab} \Rightarrow ab = 16 \quad \text{(i)}$$

$$H.M = \frac{2ab}{a+b} \Rightarrow \frac{16}{5} = \frac{2ab}{a+b}$$

$$\Rightarrow \frac{16}{5} = \frac{2(16)}{a+b} \Rightarrow \frac{1}{5} = \frac{2}{a+b} \Rightarrow a+b = 10 \quad \text{(ii)}$$

$$\text{From (ii) } a = 10 - b \quad \text{(iii)}$$

$$\text{Eq. (i) Becomes } (10-b)b = 16 \Rightarrow 10b - b^2 - 16 = 0 \Rightarrow b^2 - 10b + 16 = 0$$

$$\Rightarrow b^2 - 2b - 8b + 16 = 0$$

$$\Rightarrow b(b-2) - 8(b-2) = 0$$

$$\Rightarrow (b-2)(b-8) = 0$$

$$\Rightarrow b-2 = 0 \quad \text{or} \quad b-8 = 0$$

$$\Rightarrow b = 2 \quad \text{or} \quad b = 8$$

$$\text{When } b = 2 \text{ then } a = 10 - 2 = 8$$

$$\text{When } b = 8 \text{ then } a = 10 - 8 = 2$$

Numbers are 8, 2 or 2, 8

Q.10. If $\frac{b+c-a}{a}, \frac{c+a-b}{b}, \frac{a+b-c}{c}$ are in A.P., show that a, b, c are in H.P.

Sol: Given that $\frac{b+c-a}{a}, \frac{c+a-b}{b}, \frac{a+b-c}{c}$ are in AP

i.e. Common difference remain constant

$$\frac{c+a-b}{b} - \frac{b+c-a}{a} = \frac{a+b-c}{c} - \frac{c+a-b}{b}$$

$$2\left(\frac{c+a-b}{b}\right) = \frac{b+c-a}{a} + \frac{a+b-c}{c}$$

Multiplying both sides by abc

$$2ac(c+a-b) = bc(b+c-a) + ab(a+b-c)$$

$$2ac^2 + 2a^2c - 2abc = b^2c + bc^2 - abc + a^2b + ab^2 - abc$$

$$= b^2c + bc^2 + a^2b + ab^2 - 2abc$$

$$2ac^2 + 2a^2c = a^2b + ab^2 + b^2c + bc^2$$

$$2ac(a+c) = ab(a+b) + bc(b+c)$$

Divide by abc :

$$\frac{2(a+c)}{b} = \frac{a+b}{c} + \frac{b+c}{a}$$

$$\frac{2a}{b} + \frac{2c}{b} = \frac{a}{c} + \frac{b}{c} + \frac{b}{a} + \frac{c}{a}$$

$$\frac{2}{b} = \frac{1}{a} + \frac{1}{c} \Rightarrow \frac{1}{b} + \frac{1}{b} = \frac{1}{a} + \frac{1}{c} \Rightarrow \frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}$$

It means $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P. therefore a, b, c are in H.P.

a, b, c are in H.P.

Q.11. If a, b, c, d are in H.P., show that $3(a-b)(c-d) = (b-c)(a-d)$.

Sol: Given that a, b, c, d are in HP.

Then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}$ are in AP.

Consider

$$\frac{1}{a} = x, \frac{1}{b} = x+k, \frac{1}{c} = x+2k, \frac{1}{d} = x+3k$$

Then

$$a = \frac{1}{x}, b = \frac{1}{x+k}, c = \frac{1}{x+2k}, d = \frac{1}{x+3k}$$

we have to prove: $3(a-b)(c-d) = (b-c)(a-d)$

$$\text{L.H.S} = 3(a-b)(c-d)$$

$$= 3\left(\frac{1}{x} - \frac{1}{x+k}\right)\left(\frac{1}{x+2k} - \frac{1}{x+3k}\right) = 3\left(\frac{x+k-x}{x(x+k)}\right)\left(\frac{x+3k-x-2k}{(x+2k)(x+3k)}\right) = 3\left(\frac{k}{x(x+k)}\right)\left(\frac{k}{(x+2k)(x+3k)}\right)$$

$$= \frac{3k^2}{x(x+k)(x+2k)(x+3k)} \quad (i)$$

$$\text{R.H.S} = (b-c)(a-d)$$

$$= \left(\frac{1}{x+k} - \frac{1}{x+2k} \right) \left(\frac{1}{x} - \frac{1}{x+3k} \right) = \left(\frac{x+2k-x-k}{(x+k)(x+2k)} \right) \left(\frac{x+3k-x}{x(x+3k)} \right) = \left(\frac{k}{(x+k)(x+2k)} \right) \left(\frac{3k}{x(x+3k)} \right)$$

$$= \frac{3k^2}{x(x+k)(x+2k)(x+3k)} \quad (ii)$$

From (i) & (ii): $\Rightarrow L.H.S = R.H.S$ or $3(a-b)(c-d) = (b-c)(a-d)$

Q.12. If between any two numbers there are inserted two A.M., A_1, A_2 , two G.Ms G_1, G_2 and two H.Ms. H_1, H_2 ; show that $\frac{A_1 + A_2}{G_1 G_2} = \frac{H_1 + H_2}{H_1 H_2}$.

Sol: If A_1 & A_2 are two arithmetic means inserted between a & b , then a, A_1, A_2, b are in A.P.

$$\text{Here } a_1 = a \text{ and } a_4 = a_1 + 3d = b \Rightarrow d = \frac{b-a}{3}$$

$$A_1 = a + d$$

$$\text{So, } A_1 = a + \frac{b-a}{3} = \frac{3a+b-a}{3} = \frac{2a+b}{3}$$

$$A_2 = a + 2d$$

$$A_2 = a + 2\left(\frac{b-a}{3}\right) = \frac{3a+2b-2a}{3} = \frac{a+2b}{3}$$

$$\text{Therefore, } A_1 + A_2 = \frac{2a+b}{3} + \frac{a+2b}{3} = \frac{3a+3b}{3} = a+b.$$

If G_1 & G_2 are two geometric means inserted between a & b , then a, G_1, G_2, b are in G.P.

Let the common ratio be r .

$$G_1 = ar$$

$$G_2 = ar^2$$

$$b = ar^3 \Rightarrow r^3 = \frac{b}{a} \Rightarrow r = \left(\frac{b}{a}\right)^{\frac{1}{3}}$$

$$\text{So, } G_1 = a \left(\frac{b}{a}\right)^{\frac{1}{3}} = a^{\frac{2}{3}} b^{\frac{1}{3}}$$

$$G_2 = a \left(\frac{b}{a}\right)^{\frac{2}{3}} = a^{\frac{1}{3}} b^{\frac{2}{3}}$$

$$\text{Therefore, } G_1 \cdot G_2 = \left(a^{\frac{2}{3}} b^{\frac{1}{3}}\right) \left(a^{\frac{1}{3}} b^{\frac{2}{3}}\right) = a^{\frac{2}{3} + \frac{1}{3}} b^{\frac{1}{3} + \frac{2}{3}} = a^1 b^1 = ab.$$

If H_1 & H_2 are two harmonic means inserted between a & b , then a, H_1, H_2, b are in H.P., which means

$\frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{b}$ are in A.P. Let the common difference be d' .

$$\frac{1}{H_1} = \frac{1}{a} + d'$$

$$\frac{1}{H_2} = \frac{1}{a} + 2d'$$

$$\frac{1}{H_3} = \frac{1}{a} + 3d' \quad \Rightarrow \quad 3d' = \frac{1}{b} - \frac{1}{a} = \frac{a-b}{ab} \quad \Rightarrow \quad d' = \frac{a-b}{3ab}$$

$$\text{So, } \frac{1}{H_1} = \frac{1}{a} + \frac{a-b}{3ab} = \frac{3b+a-b}{3ab} = \frac{a+2b}{3ab} \quad \Rightarrow \quad H_1 = \frac{3ab}{a+2b}$$

$$\frac{1}{H_2} = \frac{1}{a} + 2\left(\frac{a-b}{3ab}\right) = \frac{3b+2a-2b}{3ab} = \frac{2a+b}{3ab} \Rightarrow H_2 = \frac{3ab}{2a+b}$$

$$\text{Therefore, } H_1 + H_2 = \frac{3ab}{a+2b} + \frac{3ab}{2a+b} = 3ab\left(\frac{1}{a+2b} + \frac{1}{2a+b}\right) = 3ab\left(\frac{2a+b+a+2b}{(a+2b)(2a+b)}\right)$$

$$= 3ab\left(\frac{3a+3b}{2a^2+ab+4ab+2b^2}\right) = \frac{9ab(a+b)}{2a^2+5ab+2b^2}$$

$$\text{And, } H_1 \cdot H_2 = \left(\frac{3ab}{a+2b}\right)\left(\frac{3ab}{2a+b}\right) = \frac{9a^2b^2}{(a+2b)(2a+b)} = \frac{9a^2b^2}{2a^2+5ab+2b^2}$$

$$\Rightarrow \frac{H_1 + H_2}{H_1 \cdot H_2} = \frac{\frac{9ab(a+b)}{2a^2+5ab+2b^2}}{\frac{9a^2b^2}{2a^2+5ab+2b^2}} = \frac{9ab(a+b)}{2a^2+5ab+2b^2} \times \frac{2a^2+5ab+2b^2}{9a^2b^2} = \frac{a+b}{ab} = \frac{A_1 + A_2}{G_1 G_2} \quad \text{Hence Proved.}$$

Q.13 The H.M. of two numbers is 4. The A.M., A and the G.M., G satisfy the relation $2A + G^2 = 27$. Find the numbers.

Sol: Let the two numbers be a and b .

$$\text{The harmonic mean (H.M.) is given by: } H = \frac{2ab}{a+b}$$

$$\text{We are given } H = 4, \text{ so } \frac{2ab}{a+b} = 4 \Rightarrow 2ab = 4(a+b) \Rightarrow ab = 2(a+b)$$

$$\text{The geometric mean (G.M.) is given by: } G = \sqrt{ab}, \text{ so } G^2 = ab$$

$$\text{We are given the relation } 2A + G^2 = 27.$$

Substituting the expressions for A and G^2 :

$$2\left(\frac{a+b}{2}\right) + ab = 27 \Rightarrow a+b+ab = 27$$

Now we have a system of two equations with two variables:

$$\text{i) } ab = 2(a+b) \text{ -----(i)}$$

$$\text{ii) } a+b+ab = 27 \text{ -----(ii)}$$

$$(a+b) + 2(a+b) = 27 \text{ Put (i) in (ii)}$$

$$3(a+b) = 27$$

$$a+b = 9 \text{ -----(iii)} \Rightarrow a = 9 - b$$

Now substitute the value of $a+b$ back into eq (i):

$$ab = 2(9)$$

$$ab = 18 \text{ -----(iv)}$$

Put value of $a = 9 - b$ in (iv)

$$(9-b)b = 18 \Rightarrow 9b - b^2 = 18$$

$$b^2 - 9b + 18 = 0$$

$$b^2 - 3b - 6b + 18 = 0$$

$$b(b-3) - 6(b-3) = 0$$

$$(b-3)(b-6) = 0$$

$$b-3 = 0 \text{ as } b-6 = 0$$

$$a = 9 - b \text{ or } a = 9 - b$$

$$a = 9 - 3 = b \text{ or } a = 9 - 6 = 3$$

Two numbers are 3,6 or 6,3

Q.14 First three of the four numbers a, b, c, d are in A.P., and the next three are in H.P., show that $ad = bc$.

Sol: Given that a, b, c are in arithmetic progression (A.P.), we have:

$$b - a = c - b$$

$$2b = a + c \quad (i)$$

Given that b, c, d are in harmonic progression (H.P.), their reciprocals $\frac{1}{b}, \frac{1}{c}, \frac{1}{d}$ are in arithmetic progression.

$$\text{Thus: } \frac{1}{c} - \frac{1}{b} = \frac{1}{d} - \frac{1}{c}$$

$$\frac{b-c}{bc} = \frac{c-d}{cd} \quad \text{Cross-multiplying gives:}$$

$$cd(b-c) = bc(c-d) \quad \text{Dividing both sides by } c \text{ (assuming } c \neq 0 \text{):}$$

$$d(b-c) = b(c-d)$$

$$db - dc = bc - bd$$

$$2bd = bc + dc \quad (ii)$$

From equation (i), we have $c = 2b - a$ (iii)

Substitute this into equation (ii):

$$2bd = b(2b - a) + d(2b - a)$$

$$2bd = 2b^2 - ab + 2bd - ad \Rightarrow 2b^2 - ab + 2bd - ad - 2bd = 0$$

$$ad = 2b^2 - ab$$

$$ad = b(2b - a)$$

Put $2b - a = c$ we get $ad = bc$

Thus, we have shown that $ad = bc$

Q.15 If a, b, c are in G.P., show that $\log_a x, \log_b x, \log_c x$ are in H.P.

Sol Given that a, b, c are in G.P.

$$\frac{b}{a} = \frac{c}{b}$$

$$\Rightarrow b^2 = ac \quad \text{.....(i)}$$

We have to show $\log_a x, \log_b x, \log_c x$ from an HP.

It is enough to show that $\frac{1}{\log_a x}, \frac{1}{\log_b x}, \frac{1}{\log_c x}$ from an AP.

$$\frac{1}{\log_b x} - \frac{1}{\log_a x} = \frac{1}{\log_c x} - \frac{1}{\log_b x}$$

$$\frac{\log b}{\log x} - \frac{\log a}{\log x} = \frac{\log c}{\log x} - \frac{\log b}{\log x} \Rightarrow \frac{\log b - \log a}{\log x} = \frac{\log c - \log b}{\log x}$$

$$\log b - \log a = \log c - \log b$$

$$\log \frac{b}{a} = \log \frac{c}{b}$$

$$\frac{b}{a} = \frac{c}{b}$$

$$b^2 = ac \Rightarrow b^2 = b^2 \quad (\text{Use I})$$

Common difference is same so $\frac{1}{\log a^x}, \frac{1}{\log b^x}$ and $\frac{1}{\log c^x}$ are in A.P. hence $\log a^x, \log b^x, \log c^x$ are in H.P.

Q.16 If a, b, c are in H.P., show that

i) $\frac{a-b}{b-c} = \frac{a}{c}$

ii) $(a-c)^2 = (a+c)(a-2b+c)$

Sol (i):

If a, b, c are in H.P., then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P., which implies:

$$\frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}$$

$$\frac{a-b}{ab} = \frac{b-c}{bc}$$

Multiply both sides by abc

$$abc \left(\frac{a-b}{ab} \right) = abc \left(\frac{b-c}{bc} \right)$$

$$c(a-b) = a(b-c)$$

$$\Rightarrow \frac{a-b}{b-c} = \frac{a}{c} \text{ hence proved.}$$

ii) To show $(a-c)^2 = (a+c)(a-2b+c)$

Given a, b, c are in H.P then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P

$$\frac{1}{c} - \frac{1}{b} = \frac{1}{b} - \frac{1}{a} \Rightarrow \frac{1}{b} + \frac{1}{b} = \frac{1}{c} + \frac{1}{a}$$

$$\frac{2}{b} = \frac{a+c}{ac} \Rightarrow \frac{b}{2} = \frac{ac}{a+c} \Rightarrow b = \frac{2ac}{a+c}$$

$$\text{So } b = \frac{2ac}{a+c}$$

Substitute b into the RHS:

$$\text{R.H.S.} = (a+c)(a-2b+c) = (a+c) \left(a - 2 \left(\frac{2ac}{a+c} \right) + c \right) = (a+c) \left(\frac{a(a+c) - 4ac + c(a+c)}{a+c} \right)$$

$$= a^2 + ac - 4ac + ca + c^2 = a^2 - 2ac + c^2 = (a-c)^2 = \text{L.H.S}$$

Hence Proved that $(a-c)^2 = (a+c)(a-2b+c)$

Q.17 If $2+x, 5+x$ & $9+x$ are in H.P., find the value of x .

Sol: Given that $2+x, 5+x, & 9+x$ are in H.P., their reciprocals are in A.P.:

$\frac{1}{2+x}, \frac{1}{5+x}, \frac{1}{9+x}$ are in A.P.

Therefore:

$$\frac{1}{5+x} - \frac{1}{2+x} = \frac{1}{9+x} - \frac{1}{5+x}$$

$$\frac{(2+x)-(5+x)}{(5+x)(2+x)} = \frac{(5+x)-(9+x)}{(9+x)(5+x)} \Rightarrow \frac{2+x-5-x}{2+x} = \frac{5+x-9-x}{9+x}$$

$$\frac{-3}{2+x} = \frac{-4}{9+x}$$

$$-3(9+x) = -4(2+x) \Rightarrow -27 - 3x = -8 - 4x$$

$$-3x + 4x = -8 + 27 \Rightarrow x = 19$$

Q.18 If the roots of the equation $a(b-c)x^2 + b(c-a)x + c(a-b) = 0$ are equal, prove that a, b, c are in H.P.

Sol Given $a(b-c)x^2 + b(c-a)x + c(a-b) = 0$ ——— (i) has equal roots

If we put $x = 1$ which satisfies (i) and roots are equal so both roots are 1, 1

$$\text{Product of roots} = \alpha\beta = \frac{C}{A}$$

$$\Rightarrow (1)(1) = \frac{C(a-b)}{a(b-c)}$$

$$\text{Product of roots: } \frac{c(a-b)}{a(b-c)} = 1$$

$$c(a-b) = a(b-c)$$

$$ca - cb = ab - ac$$

$$2ac = ab + cb$$

Dividing by abc :

$$\frac{2ac}{abc} = \frac{ab}{abc} + \frac{cb}{abc}$$

$$\frac{2}{b} = \frac{1}{c} + \frac{1}{a} \Rightarrow \frac{1}{b} + \frac{1}{b} = \frac{1}{c} + \frac{1}{a} \Rightarrow \frac{1}{c} - \frac{1}{b} = \frac{1}{b} - \frac{1}{a} \text{ Which shows that } \frac{1}{a}, \frac{1}{b}, \frac{1}{c} \text{ are in A.P.}$$

Hence a, b, c are in H.P.