Q1. Sum the following series up to n terms

i. $1\times3+2\times5+3\times7+\ldots$

Sol: First factor: $1,2,3,4,\ldots,n$ Here $a_1=1, d=1, T_k=a_1+(k-1)d=1+(k-1)1=1+k-1=k$

2^{nd} Factor: $3,5,7,9,\ldots$ Here $a_1=3, d=2, T_k=a_1+(k-1)d=3+(k-1)2=3+2k-2=2k+1$

General term T_k is

$$T_k=k\cdot(2k+1)$$

Sum of the first n term is

$$S_n=\sum_{k=1}^{n}k(2k+1)=\sum_{k=1}^{n}(2k^2+k)$$

$$S_n=2\sum_{k=1}^{n}k^2+\sum_{k=1}^{n}k$$

$$\sum_{k=1}^{n}k=\frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n}k^2=\frac{n(n+1)(2n+1)}{6}$$

Now use the formulae

$$S_n=2\cdot\frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}$$

$$=\frac{n(n+1)}{6}\cdot[2(2n+1)+3]=\frac{n(n+1)}{6}\cdot(4n+5)$$

$$S_n=\frac{n(n+1)(4n+5)}{6}$$

ii. $1\times5+2\times8+3\times11+\ldots$

Sol: First factor: $1,2,3,4,\ldots,n$ Here $a=1,d=1,T_k=a_1+(k-1)d=1+(k-1).1=1+k-1=k$

2^{nd} Factor: $5,8,11,\ldots$ Here $a=5,d=3,T_k=a_1+(k-1)d=5+(k-1)3=5+3k-3=3k+2$

General term T_k is $T_k=k\cdot(3k+2)$

Sum of the first n term is

$$S_n=\sum_{k=1}^{n}(3k^2+2k)=3\sum_{k=1}^{n}k^2+2\sum_{k=1}^{n}k$$

$$\sum_{k=1}^{n}k=\frac{n(n+1)}{2}$$

Now by putting

$$\sum_{k=1}^{n}k^2=\frac{n(n+1)(2n+1)}{6}$$

$$S_n=3\cdot\frac{n(n+1)(2n+1)}{6}+2\cdot\frac{n(n+1)}{2}$$

$$=\frac{n(n+1)}{6}\cdot[3(2n+1)+6]=\frac{n(n+1)}{6}\cdot(6n+9)=\frac{n(n+1)(6n+9)}{6}S_n=\frac{3n(n+1)(2n+3)}{6}=\frac{n(n+1)(2n+3)}{2}$$

iii. $1\times2+2\times5+3\times8+\ldots$

Sol: First factor: $1,2,3,4,\ldots,n$ Here $a_1=1,d=1,T_k=a_1+(k-1)d=1+(k-1).1=\not{1}+k-\not{1}=k$

2^{nd} Factor: $2,5,8,\ldots$ here $a_1=2,d=3,T_k=a_1+(k-1)d=2+(k-1)3=2+3k-3=3k-1$

General term T_k is

$T_k=k\cdot(3k-1)$

Sum of the first n term is $S_n=\sum_{n=1}^{n}(3k^2-k)=3\sum_{k=1}^{n}k^2-\sum_{k=1}^{n}k$

$$\sum_{k=1}^{n}k=\frac{n(n+1)}{2}$$

Now by putting

$$S_n=3\cdot\frac{n(n+1)(2n+1)}{6}-\frac{n(n+1)}{2}=\frac{n(n+1)}{6}\cdot[3(2n+1)-3]=\frac{n(n+1)}{6}\cdot(6n)$$

$$\sum_{k=1}^{n}k^2=\frac{n(n+1)(2n+1)}{6}$$

$$S_n=n(n+1)(n)=n^2(n+1)$$

iv. $1\times3\times5+2\times4\times6+3\times5\times7+\ldots$

Sol: First Factor: $1,2,3,\ldots..n$ here $a_1=1,d=1,T_k=a_1+(k-1)d=1+(k-1)1=\not{1}+k-\not{1}=k$

2^{nd} Factor: $3,4,5,\ldots..n$ Here $a_1=3,d=1,T_k=a_1+(k-1)d=3+(k-1)1=3+k-1=k+2$

3^{rd} Factor: $5,6,7,\ldots..n$ Here $a_1=5,d=1,T_k=a_1+(k-1)d=5+(k-1)=5+k-1=k+4$

$$T_k=k\cdot(k+2)(k+4)$$

$$S_n=\sum_{k=1}^{n}k(k+2)(k+4)=k(k^2+6k+8)=k^3+6k^2+8k$$

$$S_n=\sum_{k=1}^{n}k(k^2+6k+8)$$

$$S_n=\sum_{k=1}^{n}k^3+6\sum_{k=1}^{n}k^2+8\sum_{k=1}^{n}k$$

Now by putting

$$S_n=\left[\frac{n(n+1)}{2}\right]^2+6\cdot\frac{n(n+1)(2n+1)}{6}+8\cdot\frac{n(n+1)}{2}$$

$$S_n=\frac{n^2(n+1)^2}{4}+n(n+1)(2n+1)+4n(n+1)=n(n+1)\left[\frac{n(n+1)}{4}+2n+1+4\right]=n(n+1)\left[\frac{n^2+n+8n+20}{4}\right]$$

$$=\frac{n(n+1)}{4}(n^2+9n+20)=\frac{n(n+1)}{4}(n^2+4n+5n+20)=\frac{n(n+1)}{4}[n(n+4)+5(n+4)]=\frac{n}{4}(n+1)(n+4)(n+5)$$

v. $1\times2\times4+2\times3\times7+3\times4\times10+\ldots$

Sol: 1st Factor: $1,2,3,\ldots.n$ Here $a_1=1,d=1,T_k=a_1+(k-1)d=1+(k-1)=1+k-\cancel{1}=k$

2nd Factor: $2,3,4,\ldots.,n$ Here $a_1=2,d=1,T_k=a_1+(k-1)d=2+(k-1)1=2+k-1=k+1$

3rd Factor: $4,7,10,\ldots..,n$ Here $a_1=4,d=3,T_k=a_1+(k-1)d=4+(k-1)3=4+3k-3=3k+1$

$$T_k=k\cdot(k+1)(3k+1)$$

$$T_k=k\left(3k^2+4k+1\right)=3k^3+4k^2+k$$

$$S_n=\sum_{k=1}^{n}\left(3k^3+4k^2+k\right)$$

$$S_n=3\sum_{k=1}^{n}k^3+4\sum_{k=1}^{n}k^2+\sum_{k=1}^{n}k$$

Now by putting

$$S_n=3\left[\frac{n(n+1)}{2}\right]^2+4\cdot\frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}$$

$$S_n=3\left(\frac{n(n+1)}{2}\right)^2+\frac{2n(n+1)(2n+1)}{3}+\frac{n(n+1)}{2}=\frac{n(n+1)}{2}\left[\frac{3n(n+1)}{2}+\frac{4(2n+1)}{3}+1\right]$$

$$=\frac{n(n+1)}{2}\left[\frac{9n(n+1)+8(2n+1)+6}{6}\right]=\frac{n(n+1)}{2\times6}\left(9n^2+9n+16n+8+6\right)=\frac{n(n+1)}{2\times6}\left(9n^2+25n+14\right)$$

$$=\frac{n(n+1)}{12}\left(9n^2+25n+14\right)$$

vi. $2^2+4^2+6^2+\ldots$

Sol: For $2,4,6,\ldots..n$ Here $a_1=2,d=2$

$$T_k=\left[a_1+(k-1)d\right]^2=\left[2+(k-1)2\right]^2=\left[\cancel{2}+2k-\cancel{2}\right]^2=(2k)^2$$

$$T_k=(2k)^2=4k^2$$

$$S_n=\sum_{k=1}^{n}4k^2=4\sum_{k=1}^{n}k^2$$

$$S_n=4\cdot\frac{n(n+1)(2n+1)}{6} \qquad \sum_{k=1}^{n}k^2=\frac{n(n+1)(2n+1)}{6}$$

$$S_n=\frac{2n(n+1)(2n+1)}{3}$$

vii. $3^2+6^2+9^2+\ldots$

Sol: For $3,6,9,\ldots\ldots n$

Here $a_1=3,d=3$

$$T_k=\left(a_1+(k-1)d\right)^2=\left(3+(k-1)3\right)^2=\left(3+3k-3\right)^2$$

$$T_k=(3k)^2=9k^2$$

$$S_n=\sum_{k=1}^{n}9k^2=9\sum_{k=1}^{n}k^2$$

$$S_n=9\cdot\frac{n(n+1)(2n+1)}{6} \qquad \sum_{k=1}^{n}k^2=\frac{n(n+1)(2n+1)}{6}$$

$$S_n=\frac{3n(n+1)(2n+1)}{2}$$

viii. $4\times1^2+7\times2^2+10\times3^2+\ldots$

Sol: 1^{st} Factor $4,7,10,\ldots..n$

$a_1=4,d=3,T_k=a_1+(k-1)d=4+(k-1)3=4+3k-3=3k+1$

2^{nd} Factor $1,2,3,\ldots..,n$

$a_1=1,d=1,T_k=[a_1+(k-1)d]^2=[1+(k-1).1]^2=[\not{1}+k-\not{1}]^2=k^2$

$T_k=(3k+1)k^2=3k^3+k^2$

$$S_n=\sum_{k=1}^{n}(3k^3+k^2)=3\sum_{k=1}^{n}k^3+\sum_{k=1}^{n}k^2$$

$$S_n=3\left(\frac{n(n+1)}{2}\right)^2+\frac{n(n+1)(2n+1)}{6}=\frac{n(n+1)}{2}\left[\frac{3n(n+1)}{2}+\frac{2n+1}{3}\right]=\frac{n(n+1)}{2}\left[\frac{9n(n+1)}{6}+4n+2\right]$$

$$=\frac{n(n+1)}{2}\left[9n^2+9n+4n+2\right]=\frac{n(n+1)}{2}\left[9n^2+13n+2\right]$$

ix. $3+(3+7)+(3+7+11)+\ldots$

The series is:

$3+(3+7)+(3+7+11)+\ldots$

Here $a_1=3$, $d=4$, $n=r$ then

$a_r=3+(r-1)4=4r-1$

The sum of the n^{th} bracket is:

$$S_n=\sum_{r=1}^{n}(4r-1)=4\sum_{r=1}^{n}r-\sum_{r=1}^{n}1=4\frac{n(n+1)}{2}-n=2n^2+2n-n=2n+n$$

The sum of the series up to n terms is:

$$S=\sum_{i=1}^{n}(2i^2+i)=2\sum_{i=1}^{n}i^2+\sum_{i=1}^{n}i$$

$$=2\frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}=\frac{n(n+1)(2n+1)}{3}+\frac{n(n+1)}{2}=\frac{n(n+1)}{6}(4n+2+3)=\frac{n(n+1)(4n+5)}{6}$$

x. $1^2+(1^2+2^2)+(1^2+2^2+3^2)+\ldots$

Sol: $\sum_{k=1}^{n}\left(\sum_{j=1}^{k}j^2\right)$

$$\sum_{k=1}^{n}j^2=\frac{k(k+1)(2k+1)}{6}$$

$$S_n=\sum_{k=1}^{n}\frac{k(k+1)(2k+1)}{6}$$

$$S_n=\frac{1}{6}\sum_{k=1}^{n}k(k+1)(2k+1)$$

$$S_n=\frac{1}{6}\sum_{k=1}^{n}(2k^3+3k^2+k)$$

$$S_n=\frac{1}{6}\left[2\sum_{k=1}^{n}k^3+3\sum_{k=1}^{n}k^2+\sum_{k=1}^{n}k\right]$$

$$S_n=\frac{1}{6}\left[2\left(\frac{n(n+1)}{2}\right)^2+3\cdot\frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}\right]$$

$$=\frac{n(n+1)}{2\times 6}\left[\frac{2n(n+1)}{2}+2n+1+1\right]=\frac{n(n+1)}{12}\left[n^2+n+2n+2\right]=\frac{n(n+1)}{12}(n^2+3n+2)$$

$$=\frac{n(n+1)}{12}(n^2+n+2n+2)=\frac{n(n+1)}{12}(n(n+1)+2(n+1))=\frac{n(n+1)(n+1)(n+2)}{12}=\frac{n(n+1)^2(n+2)}{12}$$

Q2. Sum the series

i. $1^2-2^2+3^2-4^2+\ldots+(2n-1)^2-(2n)^2$

Sol: $T_n=(2n-1)^2-(2n)^2=(4n^2-4n+1-4n^2)=-4n+1$

So $T_k=-4k+1$

$$S_n=\sum_{k=1}^{n}(-4k+1)=-4\sum_{k=1}^{n}k+\sum_{k=1}^{n}1$$

$$=-4\cdot\frac{n(n+1)}{2}+n=-2n(n+1)+n=-2n^2-2n+n=-2n^2-n$$

$$S_n=-2n^2-n=-n(2n+1)$$

ii. $\frac{1^2}{1}+\frac{1^2+2^2}{2}+\frac{1^2+2^2+3^2}{3}+\ldots$ **to n terms**

Sol: Here $T_n=\dfrac{\frac{n(n+1)(2n+1)}{6}}{n}=\frac{1}{n}\left[\frac{n(2n^2+n+2n+1)}{6}\right]=\frac{1}{6}(2n^2+3n+1)$

So $T_k=\frac{1}{6}(2k^2+3k+1)$

$$S_n=\sum_{k=1}^{n}T_k=\frac{1}{6}\sum_{k=1}^{n}(2k^2+3k+1)$$

$$=\frac{1}{6}\left[2\sum_{k=1}^{n}k^2+3\sum_{k=1}^{n}k+n\right]=\frac{1}{6}\left[\frac{2n(n+1)(2n+1)}{6}+3.\frac{n(n+1)}{2}+n\right]=\frac{n}{6}\left[\frac{(n+1)(2n+1)}{3}+\frac{3(n+1)}{2}+1\right]$$

$$=\frac{n}{6}\left[\frac{2(2n^2+3n+1)+9(n+1)+6}{6}\right]=\frac{n}{36}(4n^2+6n+2+9n+9+6)=\frac{n}{36}(4n^2+15n+15)$$

Q3. Find the sum of n terms of the series whose n^{th} terms are given

i. $5n^2+2n+3$

Sol: $T_n=5n^2+2n+3$

$T_k=5k^2+2k+3$

$$S_n=\sum_{k=1}^{n}T_k=\sum_{k=1}^{n}(5k^2+2k+3)$$

$$S_n=5\sum_{k=1}^{n}k^2+2\sum_{k=1}^{n}k+\sum_{k=1}^{n}3=5\cdot\frac{n(n+1)(2n+1)}{6}+2\cdot\frac{n(n+1)}{2}+3n$$

$$S_n=\frac{5n(n+1)(2n+1)}{6}+n(n+1)+3n=n\left[\frac{5(n+1)(2n+1)}{6}+n+1+3\right]$$

$$=n\left[\frac{5(2n^2+3n+1)+6n+24}{6}\right]=\frac{n}{6}\left[10n^2+15n+5+6n+24\right]=\frac{n}{6}(10n^2+21n+30)$$

ii. n^2+2n-3

Sol: $T_n=n^2+2n-3$ so $T_k=k^2+2k-3$

$$S_n=\sum_{k=1}^{n}\left(k^2+2k-3\right)$$

$$S_n=\sum_{k=1}^{n}k^2+2\sum_{k=1}^{n}k-\sum_{k=1}^{n}3=\frac{n(n+1)(2n+1)}{6}+2\frac{n(n+1)}{2}-3n$$

$$=\frac{n(n+1)(2n+1)}{6}+n(n+1)-3n=n\left(\frac{(n+1)(2n+1)}{6}+n+1-3\right)=n\left(\frac{2n^2+3n+1+6n-12}{6}\right)$$

$$S_n=\frac{n}{6}\left(2n^2+9n-11\right)=\frac{2n^3+9n^2-11n}{6}$$

Q4. Given n^{th} term of the series, find the sum to $2n$ terms

i. $3n^2+5n+2$

Sol: $T_n=3n^2+5n+2$, $T_k=3k^2+5k+2$

$$S_n=\sum_{k=1}^{n}\left(3k^2+5k+2\right)$$

$$S_n=3\sum_{k=1}^{n}k^2+5\sum_{k=1}^{n}k+\sum_{k=1}^{n}2=3\cdot\frac{n(n+1)(2n+1)}{6}+5\cdot\frac{n(n+1)}{2}+2n$$

$$S_n=\frac{3n(n+1)(2n+1)}{6}+\frac{5n(n+1)}{2}+2n=n\left[\frac{(n+1)(2n+1)}{2}+\frac{5(n+1)}{2}+2\right]$$

$$=n\left[\frac{2n^2+3n+1}{2}+\frac{5n+5}{2}+2\right]=\frac{n}{2}\left[2n^2+3n+1+5n+5+4\right]=\frac{n}{2}\left(2n^2+8n+10\right)=n\left(n^2+4n+5\right)$$

Replace n by $2n$

$$S_{2n}=2n\left[(2n)^2+4(2n)+5\right]=2n\left(4n^2+8n+5\right)=n\left(8n^2+16n+10\right)$$

ii. n^2+n-2

Sol: $T_n=n^2+n-2$, $T_k=k^2+k-2$

$$S_n=\sum_{k=1}^{n}\left(n^2+n-2\right)$$

$$S_n=\sum_{k=1}^{n}k^2+\sum_{k=1}^{n}k-\sum_{k=1}^{n}2=\frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}-2n$$

$$=n\left(\frac{(n+1)(2n+1)}{6}+\frac{n+1}{2}-2\right)=n\left(\frac{2n^2+3n+1}{6}+\frac{n+1}{2}-2\right)=n\left(\frac{2n^2+3n+1+3n+3-12}{6}\right)=\frac{n}{6}\left(2n^2+6n-8\right)$$

$$S_n=\frac{n}{3}\left(n^2+3n-4\right)$$

Replace n by $2n$

$$S_{2n}=\frac{2n}{3}\left((2n)^2+3(2n)-4\right)$$

$$S_{2n}=\frac{2n}{3}\left(4n^2+6n-4\right)$$