Exercise 6.11

Q1: **A sum of Rs. 10400 is paid off in 40 installment such that each installment is Rs.10 preceding installment. Calculate the value of the first installment.**

Sol: Total sum $S_{40} = 10400$ to find $= a$

N. of installments $n = 40$

Difference $d = 10$

Sum of an $A \cdot P = S_n = \frac{n}{2}[2a+(n-1)d] \longrightarrow (A)$

Put value in eq (A) then we get

$$S_{40} = \frac{40}{2}[2a+(40-1)10]$$

$$10400 = \frac{40}{2}[2a+(40-1)10]$$

$$10400 = 20[2a+(39)10]$$

$$\frac{10400}{20} = 2a+390 \Rightarrow 520 = 2a+390$$

$$520-390 = 2a \Rightarrow a = 65$$

So first installment is Rs.65.

Q2: **An investor invests Rs. 150000 at an annual compound interest rate of 6% for 8 years. Find the total amount will he get after 8 years.**

Sol: Tine Period $t = 8$ years

Annual Interest rate $r = 6\% = 0.06$

Formula for compound interest: $A = P(1+4)^t \longrightarrow (A)$

P = Principal Amount To find amount A = ?

Put value in eq (A) then we get

$$A = 150000(1+0.06)^8$$

$$A \approx 329077.2$$

Q3: **The population of a town is 4084101 at present and five years ago it was 3200000. Find its rate of increase if it increased geometrically.**

Sol: Present population $= 4084101$

Population five years ago $= 3200,000$.

Formula for geometric increase is: $P_n = P_0(1+r)^n \longrightarrow (A)$

- P_n is population after n years.
- P_0 is the initial population.
- r is the rate of increase.
- n is number of years.

Put value in eq (A) then we get

$$4084101 = 3200000(1+r)^5$$

$$\frac{4084101}{3200000} = (1+r)^5$$

$$1.276256875 = (1+r)^5$$

Take 5^{th} root of both sides.

$$(1.276256875)^{\frac{1}{5}} = 1+r \Rightarrow 1+r = 1.05 \Rightarrow r = 1.05-1 \Rightarrow r = 0.05$$

Convert the rate to percentage

Multiplying the rate by 100

$$0.05 \times 100 = 5\%$$

Hence population is increasing 5%.

Q4: **Determine the total worth of a yearly Rs.5000 investment after 20 years if the interest rate is 5% compounded annually.**

Sol: number of years $= 20$

Interest rate $= 5\% = 0.05$

Investment $= 5000$

Formula is: $F_{\upsilon} = P \cdot \frac{(1+r)^n - 1}{r} \longrightarrow (A)$

- P_{υ} is the future value.
- P is the periodic payment.
- Rs the interest rate per period.
- n is the number of periods.

$P = 5000$

$r = 0.05$

$n = 20$

Put value in eq (A) then we get

$F_{\upsilon} = 5000 \cdot \frac{(1+0.05)^{20} - 1}{0.05}$ $F_{\upsilon} = 500 \times 33.065954$

$F_{\upsilon} \approx 165.32977$

Q5: **A water tank has a leakage. Each week, the tank loses 5 gallons of water due to the leakage. Initially, the tank is full and contains 2000 gallons.**

i. How many gallons are in the tank 20 weeks later? **ii. How many weeks until the tank is half-full?**

iii. How many weeks until the tank is empty?

Sol: The tank initially grain 2000 gallons of water.

The Tank loses 5 gallons of water per week.

a) **Calculate amount of water after 20 week**

Calculate total water lost in 20 weeks $= (20)(5) = 100\, gallons$

Subtract lost of water from initial amount

$2000 - 100 = 1900$ gallons.

b) **Calculate half of tanks capacity.** $\frac{2000}{2} = 1000$ **gallons.**

Calculate amount of water to be lost $2000 - 1000 = 1000$ gallons.

Calculate number of weeks to lose 1000 gallons $\frac{1000}{5} = 200$ weeks.

c) **Calculate number of weeks to lose 2000 gallons.**

$= \frac{2000}{5} = 400$ weeks

After 20 weeks there are 1900 gallons in the tank.

Q6: **A drug company has manufactured 7 million doses of a vaccine to date. They promise additional production at a rate of 1.4 million doses/month over the next year.**

Sol:

i) **How many doses of the vaccine, in total, will have been produced after a year?**

The company has already produced 7 million doses.

Over the next year (which is 12 months), they will produce an additional 1.4 million doses / month $\times$ 12 months.

Additional doses produced in a year $= 1.4 \times 12 = 16.8$ million doses.

Total doses after a year = *initial doses* + additional doses

Total doses after a year 7 million + 16.8 million = 23.8 million doses.

So, after a year, a total of 23.8 million doses of the vaccine will have been produced.

ii) The general term a_n, describes the total number of doses of the vaccine produced. Describe the meaning of the variable n in the context of this problem. Find the general term a_n

Sol: In this problem, the variable n represents the number of months that have passed since the initial 7 million doses were manufactured. Here, n can take on values $0,1,2,3,...,12$, representing the start and each subsequent month over the next year.

To find the general term a_n, we consider the initial production and the production over n months:

- At $n = 0$ (the current time), the total doses produced is $a_0 = 7$ million.
- After n months, the additional doses produced will be $1.4 \times n$ million.

Therefore, the general term a_n representing the total number of doses produced after n months is:

$a_n = 7 + 1.4n$

Where a_n is in millions of doses, and n is the number of months since the initial production.

iii) Find the value of a_{10}, and interpret its meaning in words.

To find the value of a_{10}, we substitute $n = 10$ into the general term we found in part (**b**):

$$a_{10} = 7 + 1.4 \times 10$$
$$a_{10} = 7 + 14$$
$$a_{10} = 21$$

Interpreting this value in words: $a_{10} = 21$

means that after 10 months from the current time, the total number of doses of the vaccine produced will be 21 million.

Q7: At a toll booth, the number of vehicles passing through during the first minute is 100. Due to road congestion, each minute only 80% of the vehicles from the previous minute manage to pass.

Sol:

i) Represent the number of vehicles passing each minute as a sequence.

Let a_n be the number of vehicles passing through the toll booth during the n-th minute.

- During the first minute $n = 1$, the number of vehicles is $a_1 = 100$.
- During the second minute $(n = 2)$, 80% of the vehicles from the first minute pass through, so $a_2 = 0.80 \times a_1 = 0.80 \times 100 = 80$.
- During the third minute $(n = 3)$, 80% of the vehicles from the second minute pass through, so $a_3 = 0.80 \times a_2 = 0.80 \times 80 = 64$.
- And so on.

We can see that the number of vehicles passing through each minute forms a geometric sequence with the first term $a_1 = 100$ and a common ratio $r = 0.80$.

The sequence representing the number of vehicles passing each minute is:

$100, 80, 64, 51.2, ...$

The general term for this sequence is given by:

$$a_n = a \times r^{(n-1)}$$
$$a_n = 100 \times (0.80)^{(n-1)}$$

ii) Find the total number of vehicles that pass through in 15 minutes.

To find the total number of vehicles that pass through in the first 15 minutes, we need to find the sum of the first 15 terms of this geometric sequence. The formula for the sum of the first n terms of a geometric sequence is:

$$S_n = \frac{a_1(1 - r^n)}{1 - r}$$

In this case, $a_n = 100, r = 0.80, \text{and } n = 15$.

Plugging these values into the formula:

$$S_{15} = \frac{100\left(1-(0.80)^{15}\right)}{1-0.80}$$

$$S_{15} = \frac{100(1-0.03518437208883196)}{0.20} = \frac{100(0.964815627911168)}{0.20}$$

$$S_{15} = \frac{96.4815627911168}{0.20}$$

$$S_{15} \approx 482.407813955584$$

Since the number of vehicles must be a whole number, we can round this to approximately 482 vehicles.

So, the total number of vehicles that pass through in 15 minutes is approximately 482.

iii) What is the maximum number of vehicles that can pass in the long run (as time *t* approaches infinity)?

To find the maximum number of vehicles that can pass in the long run, we need to find the sum of the infinite geometric series is:

$$S_{\infty} = \frac{a_1}{1-r}$$

This formula is valid only when the absolute value of the common ratio $|r|$ is less than 1. In this case, $r = 0.80, \text{and } |0.80| < 1$, so the formula applies.

Plugging in the values $a_1 = 100$ and 0.80:

$$S_{\infty} = \frac{100}{1-0.80}$$

$$S_{\infty} = \frac{100}{0.20}$$

$$S_{\infty} = 500$$

Therefore, the maximum number of vehicles that can pass through the toll booth in the long run (as time approaches infinity) is 500.

Q8: A sum of Rs. 5000 is inverted at 8% simple interest per year. Calculate the interest at the end of each year. Do these interests form an A.P.? If so find the interest at the end of 20 years making use of this fact.

Sol: Given:

Principal (*P*) = Rs. 5000

Rate of simple interest $(R) = 8\%$ per year

Calculating the interest at the end of each year (I):

The formula for simple interest for one year is: $I = \frac{P \times R \times T}{100}$

Where Time $(T) = 1$ year.

- Year 1:

$$I_1 = \frac{500 \times 8 \times 1}{100} = \frac{40000}{100} = 400$$

- Year 2:

$$I_2 = \frac{500 \times 8 \times 1}{100} = 400$$

- Year 3:

$$I_3 = \frac{500 \times 8 \times 1}{100} = 400$$

The sequence of interests is 400, 400, 400,...

Checking if the interests form an Arithmetic Progression (A.P.):

A sequence is an A.P. if the difference between consecutive terms is constant. This constant difference is called the common difference (*d*).

Here, the terms are $a_1 = 400, a_2 = 400, a_3 = 400, \ldots$

The common difference *d* is:

$$d = a_2 - a_1 = 400 - 400 = 0$$
$$d = a_3 - a_2 = 400 - 400 = 0$$

Since the common difference is constant $(d = 0)$, the interests form an Arithmetic Progression.

Finding the interest at the end of 20 years using the A.P. formula:

The formula for the n^{th} term (a_n) of an A.P. is:

$$a_n = a + (n+1)d$$

Where:

- *a* is the first term $(a = 400)$
- *n* is the number of years $(n = 20)$
- *d* is the common difference $(d = 0)$

Substituting the values:

$$a_{20} = 400 + (20-1) \times 0$$
$$a_{20} = 400 + (19) \times 0$$
$$a_{20} = 400 + 0$$
$$a_{20} = 400$$

Thus, the interest at the end of 20 years is Rs. 400.

Total interest for 20 years $400 \times 20 = 8000$

Q9: **A machine is purchased for Rs. 20,000. Depreciates at 6% per annum for the first four years and after that 8% per annum for the next six years. Depreciation being calculated on diminishing value. Find the value of the machine after a period of 10 year.**

Sol: Initial Value: Rs. 20,000

Depreciation for the First Four Years (6% per annum on diminishing value):

- **End of Year 1:**

  Depreciation $= 6\% \text{ of } 20000 = \frac{6}{100} \times 20000 = 1200$

  Value after Year $1 = 20000 - 1200 = Rs.18,800$

- **End of Year 2:**

  Depreciation $= 6\% \text{ of } 18800 = \frac{6}{100} \times 18800 = 1228$

  Value after Year $2 = 18800 - 1228 = Rs.17,672$

- **End of Year 3:**

  Depreciation $= 6\% \text{ of } 17672 = \frac{6}{100} \times 17672 = 1060.32$

  Value after Year $3 = 17672 - 1228 = Rs.16,611.68$

- **End of Year 4:**

  Depreciation $= 6\% \text{ of } 16611.68 = \frac{6}{100} \times 16611.68 = 996.7008$

  Value after Year $4 = 18800 - 16611.68 = \text{Rs.}15,614.9792$

Value after 4 years $= \text{Rs}.15{,}614.98$ (approximately)

Depreciation for the Next Six Years (8% per annum on diminishing value):

Now, the value at the beginning of this period is $\text{Rs}.15{,}614.9792$.

- **End of Year 5:**

Depreciation $= 8\%$ of $15614.9792 = \frac{8}{100} \times 15614.9792 = 1249.198336$

Value after Year $5 = 15614.9792 - 1249.198336 = \text{Rs}.14{,}365.780864$

- **End of Year 6:**

Depreciation $= 8\%$ of $14{,}365.780864 = \frac{8}{100} \times 14{,}365.780864 = 1149.26246912$

Value after Year $6 = 14365.780864 - 1149.26246912 = \text{Rs}.13{,}216.51839488$

- **End of Year 7:**

Depreciation $= 8\%$ of $13216.51839488 = \frac{8}{100} \times 13216.51839488 = 1057.321471590$

Value after Year $7 = 13216.51839488 - 1057.321471590 = \text{Rs}.12{,}159.1969232896$

- **End of Year 8:**

Depreciation $= 8\%$ of $12159.1969232896 = \frac{8}{100} \times 12159.1969232896 = 972.7235753168$

Value after Year $8 = 12159.1969232896 - 972.7235753168 = \text{Rs}.11{,}186.461169426432$

- **End of Year 9:**

Depreciation $= 8\%$ of $11186.461169426432 = \frac{8}{100} \times 11186.461169426432 = 894.91689355411456$

Value after Year $9 = 11186.461169426432 - 894.91689355411456 = \text{Rs}.10{,}291.54427231744$

- **End of Year 10:**

Depreciation $= 8\%$ of $10291.54427231744 = \frac{8}{100} \times 10291.54427231744 = 823.3253420697854$

Value after Year $10 = 10291.54427231744 - 823.3253420697854 = \text{Rs}.9468.22073380253204$

Therefore, the value of the machine after a period of 10 years is approximately $\text{Rs}.9468.22$.

Q10: Two cars start together in the same direction from the same place. The first goes with uniform speed of $20km/h$. The second goes at a speed of $12km/h$ in the first hour and increases the speed by $1km/h$ each succeeding hour. After how many hours will the second car overtake the first car if both cars go non-stop?

Sol: **Car 1:**

Speed $= 20km/h$

Distance after time t: $D_1 = 20t$

Car 2:

Speed in hour n: $v_n = 12 + (n-1) \times 1 = 11 + n$

Distance after time t: $D_2 = \sum_{n=1}^{t}(11+n) = \sum_{n=1}^{t} 11 + \sum_{n=1}^{t} n = 11t + \frac{t(t+1)}{2}$

Overtaking condition: $D_1 = D_2$

$$20t = 11t + \frac{t(t+1)}{2}$$

Multiply by 2:

$$40t = 22t + t(t+1)$$

$$40t = 22t + t^2 + t$$

$$40t = t^2 + 23t$$

$$t^2 - 17t = 0$$

$t(t-17)=0$
Solutions: $t=0$ or $t=17$
Since $t=0$ is the starting time, the overtaking occurs at $t=17$ hours.
Final Answer: 17

Q11: **150 workers were engaged to finish a piece of work in a certain number of days. Five workers dropped the second day, five more workers dropped the third day and so on. It takes 10 more days to finish the work now. Find the number of days in which the work was completed.**

Sol: Sum of arithmetic series $S_n=\frac{11}{2}(2a+(n-1)d)$

n is number of terms.
a is first term.
d is difference.
Let x be the original number of days the total work is $150x$.
Workers is completed in $x+10$ days same as $150,145,140,.....$

Then sum is $\frac{x+10}{2}(2(150)+(x+10-1)(-5))$

Equate two expressions for total work.

$$150x=\frac{x+10}{2}(300-5(x+9))$$
$$300x=(x+10)(300-5x-45)$$
$$300x=(x+10)(255-5x)$$
$$300x=255x-5x^2+2550-50x$$
$$5x^2+95x-2550=0$$
$$x^2+19x-510=0$$
$$(x-15)(x+34)=0$$
$$x=15,x=-34$$

Since number of days cannot be $(-ve)$ so $x=15$

Calculate number of days to complete the work.
The work was completed in $x+10$ days
So, $15+10=25$ days.

Q12: **A radioactive product has a half-life of 5 years. If the radioactivity level is 68 micro curies after 20 years. Determine the original level of radioactivity.**

Sol: Half Life $15+10=25$ years.
Radioactivity level after 20 years $A(20)=68$ micro curies
Tim elapsed $t=20$ years.

Radioactivity decay formula $=A(t)=A_0\cdot\left(\frac{1}{2}\right)^{\frac{t}{t_{\frac{1}{2}}}}\longrightarrow(A)$

- $A(t)$ is amount of substance at time t
- A_0 is initial amount of substance.
- $t_{\frac{1}{2}}$ Half Life

To find $A_0=?$

$$A(20)=68$$
$$t=20$$
$$t_{\frac{1}{2}}=5$$

Put value in Eq. (A)

$$68 = A_0 \cdot \left(\frac{1}{2}\right)^{\frac{20}{5}}$$

$$68 = A_0 \cdot \left(\frac{1}{2}\right)^4 = A_0 \cdot \frac{1}{16}$$

$$A_0 = 68 \cdot 16$$

$A_0 = 1088$ Micro curies

Q13: **An object moving in a line is given an initial velocity of $4.5m/s$ and a constant acceleration of 2.5 m/s². How long will it take the object to reach a velocity of $20m/s$?**

Sol: Initial velocity $v_1 = 4.5m/s$

Acceleration $a = 2.5m/s^2$

Final velocity $= v_f = 20m/s$

Find Time $t = ?$

Acc. To 1st eq. of motion.

$$v_f = v_i + at$$

$$\frac{v_f - v_i}{a} = t \longrightarrow (A)$$

Put value in Eq. (A)

$$\frac{20 - 4.5}{2.5} = t$$

$t = 6.2s$

Q14: **In an integrated circuit with an initial current of $1080mA$, the temperature in the components decreases from 20% to 17% to 14%. Assuming that each temperature decrease is caused by a decrease in the initial current, what is the value of current at fourth measurement?**

Sol: Let $t = 6.2S$ be the initial current.

$20\% \rightarrow 17\% \rightarrow 14\%$

The temperature decreases in steps of:

$t = 6.2S$

The percentage decrease in temperature per step is:

$20\% - 17\% = 3\%$

$17\% - 14\% = 3\%$

Assume each 3% temperature decrease causes a 3% decrease in the initial current.

1. ***Calculate the current decrease per step:***
   Decrease per step $= 3\%$ of $I_0 = 0.03 \times 1080mA = 32.4mA$
2. **Determine the number of decreases to reach the fourth measurement:**
   First measurement: Initial current
   Second measurement: After 1st decrease
   Third measurement: After 2^{nd} decrease
   Fourth measurement: After 3^{rd} decrease
   Therefore, there are 3 decreases from the initial current.
3. **Calculate the total current decrease:**
   Total decrease = Number of decreases × Decrease per step
   Total decrease $= 3 \times 32.4mA = 97.2mA$
4. **Calculate the current at the fourth measurement (I_4):**

$$I_4 = I_0 - \text{Total decrease}$$

$$I_4 = 1080mA - 97.2mA$$

$$I_4 = 982.8mA$$

Q15: ***Show that the amount of a certain sum of money at compound interest of $r\%$ per year for n years form a G.P.***

Sol: Let P be the principal amount.

Let r be the annual interest rate as a percentage.

Let R be the annual interest rate as a decimal, so $R = \frac{r}{100}$.

Let A_n be the amount of money at the end of n years.

Amount at the end of Year 0 (initial amount):

$$A_0 = P$$

Amount at the end of Year 1:

$$A_1 = P + P \cdot R = P(1+R)$$

Amount at the end of Year 2:

The principal for Year 2 is A_1.

$$A_2 = A_1 + A_1 \cdot R = A_1(1+R)$$

Substitute $A_1 = P(1+R)$:

$$A_2 = P(1+R)(1+R) = P(1+R)^2$$

Amount at the end of Year 3:

The principal for Year 3 is A_2.

$$A_3 = A_2 + A_2 \cdot R = A_2(1+R)$$

Substitute $A_2 = P(1+R)^2$:

$$A_3 = P(1+R)^2(1+R) = P(1+R)^3$$

Generalizing, the amount at the end of n years is:

$$A_n = P(1+R)^n$$

Consider the sequence of amounts:

$$A_0, A_1, A_2, A_3, \ldots, A_n, \ldots$$

$$P, P(1+R), P(1+R)^2, P(1+R)^3, \ldots, P(1+R)^n, \ldots$$

To show this sequence is a Geometric Progression (G.P.), we need to show that the ratio of any term to its preceding term is a constant. Let this common ratio be k.

Calculate the ratio of consecutive terms:

$$\frac{A_1}{A_0} = \frac{P(1+R)}{P} = (1+R)$$

$$\frac{A_2}{A_1} = \frac{P(1+R)^2}{P(1+R)} = (1+R)$$

$$\frac{A_3}{A_2} = \frac{P(1+R)^3}{P(1+R)^2} = (1+R)$$

In general, for any $n \geq 1$:

$$\frac{A_n}{A_{n-1}} = \frac{P(1+R)^n}{P(1+R)^{n-1}} = (1+R)$$

Since the ratio of any term to its preceding term is the constant $(1+R)$, the sequence of amounts froms a Geometric Progression with a common ratio $k = (1+R)$.