1. **Use mathematical induction to prove the following formulae for every positive integer n**

i) $\log x^n = n\log x$ **, where** x **is positive**

Sol: Let $S(n)$ be the give statement $S(n): \log x^n = n\log x$

Base case for $n=1$

$S(1): \log x^1 = \log x$

$1\log x = \log x$

$\log x = \log x$ $\quad S(1)$ is true base case is satisfied

Inductive Hypothesis:

Let $S(k)$ is true for $n = k \in N$

$S(k): \log x^k = k\log(x)$

For $n = k+1$

$S(k+1): \log x^{k+1} = (k+1)\log(x)$

$\log x^{k+1} = \log(x^k.x)$

$= \log x^k + \log x = k\log(x) + \log x = (k+1)\log(x)$ Hence proved.

ii) $2+5+8+...+(3n-1) = \frac{n}{2}(3n+1)$

Sol: Let $S(n)$ be the given statement then $S(n): 2+5+8+.....+(3n-1) = \frac{n}{2}(3n+1)$

Base case for: For $n=1$

$S(1): (3(1)-1) = \frac{1}{2}(3(1)+1) \Rightarrow 2 = 2$

Inductive Hypothesis:

Assume $S(k): 2+5+8+...+(3k-1) = \frac{k}{2}(3k+1)$ (i)

Inductive Step: For $n = k+1$

Prove $2+5+8+...+(3(k+1)-1) = \frac{(k+1)}{2}(3(k+1)+1)$ (ii)

Add $(3(k+1)-1)$ both sides of (i)

$[2+5+8+...+(3k-1)] + (3(k+1)-1) = \frac{k}{2}(3k+1) + (3(k+1)-1)$

$$=\frac{k}{2}(3k+1)+(3k+2)=\frac{k(3k+1)+2(3k+2)}{2}=\frac{3k^2+k+6k+4}{2}=\frac{3k^2+7k+4}{2}$$

$$=\frac{3k^2+3k+4k+4}{2}=\frac{3k(k+1)+4(k+1)}{2}=\frac{(k+1)(3k+3+1)}{2}=\frac{(k+1)[3(k+1)+1]}{2}$$

Which is (ii) and is true for $n=k+1$

iii) $\mathbf{2+(2+5)+(2+5+8)+\ldots+\frac{n}{2}(3n+1)=\frac{n}{4}(n+1)^2}$

Sol: Let $S(n)$ be the given statement $S(n):2+(2+5)+(2+5+8)+\ldots+\frac{n}{2}(3n+1)=\frac{n}{4}(n+1)^2$

for $n=1$

$S(1):\frac{1}{2}(3(1)+1)=\frac{1}{4}(1+1)^2$

$\Rightarrow 2=1$

$S(1)$ not satisfied formula is incorrect

iv) $\mathbf{2+6+18+\ldots+2\times3^{n-1}=3^n-1}$

Let $S(n)$ be the given statement

$S(n):2+6+18+\ldots+2\times3^{n-1}=3^n-1$

Base Case: For $n=1$

$S(1):\ 2\times3^{1-1}=3^1-1$

$\Rightarrow 2\times3^0=3-1\Rightarrow 2\times1=2\Rightarrow 2=2$. So, base case is satisfied.

Induction of Hypothesis:

Suppose that formula is true for $n=k\in N$

$S(k):2+6+18+\ldots+2\times3^{k-1}=3^k-1 \qquad (i)$

Now we prove it for $n=k+1$

$S(k+1):2+6+18+\ldots+2\times3^{\overline{k+1}-1}=3^{k+1}-1$ (ii)

Adding $2\times3^{k+1-1}$ on both sides of

$2+6+18+\ldots+2\times3^{k-1}+2\times3^{k+1-1}=3^k-1+2\times3^{k+1-1}$

$2+6+18+\ldots+2\times3^{k+1-1}=3^k-1+2\times3^k$

$2+6+18+\ldots+2\times3^{k+1-1}=3^k+2\times3^k-1$

$2+6+18+\ldots+2\times3^{k+1-1}=3^k(1+2)-1$

$2+6+18+\ldots+2\times3^{k+1-1}=3^k.3-1$

$2+6+18+\ldots+2\times3^{\overline{k+1}-1}=3^{\overline{k+1}}-1$ Which is (ii)

It is true for $n=k+1$

Hence given formula is true for all positive integer n.

v) $\mathbf{1\times3+2\times5+3\times7+\ldots+n\times(2n+1)=\frac{n(n+1)(4n+5)}{6}}$

Sol: Let $S(n)$ be the give statement $S(n):1\times3+2\times5+3\times7+\ldots+n\times(2n+1)=\frac{n(n+1)(4n+5)}{6}$

Base Case: For $n=1$

$S(1):\ 1\times(2(1)+1)=\frac{1(1+1)(4(1)+5)}{6}\Rightarrow1\times(2+1)=\frac{1(2)(9)}{6}\Rightarrow1\times3=\frac{2\times3\times3}{2\times3}\Rightarrow3=3$

So, base case is satisfied.

Induction of Hypothesis:

Suppose that formula is true for $n = k \in N$

$$S(k): 1\times3+2\times5+3\times7+...+k\times(2k+1)=\frac{k(k+1)(4k+5)}{6} \qquad (i)$$

For $n = k+1$

$$S(k+1): 1\times3+2\times5+3\times7+...+(k+1)\times(2\overline{k+1}+1)$$

$$=\frac{(k+1)(\overline{k+1}+1)(4\overline{k+1}+5)}{6} \qquad (ii)$$

Adding $(k+1)\times[2(k+1)+1]$ on both sides of (i)

$$1\times3+2\times5+3\times7+...+k\times(2k+1)+(k+1)\times[2(k+1)+1]$$

$$=\frac{k(k+1)(4k+5)}{6}+(k+1)\times[2(k+1)+1]$$

$$=\frac{k(k+1)(4k+5)}{6}+(k+1)\times(2k+2+1)=\frac{k(k+1)(4k+5)}{6}+(k+1)\times(2k+3)$$

$$=\frac{k(k+1)(4k+5)}{6}+(k+1)\times(2k+3)=\frac{(k+1)[k(4k+5)+6(2k+3)]}{6}$$

$$=\frac{(k+1)(4k^2+5k+12k+18)}{6}=\frac{(k+1)(4k^2+17k+18)}{6}$$

$$=\frac{(k+1)(4k^2+8k+9k+18)}{6}=\frac{(k+1)[4k(k+2)+9(k+2)]}{6}$$

$$=\frac{(k+1)(k+2)(4k+9)}{6}=\frac{(k+1)(k+2)(4k+4+5)}{6}$$

$$=\frac{(k+1)(\overline{k+1}+1)(4(\overline{k+1})+5)}{6}$$ Which is (ii)

It is true for $n = k+1$

Hence given formula is true for all positive integer n.

vi) $$\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+...+\frac{1}{n(n+1)}=1-\frac{1}{n+1}$$

Sol: Let $S(n)$ be the give statement

$$S(n): \frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+...+\frac{1}{n(n+1)}=1-\frac{1}{n+1}$$

Base Case: For $n = 1$

$$S(1): \frac{1}{1(1+1)}=1-\frac{1}{1+1} \Rightarrow \quad \frac{1}{2}=1-\frac{1}{2} \Rightarrow \quad \frac{1}{2}=\frac{1}{2}$$

So, base case is satisfied.

Induction of Hypothesis:

Suppose that formula is true for $n = k \in N$

$$S(k): \frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+...+\frac{1}{k(k+1)}=1-\frac{1}{k+1} \qquad (i)$$

For $n = k+1$

$$S(k+1):\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+...+\frac{1}{(k+1)(\overline{k+1}+1)}=1-\frac{1}{\overline{k+1}+1}\text{ —— }(ii)$$

Adding $\frac{1}{(k+1)(k+1+1)}$ both sides of (i)

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+...+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+1+1)}$$

$$=1-\frac{1}{(k+1)}+\frac{1}{(k+1)(k+1+1)}=1-\frac{1}{(k+1)}+\frac{1}{(k+1)(k+2)}$$

$$=1+\frac{1}{(k+1)(k+2)}-\frac{1}{(k+1)}=1+\frac{1-(k+2)}{(k+1)(k+2)}=1+\frac{1-k-2}{(k+1)(k+2)}$$

$$=1+\frac{-1-k}{(k+1)(k+2)}=1+\left[\frac{-(1+k)}{(k+1)(k+2)}\right]=1-\frac{(1+k)}{(k+1)(k+2)}$$

$$=1-\frac{1}{\overline{k+1}+1}\quad \text{Which is (ii)}$$

It is true for $n=k+1$

Hence given formula is the true for all positive integer n

vii) $r+r^2+r^3+...+r^n=\frac{r(1-r^n)}{1-r},(r\neq 1)$

Sol: Let $S(n)$ be the give statement

$$r+r^2+r^3+...+r^n=\frac{r(1-r^n)}{1-r},(r\neq 1)$$

Base case: For $n=1$

$$S(1):\ r^1=\frac{r(1-r^1)}{1-r}\quad \Rightarrow r=r$$

So, base case is satisfied

Induction of Hypothesis:

Suppose that formula is true for $n=k\in N$

$$S(k):r+r^2+r^3+...+r^k=\frac{r(1-r^k)}{1-r}\qquad (i)$$

For $n=k+1$

$$S(k+1):r+r^2+r^3+...+r^k+r^{k+1}=\frac{r(1-r^{k+1})}{1-r}\qquad (ii)$$

Adding r^{k+1} on both sides of (i)

$$=r+r^2+r^3+...+r^4+r^{4+1}=\frac{r(1-r^{k+1})}{1-r}+r^{4+1}=\frac{r(1-r^k)+r^{k+1}(1-r)}{1-r}=\frac{r-r.r^k+r^{k+1}-r.r^{k+1}}{1-r}$$

$$=\frac{r-r^k+r^{k+1}-r.r^{k+1}}{1-r}=\frac{r-r.r^{k+1}}{1-r}=\frac{r(1-r^{\overline{k+1}})}{1-r}\text{ which is (ii)}$$

It is true for $n=k+1$

Hence given formula is true for all positive integer n

viii) $\boldsymbol{a+(a+d)+(a+2d)+\ldots.+[a+(n-1)d]=\frac{n}{2}[2a+(n-1)d]}$

Sol: Let $S(n)$ be the give statement

$$S(n): a+(a+d)+(a+2d)+\ldots+[a+(n-1)d]=\frac{n}{2}[2a+(n-1)d]$$

Base case: For $n=1$

$$S(1): [a+(1-1)d]=\frac{1}{2}[2a+(1-1)d]$$

$$\Rightarrow a+0=\frac{1}{2}[2a+0] \Rightarrow a=\frac{2a}{2} \Rightarrow a=a$$

So, base case is satisfied.

Induction of Hypothesis:

Suppose that formula is true for $n=k\in N$

$$S(k): a+(a+d)+(a+2d)+\ldots+[a+(k-1)d]=\frac{k}{2}[2a+(k-1)d] \qquad (i)$$

$n=k+1$

$$S(k+1): a+(a+d)+(a+2d)+\ldots+[a+(\overline{k+1}-1)d]$$

$$=\frac{(k+1)}{2}[2a+(\overline{k+1}-1)d] \qquad (ii)$$

Adding $[a+(k+1-1)d]$ on both sides of (i)

$$a+(a+d)+(a+2d)+\ldots+[a+(k-1)d]+[a+(k+1-1)d]$$

$$=\frac{k}{2}[2a+(k-1)d]+[a+(k+1-1)d]=\frac{k}{2}[2a+(k-1)d]+[a+kd]$$

$$=\frac{2ak+k^2d-kd+2a+2kd}{2}=\frac{2ak+k^2d+2a+kd}{2}$$

$$=\frac{2ak+2a+k^2d+kd}{2}=\frac{2a(k+1)+kd(k+1)}{2}$$

$$=\frac{(k+1)+(2a+kd)}{2}=\frac{(k+1)+[2a+(k+1-1)d]}{2}$$

$$=\frac{\overline{(k+1)}}{2}\frac{[2a+(\overline{k+1}-1)d]}{2} \text{ Which is (ii)}$$

It is true for $n=k+1$

Hence given formula is true for all positive integer n.

ix) $\boldsymbol{a_n=a_1+(n-1)d}$ **when** $\boldsymbol{a_1, a_1+d, a_1+2d, \ldots\ldots}$ ***form an A.P***

Sol: Let $S(n)$ be the given statement

$$a_n=a_1+(n-1)d$$

Base Case: For $n=1$

$$S(1): a_1=a_1+(1-1)d$$

$\Rightarrow \quad a_1=a_1+0$

$\Rightarrow \quad a_1=a_1$

So, base case is satisfied

Induction of Hypothesis:

Suppose that formula is true for $n=k\in N$

$S(k): a_k = a_1 + (k-1)d \qquad (i)$

For n=k+1

$S(k+1): a_{k+1} = a_1 + (\overline{k+1}-1)d \qquad (ii)$

Adding 'd' on both sides of (i)

$a_k + d = a_1 + (k-1)d + d$

$a_{k+1} = a_1 + (k-1+1)d$

or $\quad a_{\overline{k+1}} = a_1 + (\overline{k+1}-1)d \quad$ *which is (ii)*

It is true for $n = k+1$

Hence given formula is true for all positive integer n.

x) $\boldsymbol{a_n = a_1 r^{n-1}}$ **when** $\boldsymbol{a_1, a_1 r, a_1 r^2, \ldots}$ ***Form a G.P***

Sol: Let $S(n)$ be the give statement

$S(n): a_n = a_1 r^{n-1}$

Base case: For $n = 1$

$S(1): a_1 = a_1 r^{1-1}$

$\Rightarrow a_1 = a_1 r^0$

$\Rightarrow a_1 = a_1 .1$

$\Rightarrow a_1 = a_1$

So, base case is satisfied.

Induction of Hypothesis:

Suppose that formula is true for $n = k \in N$

$S(k): \ a_k = a_1 r^{k-1} \qquad (i)$

For $n = k+1$

$S(k+1): \ a_{k+1} = a_1 r^{k+1-1} \qquad (ii)$

Multiply by 'r' on both sides of (i)

$\Rightarrow a_k . r = a_1 r^{k-1}.r \qquad (i)$

$\Rightarrow a_{k+1} = a_1 r^{k-1+1}$

$\Rightarrow a_{\overline{k+1}} = a_1 r^{\overline{k+1}-1}$ *which is (ii) and*

It is true for $n = k+1$

Hence given formula is true for all positive integer n.

xi) $\binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \ldots + \binom{n+2}{3} = \binom{n+3}{4}$

Sol: Let $S(n)$ be the give statement

$S(n): \binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \ldots + \binom{n+2}{3} = \binom{n+3}{4}$

Base case: For $n = 1$

$S(1): \binom{1+2}{3} = \binom{1+3}{4} \Rightarrow \binom{3}{3} = \binom{4}{4} \qquad \left(\binom{n}{n} = {}^n C_n = 1\right)$

$\Rightarrow 1 = 1$

So, base case is satisfied.

Induction of Hypothesis:

Suppose that formula is true for $n=k\in N$

$$S(k):\binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\ldots+\binom{k+2}{3}=\binom{k+3}{4} \qquad (i)$$

For $n=k+1$

$$S(k+1):\binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\ldots+\binom{\overline{k+1}+2}{3}=\binom{\overline{k+1}+3}{4} \qquad (ii)$$

Adding $\binom{k+1+2}{3}$ on both sides of (i)

$$\binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\ldots+\binom{k+2}{3}+\binom{k+1+2}{3}=\binom{k+3}{4}+\binom{k+1+2}{3}$$

$$=\binom{k+3}{4}+\binom{k+1+2}{3}=\binom{k+3}{4}+\binom{k+3}{3}$$

$\therefore \quad {}^{n}C_{r}+{}^{n}C_{r-1}={}^{n+1}C_{r}$ or $\binom{n}{r}+\binom{n}{r-1}=\binom{n+1}{r}$

$$=\binom{k+3+1}{4}=\binom{\overline{k+1}+3}{4}$$

$$\binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\ldots+\binom{\overline{k+1}+2}{3}=\binom{\overline{k+1}+3}{4} \text{ which is } (ii)$$

It is true for $n=k+1$

Hence given formula is true for all positive integer n.

xii) The sum of first n odd natural is n^2

Sol: Let $S(n)$ be the give statement

$$S(n):1+3+5+\ldots+(2n-1)=n^2$$

Base case: For $n=1$

$$S(1):2(1)-1=1^2$$

$$1=1$$

Inductive Hypothesis:

Suppose the formula is true for $n=k\in N$

$$S(k):1+3+5+\ldots+(2k-1)=k^2 \_\_\_\_(i)$$

For $n=k+1$

$$S(k+1):1+3+5+\ldots+(2(k+1)-1)=(k+1)^2 \_\_\_\_(ii)$$

Add $(2(k+1)-1)$ both sides of (i)

$$[1+3+5+\ldots+(2k-1)]+(2(k+1)-1)=k^2+(2(k+1)-1)$$

$$=k^2+(2k+1)=(k+1)^2 \text{ Which is } (ii)$$

It is true for $n=k+1$

2. Prove by mathematical induction that for all positive integral values of n:

i) n^2+n is divisible by 2

Sol: n^2+n is divisible by 2

Let $S(n)$ be the give statement

$S(n): n^2+n$ is divisible of 2

Base Case: For $n=1$

$=1^2+1=2$, which is divisible by 2

So, base case is satisfied.

Induction of Hypothesis:

Suppose that statement is true for $n=k\in N$

$S(k): k^2+k$ is divisible by 2

$\Rightarrow k^2+k=2Q---(i)$

Where Q is some quotient

For $n=k+1$

$S(k+1):(k+1)^2+(k+1)=k^2+2k+1+k+1$

$=k^2+k+2k+2$

$=(k^2+k)+2k+2$

From eq (i), we get

$=2Q+2(k+1)$

$=2[Q+k+1]$

Where $Q+k+1$ is some quotient

So, $(k+1)^2+(k+1)$ is divisible by 2

Which is true for $n=k+1$

Hence given statement is true for all positive integral values of n.

ii) 5^n-2^n is divisible by 3

Sol: 5^n-2^n is divisible by 3

Let $S(n)$ be the give statement

$S(n): 5^n-2^n$ is divisible by 3

Base Case: for $n=1$

$S(1): 5^1-2^1=3$ is divisible by 3 so base case is satisfied

Suppose the statement is true for $n=k\in N$

$S(k): 5^k-2^k$ is divisible by 3 (i)

For $n=k+1$

$S(k): 5^{k+1}-2^{k+1}=5.5^k-2.2^k=(3+2).5^k-2.2^k$

$=3.5^k+2.5^k-2.2^k=3.5+2(5^k-2^k)$

Both factors are divisible by 3 so

$=3Q$ It is true form $n=k+1$

Hence proved for all $n\in N$

iii) $8\times 10^n-2$ is divisible by 6

Sol: Let $S(n): 8\times 10^n-2$ is divisible by 6

Base Case: For $n=1$

$S(1): 8\times 10^1-2=80-2=78=6\times 13$

Which is divisible by 6

So, base case is satisfied.

Induction of hypothesis:

Suppose that statement is true for $n=k\in N$

$S(k): 8\times 10^k-2$ is divisible by 6

$\Rightarrow 8\times 10^k-2=6Q \quad (i)$

Where Q is some quotient

Now we prove it for $n=k+1$

Take $8\times 10^{k+1}-2=8\times 10^k\times 10-2$

$=8\times 10^k\times 10-2\times 10+20-2$

$=10(8\times 10^k-2)+18$

From (i) we get

$=10(6Q)+18$

Both factor of R.H.S are divisible by 6.

Hence it is divisible by 6

Which is true for $n=k+1$

Hence given statement is true for all positive integral values of n

3. Prove that $\sum_{k=1}^{n} r^k=\frac{r^{n+1}-r}{r-1}$, whenever n is a positive integer.

Note: Wrong Statement is given in Textbook. We solve it with correction

Sol: Let $S(n)$ be the give statement

$S(n): \sum_{k=1}^{n} r^k=\frac{r^{n+1}-r}{r-1}$

$S(n): r+r^2+r^3+...+r^n=\frac{r^{n+1}-r}{r-1}$

Base Case: For $n=1$

$S(1): r^1=\frac{r^{1+1}-r}{r-1}$

$r=\frac{r^2-r}{r-1}$

$r=\frac{r(r-1)}{r-1}\Rightarrow r=r$

So base case is satisfied.

Induction of Hypothesis:

Suppose that formula is true for $n=k\in N$

$S(k): r+r^2+r^3+...+r^k = \frac{r^{k+1}-r}{r-1}$ (i)

For $n = k+1$

$S(k+1): r+r^2+r^3+...+r^{k+1} = \frac{r^{k+1+1}-r}{r-1}$ (ii)

Adding r^{k+1} on both sides of (i)

$$r+r^2+r^3+...+r^k+r^{k+1} = \frac{r^{k+1}-r}{r-1}+r^{k+1}$$

$$= \frac{r^{k+1}-r+r^{k+1}(r-1)}{r-1}$$

$$= \frac{\cancel{r^{k+1}}-r+r.r^{k+1}-\cancel{r^{k+1}}}{r-1}$$

$$= \frac{-r+r.r^{k+1}}{r-1} = \frac{r.r^{k+1}-r}{r-1}$$

$$= \frac{r^{\overline{k+1}+1}-r}{r-1}$$

i.e. $r+r^2+r^3+...+r^{\overline{k+1}} = \frac{r^{\overline{k+1}+1}-r}{r-1}$ *which is (ii)*

It is true for $n = k+1$

Hence given formula is true for all positive integer n.

4. **$x-y$ is a factor of x^n-y^n for all positive integral values of n $(x \neq y)$.**

Sol: Let $S(n)$ be the give statement

$S(n): x-y$ is a factor of x^n-y^n

Base Case: For $n=1$

$S(1): x^1-y^1 = x-y$

Which is a factor of x^n-y^n

So, base case is satisfied.

Induction of Hypothesis:

Suppose that statement is true for $n = k \in N$

$S(k): x-y$ is a factor of x^k-y^k

i.e $x^k-y^k = (x-y)Q$ (i)

Where Q is some quotient.

Now we prove it for $n = k+1$

Take $x^{k+1}-y^{k+1} = x.x^k-y.y^k$

$$= x.x^k-x^ky+x^k.y-y.y^k$$

$$= x^k(x-y)+y(x^k-y^k)$$

From (i), we get

$$= x^k(x-y)+y(x-y)Q$$

$$= (x-y)[x^k+yQ]$$

Where x^k+yQ is some quotient

Clearly $x-y$ is a factor of $x^{k+1}-y^{k+1}$

Which is true for $n = k+1$

Hence given statement is true for all positive value of n

5. **$n! > 2^n - 1$ for integral values of $n \geq 4$.**

Sol: Let $S(n)$ be the give statement

$S(n): n! > 2^n - 1$ for integral values of $n \geq 4$

Base Case: For $n = 4$

$S(1): 4! > 2^4 - 1$

$\therefore 4! = 4\times3\times2\times1 = 24$

$\Rightarrow 24 > 16-1$

$\Rightarrow 24 > 15$

So base case is satisfied.

Induction of hypothesis:

Suppose that statement is true for $n = k \in N$

$S(k): k! > 2^k - 1$

For integral values of $k \geq 4$ (i)

Now we prove it for $n = k+1$

Multiplying eq (i) by $k+1$ on both sides

$(k+1)k! > (k+1)(2^k-1)$

$\Rightarrow (k+1)k! > k(2^k-1)+1(2^k-1)$ $\therefore 2^k-1>0$

$\Rightarrow (k+1)k! > k(2^k-1)$

$\Rightarrow (k+1)k! > 2.2^k-1$ $\therefore k>2$

$\Rightarrow \overline{(k+1)}! > 2^{\overline{k+1}}-1$

Which is true for $n = k+1$

Hence given statement is true for all positive values of $n \geq 4$

6. **$4^n > 3^n + 2^{n-1}$ for integral values of $n \geq 2$.**

Sol: Let $S(n)$ be the give statement

$S(n): 4^n > 3^n + 2^{n-1}$, $n \geq 2$

Base Case: For $n = 2$

$S(2): 4^2 > 3^2 + 2^{2-1}$

$\Rightarrow 16 > 9+2 \Rightarrow 16 > 11$

So, base case is satisfied.

Induction of hypothesis:

Suppose that statement is true for $n = k \in N \geq 2$

$S(k): 4^k > 3^k + 2^{k-1}$ (i)

Now we prove it for $n = k+1$

Multiplying eq (i) by 4 on both sides

$4.4^k > 4.(3^k+2^{k-1})$

$\Rightarrow 4^{k+1} > 4.3^k + 4.2^{k-1}$

$\Rightarrow 4^{k+1} > (3+1).3^k + 2^2.2^{k-1}$

$\Rightarrow 4^{k+1} > 3.3^k + 3^k + 2^{k-1+2}$

$\Rightarrow 4^{k+1} > 3^{k+1} + 3^k + 2^{k-1} > 3^{k+1} + 2^{k+1}$

$\Rightarrow 4^{k+1} > 3^{k+1} + 2^{k+1} > 3^{k+1} + 2^k$

$\Rightarrow 4^{\overline{k+1}} > 3^{\overline{k+1}} + 2^{\overline{k+1}-1}$

Which is true for $n = k+1$

Hence given statement is true for all positive value of $n \geq 2$.

7. $1 + nx \leq (1+x)^n$ **for** $n \geq 2$ **and** $n > -1$.

Sol: Let $S(n)$ be the give statement

$S(n): 1 + nx \leq (1+x)^n$, $n \geq 2$ and $n > -1$

Base Case: For $n = 2$

$S(2): 1 + 2x \leq (1+x)^2$

$\Rightarrow \quad 1 + 2x \leq 1 + 2x + x^2$

So, base case is defined.

Induction of Hypothesis:

Suppose that statement is true for $k \in N \geq 2$

$S(k): 1 + kx \leq (1+x)^k \quad (i)$

Multiplying eq (i) by $(1+x)$ on both sides

$(1+x)(1+x)^k \geq (1+x)(1+kx)$

$\Rightarrow \quad (1+x)^{k+1} \geq 1 + kx + x + kx^2$

$\Rightarrow \quad (1+x)^{k+1} \geq 1 + (k+1)x + kx^2$

$\Rightarrow \quad (1+x)^{k+1} \geq 1 + (k+1)x \quad \therefore kx^2 > 0$

or $\quad 1 + (k+1)x \leq (1+x)^{k+1}$

Which is true for $n = k+1$

Hence given statement is true for all positive values of $n \geq 2$

8. **Aliza invests Rs. 1000000 in a business that promises a 6% return compounded annually. Prove by mathematical induction that the amount of money after n years is** $1000000(1.06)^n$.

Sol: Let $S(n)$ be the give statement

$S(n): A(n) = 1000000(1.06)^n$

Base case: For $n = 1$

$S(1): A(1) = 1000000(1.06)^1$

$A(1) = 1000000 \times 1.06$

$A(1) = 1060000$

So, base case is satisfied.

Induction of Hypothesis:

Suppose that statement is true for $n = k \in N$

$S(k): A(k) = 1000000(1.06)^k \quad (i)$

Now we prove it for $n = k+1$

Multiply by 1.06 on both sides of (i)

$A(k) \times 1.06 = 1000000(1.06)^k \times (1.06)^1$

$A(k+1) = 1000000 \times (1.06)^{k+1}$

Which is true for $n = k+1$

Hence amount of money after n years is $1000000(1.06)^n$

9. **A bank offers an investment with an annual interest rate r. If P rupees are invested, the amount after n years is given by** $A(n) = P(1+r)$, **prove by induction that this formula hold for all** $n \geq 0$.

Sol: Let $S(n)$ be the give statement

$S(n): A(n) = P(1+r)^n$, $n \geq 0$

Base Case: For $n = 0$

$S(0): A(0) = P(1+r)^0$

$A(0) = P(1)$

$A(0) = P$

As no time pass, the amount remains same as initial investment.

Therefore, base case is satisfied.

Induction of hypothesis:

Suppose that statement is true for $n = k \geq 0$

$S(k): A(k) = P(1+r)^k \quad (i)$

Now we prove it for $n = k+1$

Multiply by $(1+r)$ on both sides of (i)

$A(k) \times (1+r) = P(1+r)^k \times (1+r)^1$

$\Rightarrow \quad A(k+1) = P(1+r)^{k+1}$

Which is true for $n = k+1$

Hence given formula $A(n) = P(1+r)^n$ holds for all $n \geq 0$.

10. **Shikander saves Rs. 500 in the first month and increases his saving by Rs. 500 every subsequent month. Using mathematical induction, determine whether his total savings will reach at least Rs. 12000 after 24 months.**

Sol: According to given condition

$S(n): 500 + 500(2) + 500(3) + \ldots + 500(n)$

$= \frac{n}{2}(500 + 500n)$

Base case for $n=1$

$$S(1): 500(1)=\frac{1}{2}(500+500(1))$$

$$\Rightarrow 500=\frac{1}{2}(1000)\Rightarrow 500=500$$

Base case is satisfied

Suppose the statement is true $n=k\in N$

$$S(k): 500+500(2)+500(3)+...+500(k)$$

$$=\frac{k}{2}(500+500k)\_\_\_\_(i)$$

For $n=k+1$

$$S(k+1): 500+500(2)+...+500(3)+...+500(k+1)$$

$$=\frac{(k+1)}{2}[500+500(k+1)] \qquad (ii)$$

Add $500(k+1)$ *both rides of* (i)

$$500+500(2)+500(3)+...+500k+500(k+1)$$

$$=\frac{k}{2}(500+500k)+500(k+1)$$

$$=\frac{k}{2}(500+500k)+500k+500$$

$$=\frac{500(k+k^2+2k+2)}{2}$$

$$=\frac{500}{2}[k(k+1)+2(k+1)]$$

$$=\frac{500}{2}[(k+1)(k+2)]$$

$$=500\frac{(k+1)}{2}[k+1+1]$$

$$=\frac{k+1}{2}[500+500(k+1)] \quad \textit{which is } (ii)$$

It is true for $k=n+1$ hence proved

11. Prove by mathematical induction that if Ali takes a loan of Rs. 2000000 and pay Rs. 50000 at the end of each year, the remaining balance after n year is $R_n=2000000-50000n$.

Sol: Given statement

$$R_n=2000000-50000n$$

Base Case: For $n=1$

$$R_1=2000000-50000\times 1$$

$$R_1=2000000-50000$$

$$R_1=1950000$$

So, base case is satisfied.

Induction of Hypothesis:

Suppose that statement is true for $n=k\in N$

i.e $R_k=2000000-50000k \qquad (i)$

Now we prove it for $n=k+1$

Subtracting 50000 on both sides, we get

$$R_k-50000=2000000-50000k-50000$$

As 50000 is reduces every (each) year

$$\therefore R_k-50000=R_{k+1}$$

So, $R_{\overline{k+1}}=2000000-50000(\overline{k+1})$

Which is true for $n=k+1$

Hence given statement is true for all $n\in N$.

12. If Salman starts saving with Rs. 5000 and saves an additional Rs. 1000 at the end of every month, derive a formula $S(n)$ for his total saving after n months. Prove the correctness of year formula using mathematical induction.

Sol: According to given statement

$$S(n)=5000+1000(n-1)$$

Base case: For $n=1$

$$S(1)=5000+1000(1-1)$$

$$S(1)=5000+1000(0)$$

$$S(1)=5000+0$$

$$S(1)=5000$$

So, base case is satisfied.

Induction of Hypothesis:

Suppose that statement is true for $n=k\in N$

i.e $S(k)=5000+1000(k-1) \qquad (i)$

Now we prove it for $n=k-1$

Adding 1000 on both sides of (i)

$$S(k)+1000=5000+1000(k-1)+1000$$

$$S(k+1)=5000+1000[k-1+1]$$

or

$$S(\overline{k+1})=5000+1000[\overline{k-1}+1]$$

Which is true for $n=k+1$

Hence given statement is true for all $n\in N$