Exercise 8.3

1. Expand the following up to 4 terms, taking the values of x such that expansion in each case is valid:

i) $(1+x)^{-1/3}$

Sol: Given $(1+x)^{-\frac{1}{3}}$

We know that $\because (1+x)^n = 1+nx+\dfrac{n(n-1)x^2}{2!}+\dfrac{n(n-1)(n-2)x^3}{3!}+\cdots$

put $n=-1/3$ we get

$$\therefore (1+x)^{-\frac{1}{3}} = 1+\left(-\frac{1}{3}\right)x+\frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)x^2}{2!}+\frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)\left(-\frac{1}{3}-2\right)x^3}{3!}+\cdots$$

$$=1-\frac{1}{3}x+\frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)x^2}{2!}+\frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)x^3}{6}+\cdots$$

$$=1-\frac{1}{3}x+\left(\frac{4}{9}\right)\left(\frac{1}{2}\right)x^2-\left(\frac{28}{27}\right)\left(\frac{1}{6}\right)x^3+\cdots = 1-\frac{1}{3}x+\frac{2}{9}x^2-\frac{14}{81}x^3+\cdots$$

The above series is valid if $|x|<1$

ii) $(4-3x)^{1/2}$

Sol: Given: $(4-3x)^{\frac{1}{2}}$

$$(4-3x)^{\frac{1}{2}} = \left[4\left(1-\frac{3}{4}x\right)\right]^{\frac{1}{2}} = 4^{\frac{1}{2}}\left(1-\frac{3}{4}x\right)^{\frac{1}{2}} = 2^{2\times\frac{1}{2}}\left[1+\left(-\frac{3}{4}x\right)\right]^{\frac{1}{2}} = 2\left[1+\left(-\frac{3}{4}x\right)\right]^{\frac{1}{2}}$$

We know that $\because$ $(1+x)^n = 1+nx+\dfrac{n(n-1)x^2}{2!}+\dfrac{n(n-1)(n-2)x^3}{3!}+\cdots$

put $x=-\frac{3}{4}x$ and $n=\frac{1}{2}$ we get

$$\because \quad 2\left[1+\left(-\frac{3}{4}x\right)\right]^{\frac{1}{2}} = 2\left[1+\left(\frac{1}{2}\right)\left(-\frac{3}{4}x\right)+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(-\frac{3}{4}x\right)^2}{2!}+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(-\frac{3}{4}x\right)^3}{3!}+\cdots\right]$$

$$=2\left[1-\frac{3}{8}x+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{9}{16}x^2\right)}{2}+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{27}{64}x^3\right)}{6}+\cdots\right]$$

$$=2\left[1-\frac{3}{8}x-\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)\left(\frac{9}{16}x^2\right)-\left(\frac{3}{8}\right)\left(\frac{1}{6}\right)\left(\frac{27}{64}x^3\right)+\cdots\right]$$

$$=2\left[1-\frac{3}{8}x-\frac{9}{128}x^2-\frac{27}{1024}x^3+\cdots\right] = 2-\frac{3}{4}x-\frac{9}{64}x^2-\frac{27}{512}x^3+\cdots$$

The above series is valid if

$$\left|\frac{3}{4}x\right|<1 \quad\Rightarrow\quad |x|<\frac{4}{3}$$

iii) $\dfrac{(1-x)^{-1}}{(1+x)^2}$

Sol: Given: $\dfrac{(1-x)^{-1}}{(1+x)^2}=(1-x)^{-1}(1+x)^{-2}$

We know that $\because (1+x)^n=1+nx+\dfrac{n(n-1)x^2}{2!}+\dfrac{n(n-1)(n-2)x^3}{3!}+\cdots$

$$(1-x)^{-1}(1+x)^{-2}=\left[1+(-1)(-x)+\frac{(-1)(-1-1)(-x)^2}{2!}+\frac{(-1)(-1-1)(-1-2)(-x)^3}{3!}+\cdots\right]$$
$$\times\left[1+(-2)(x)+\frac{(-2)(-2-1)x^2}{2!}+\frac{(-2)(-2-1)(-2-2)x^3}{3!}+\cdots\right]$$
$$=\left[1+x+\frac{(-1)(-2)x^2}{2}+\frac{(-1)(-2)(-3)(-x)^3}{6}\right]\times\left[1-2x+\frac{(-2)(-3)x^2}{2}+\frac{(-2)(-3)(-4)x^3}{6}+\cdots\right]$$
$$=\left[1+x+\frac{2x^2}{2}+\frac{6x^3}{6}+\cdots\right]\times\left[1-2x+\frac{6x^2}{2}-\frac{24x^3}{6}+\cdots\right]=\left(1+x+x^2+x^3+\cdots\right)\left(1-2x+3x^2-4x^3+\cdots\right)$$
$$=1-x+2x^2+3x^2+4x^3+x-2x^2+3x^3+x^2-2x^3+x^3+\cdots=1-x+2x^2-2x^3+\cdots$$

The above series is valid if $|x|<1$

iv) $\dfrac{\sqrt{1+2x}}{1-x}$

Sol: Given: $\dfrac{\sqrt{1+2x}}{1-x}=(1+2x)^{\frac{1}{2}}(1-x)^{-1}$

We know that

$$\because (1+x)^n=1+nx+\frac{n(n-1)x^2}{2!}+\frac{n(n-1)(n-2)x^3}{3!}+\cdots$$

$$\therefore\quad (1+2x)^{\frac{1}{2}}(1-x)^{-1}=\left[1+\left(\frac{1}{2}\right)(2x)+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)(2x)^2}{2!}+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{2}-2\right)(2x)^3}{3!}+\cdots\right]$$
$$\times\left[1+(-1)(-x)+\frac{(-1)(-1-1)(-x)^2}{2!}+\frac{(-1)(-1-1)(-1-2)(-x)^3}{3!}+\cdots\right]$$
$$=\left[1+\frac{2x}{2}+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)(4x^2)}{2}+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(8x^3)}{6}+\cdots\right]\times\left[1+x+\frac{(-1)(-2)x^2}{2}+\frac{(-1)(-2)(-3)(-x^3)}{6}+\cdots\right]$$
$$=\left[1+x-\left(\frac{4}{4}\right)\left(\frac{1}{2}\right)x^2+\left(\frac{24}{8}\right)x^3\left(\frac{1}{6}\right)+\cdots\right]\times\left[1+x+\frac{2x^2}{2}+\frac{6x^3}{6}+\cdots\right]$$
$$=\left(1+x-\frac{1}{2}x^2+\frac{1}{2}x^3+\cdots\right)\left(1+x+x^2+x^3+\cdots\right)=1+x+x^2+x^3+x+x^3-\frac{1}{2}x^2-\frac{1}{2}x^3+\frac{1}{2}x^3+\cdots$$
$$=1+2x+\left(2-\frac{1}{2}\right)x^2+2x^3+\cdots=1+2x+\frac{3}{2}x^2+2x^3+\cdots$$

The above series is valid if $|x|<\dfrac{1}{2}$

2. Find the coefficient of x^n in the expansion of:

i) $\dfrac{1+x^2}{(1+x)^2}$

Sol: Given: $\dfrac{1+x^2}{(1+x)^2}=(1+x^2)(1+x)^{-2}$

$$\because (1+x)^n = 1+nx+\frac{n(n-1)x^2}{2!}+\frac{n(n-1)(n-2)x^3}{3!}+\cdots$$

$$\therefore (1+x^2)(1+x)^{-2}=(1+x^2)\left[1+(-2)(x)+\frac{(-2)(-2-1)x^2}{2!}+\frac{(-2)(-2-1)(-2-2)x^3}{3!}+\cdots\right]$$

$$=(1+x^2)\left[1-2x+\frac{6x^2}{2}+\frac{(6)(-4)x^3}{3!}+\cdots\right]=(1+x^2)\left[1-2x+3x^2-4x^3+\cdots\right]$$

$$=(1+x^2)\left[1-2x+3x^2-4x^3+\cdots+(-1)^{n-2}(n-1)x^{n-2}+(-1)^{n-1}nx^{n-1}+(-1)^n(n+1)x^n+\cdots\right]$$

$$=1-2x+\dots.+(-1)^n(n+1)x^n+x^2-2x^3+\dots+(-1)^{n-2}(n-1)x^n+\dots$$

Term involving x^n are

$$=(-1)^n(n+1)x^n+(-1)^{n-2}(n-2)x^n=(-1)^n\left[n+1+(-1)^{-2}(n-1)\right]x^n=(-1)^n[n+1+n-1]x^n=(-1)^n(2n)x^n$$

So, the coefficient of x^n is $(-1)^n\times(2n)$.

ii) $\dfrac{(1+x)^2}{(1-x)^2}$

Sol: Given: $\dfrac{(1+x)^2}{(1-x)^2}=(1+x)^2(1-x)^{-2}$

We know that

$$\because (1+x)^n = 1+nx+\frac{n(n-1)x^2}{2!}+\frac{n(n-1)(n-2)x^3}{3!}+\cdots$$

$$\therefore (1+x)^2(1-x)^{-2}=(1+x)^2\left[1+(-2)(-x)+\frac{(-2)(-2-1)(-x)^2}{2!}+\frac{(-2)(-2-1)(-2-2)(-x)^3}{3!}+\cdots\right]$$

$$=(1+2x+x)\left[1+2x+\frac{6x^2}{2}+\frac{(6)(4x^3)}{6}+\cdots\right]=(1+2x+x^2)\left[1+2x+3x^2+4x^3+\cdots\right]$$

$$=1+2x+\dots+(n+1)x^n+2x+4x^2+\dots+2nx^n+\dots+x^2+2x^3+3x^4+\dots+(n-1)x^n+\dots$$

$$=(1+2x+x^2)\left[1+2x+3x^2+4x^3+\cdots+(n-1)x^{n-2}+nx^{n-1}+(n+1)x^n+\cdots\right]$$

$$=1+2x+\dots+(n+1)x^n+2x+4x^2+\dots+2nx^n+\dots+x^2+2x^3+3x^4+\dots+(n-1)x^n+\dots$$

The term introducing x^n is

$$=(n+1)x^n+2nx^n+(n-1)x^n=\left[n+1+2n+n-1\right]x^n=(4n)x^n$$

So, the coefficient of x^n is $4n$.

3. If x is so small that its square and higher powers can be neglected, then show that:

i) $\frac{1-x}{\sqrt{1+x}} \approx 1-\frac{3}{2}x$

Sol: L.H.S $=\frac{1-x}{\sqrt{1+x}}=(1-x)(1+x)^{-\frac{1}{2}}$

$$=(1-x)\left[1+\left(-\frac{1}{2}\right)x+neglect\right]\approx(1-x)\left[1-\frac{x}{2}\right]$$

$$\approx 1-\frac{x}{2}-x+neglect\,x^2\approx 1-\frac{3}{2}x = \text{R.H.S}$$ Hence Proved.

ii) $\frac{\sqrt{1+2x}}{\sqrt{1-x}}\approx 1+\frac{3}{2}x$

Sol: L.H.S $=\frac{\sqrt{1+2x}}{\sqrt{1-x}}=(1+2x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}$

$$=\left[1+\frac{1}{2}(2x)+neglect\right]\left[1+\left(-\frac{1}{2}\right)(-x)+neglect\right]$$

$$=(1+x)\left(1+\frac{x}{2}\right)$$

$$=1+\frac{x}{2}+x+neglect$$ (Neglecting x^2 we get)

$$\approx 1+\left(\frac{x}{2}+x\right)\approx 1+\left(\frac{x+2x}{2}\right)\approx 1+\frac{3}{2}x=R.H.S$$

Hence Proved.

iii) $\frac{(9+7x)^{1/2}-(16+3x)^{1/4}}{4+5x}\approx\frac{1}{4}-\frac{17}{384}x$

Sol: L.H.S $=\frac{(9+7x)^{1/2}-(16+3x)^{1/4}}{4+5x}$

$$=\left[(9+7x)^{\frac{1}{2}}-(16+3x)^{\frac{1}{4}}\right](4+5x)^{-1}=\left(\left[9\left(1+\frac{7}{9}x\right)\right]^{\frac{1}{2}}-\left[16\left(1+\frac{3}{16}x\right)\right]^{\frac{1}{4}}\right)\left(4\left(1+\frac{5}{4}x\right)\right)^{-1}$$

$$=\left[9^{\frac{1}{2}}\left(1+\frac{7}{9}x\right)^{\frac{1}{2}}-16^{\frac{1}{4}}\left(1+\frac{3}{16}x\right)^{\frac{1}{4}}\right]\left[4^{-1}\left(1+\frac{5}{4}x\right)^{-1}\right]$$

$$=3\left(1+\frac{1}{2}\left(\frac{7x}{9}\right)\right)-2\left(1+\frac{1}{4}\left(\frac{3}{16}x\right)\right)\frac{1}{4}\left(1+(-1)\left(\frac{5}{4}x\right)\right)(neglect\,x^2)$$

$$=\left[3\left(1+\frac{7}{18}x\right)-2\left(1+\frac{3}{64}x\right)\right]\frac{1}{4}\left(1-\frac{5}{4}x\right)=\frac{1}{4}\left[\left(3+\frac{3\times 7x}{18}\right)-\left(2+\frac{2\times 3x}{64}\right)\right]\left(1-\frac{5}{4}x\right)$$

$$=\frac{1}{4}\left[3+\frac{7}{6}x-2-\frac{3x}{32}\right]\left(1-\frac{5}{4}x\right)=\frac{1}{4}\left[1+\frac{7}{6}x-\frac{3x}{32}\right]\left(1-\frac{5}{4}x\right)$$

$$=\frac{1}{4}\left[1-\frac{5}{4}x+\frac{7}{6}x-\frac{3x}{32}\right]$$ (Neglecting x^2 term)

$$=\frac{1}{4}\left[1-\left(\frac{120x-112x+9}{96}\right)\right]=\frac{1}{4}\left[1-\frac{17}{96}x\right]=\frac{1}{4}x-\frac{17}{384}=R.H.S$$ Hence Proved.

iv) $\dfrac{\sqrt{4+x}}{(1-x)^3}\approx 2+\dfrac{25}{4}x$

Sol: $\text{L.H.S}=\dfrac{\sqrt{4+x}}{(1-x)^3}=2+\dfrac{25}{4}x$

$$=\left[4\left(1+\frac{x}{4}\right)\right]^{\frac{1}{2}}(1-x)^{-3}=4^{\frac{1}{2}}\left(1+\frac{x}{4}\right)^{\frac{1}{2}}(1-x)^{-3}$$

$$=2\left[1+\left(\frac{1}{2}\right)\left(\frac{x}{4}+neglect\right)\right]\left[1+(-3)(-x)+neglect\right]=2\left[1+\frac{x}{8}\right][1+3x]\ (neglect\ x^2)$$

$$=2\left[1+3x+\frac{x}{8}\right] \quad \text{Neglecting } x^2 \text{ term}$$

$$\approx 2\left[1+\left(3x+\frac{x}{8}\right)\right]\approx 2\left[1+\left(\frac{24x+x}{8}\right)\right]\approx 2\left[1+\frac{25x}{8}\right]\approx 2+\frac{2\times 25x}{8}\approx 2+\frac{24x}{8}=R.H.S$$

Hence Proved.

4. If x is so small that is cube and higher power can be neglected, show that:

i) $\sqrt{1-x-2x^2}\approx 1-\dfrac{1}{2}x-\dfrac{9}{8}x^2$

Sol: $\text{L.H.S}=\sqrt{1-x-2x^2}=\left[1-\left(x+2x^2\right)\right]^{\frac{1}{2}}$

$$=1-\frac{1}{2}\left(x+2x^2\right)+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(x+2x^2\right)^2}{2!} \quad \text{(Neglecting } x^3 \text{ and higher powers of } x\text{)}$$

$$=1-\frac{1}{2}\left(x+2x^2\right)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(x^2+4x^4+4x^3\right)=1-\frac{1}{2}x-\frac{2x^2}{2}-\frac{1}{4\times 2}\left(x^2\right)$$

(Neglecting x^3 term & higher powers of x)

$$\approx 1-\frac{1}{2}x-x^2+\frac{x^2}{8}\approx 1-\frac{1}{2}x-\left(x^2+\frac{x^2}{8}\right)\approx 1-\frac{1}{2}x-\left(\frac{8x^2+x^2}{8}\right)\approx 1+\frac{1}{2}x+\frac{9}{8}x^2=R.H.S$$

Hence proved.

ii) $\sqrt{\dfrac{1+x}{1-x}}\approx 1+x+\dfrac{1}{2}x^2$

Sol: $\text{L.H.S}=\sqrt{\dfrac{1+x}{1-x}}=(1+x)^{\frac{1}{2}}(1+x)^{-\frac{1}{2}}$

$$=\left[1+\frac{1}{2}(x)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)x^2}{2!}+neglect\ x^3\right]\left[1+\left(-\frac{1}{2}\right)(-x)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)x^2}{2!}+neglect\ x^3\right]$$

$$=\left[1+\frac{x}{2}+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)x^2}{2}\right]\left[1+\frac{1}{2}x+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^2}{2}\right]$$

$$=\left(1+\frac{x}{2}-\frac{1}{4\times 2}x^2\right)\left(1+\frac{x}{2}+\frac{1}{4\times 2}x^2\right)=1+\frac{x}{2}+\frac{3x^2}{8}+\frac{x}{2}+\frac{x^2}{4}-\frac{x^2}{8}$$ (Neglecting x^3 and higher power)

$$=1+\frac{x}{2}+\frac{x}{2}+\frac{3x^2}{8}+\frac{x^2}{4}-\frac{x^2}{8}=1+\frac{x+x}{2}+\frac{3x^2}{8}+\frac{x^2}{4}-\frac{x^2}{8}$$

$$=1+\frac{2x}{2}+\frac{4x^2}{8}=1+x+\frac{x^2}{2}=R.H.S$$

Hence proved

5. If x is very nearly equal 1, then prove that $px^p-qx^q=(p-q)x^{p+q}$.

Sol: Given $px^p-qx^q=(p-q)x^{p+q}$

Given x is very nearly equal to 1

Therefore, we consider $x=1+h$ $\quad(i)$

Where h is so small its square and higher powers can be neglected.

Now,

L.H.S $=px^p-qx^q$

Putting from (i), we get

$$=p(1+h)^p-q(1+h)^q$$

Using $(1+x)^n=1+nx$, neglecting x^2 and higher powers of x, we get

$$=p(1+ph)-q(1+qh)$$

$$=p+p^2h-q-q^2h$$

$$=(p-q)+p^2h-q^2h$$

$$=(p-q)+h\left(p^2-q^2\right)$$

$$=(p-q)+h(p-q)(p+q)$$

Taking $p-q$ common, we get

$$=(p-q)\left[1+h(p+q)\right] \quad (i)$$

$R.H.S=(p-q)x^{p+q}$

Put $x=1+h$

$$=(p-q)[1+h]^{p+q}$$

$$=(p-q)\left[1+(p+q)(h)+neglect\right]$$

$$=(p-q)\left[1+(p+q)h\right] \quad (ii)$$

From i and ii

L.H.S = R.H.S Hence proved

6. Identify the following series as binomial expansion and find the sum.

$$1-\frac{1}{2}\left(\frac{1}{4}\right)+\frac{1.3}{2!4}\left(\frac{1}{4}\right)^2-\frac{1.3.5}{3!8}\left(\frac{1}{4}\right)^3+\ldots$$

Sol: Let $(1+x)^n=1-\frac{1}{8}+\frac{3}{8}\left(\frac{1}{16}\right)-\frac{15}{6\times 8}\left(\frac{1}{64}\right)+\ldots \quad (i)$

$$\therefore (1+x)^n=1+nx+\frac{n(n-1)x^2}{2!}+\frac{n(n-1)(n-2)x^3}{3!}+\ldots \quad (ii)$$

Comparing (i) & (ii), R.H.S term by term, we get

$$nx=-\frac{1}{8}\ \_\_\_\_(iii)\ \&\ \frac{n(n-1)}{2!}x^2=\frac{3}{8}\left(\frac{1}{16}\right)\Rightarrow n(n-1)x^2=\frac{3}{64}\ \_\_\_\_(iv)$$

Squaring eq. (iii)

$$n^2x^2=\frac{1}{64}\ \_\_\_\_(v)$$

divide iv by v

$$\frac{n(n-1)x^2}{n^2x^2}=\frac{3/64}{1/64}\Rightarrow\frac{n-1}{n}=3$$

$$\Rightarrow n-1=3n\Rightarrow -1=2n\Rightarrow n=-1/2$$

Put in (iii)

$$\left(-\frac{1}{2}\right)x=-\frac{1}{8}\ \Rightarrow x=\frac{1}{4}$$

$$(1+x)^n=\left(1+\frac{1}{4}\right)^{-\frac{1}{2}}=\left(\frac{4+1}{2}\right)^{-\frac{1}{2}}=\left(\frac{5}{4}\right)^{-\frac{1}{2}}=\left(\frac{4}{5}\right)^{\frac{1}{2}}=\frac{4^{\frac{1}{2}}}{5^{\frac{1}{2}}}=\frac{2}{\sqrt{5}}$$ Which is required.

7. Use binomial theorem to show that $1+\frac{1}{4}+\frac{1.3}{4.8}+\frac{1.3.5}{4.8.12}+\ldots=\sqrt{2}$

Sol: Given: $(1+x)^n=1+\frac{1}{4}+\frac{3}{32}+\frac{1.3.5}{4.8.12}+\ldots$ (i)

$$\therefore(1+x)^n=1+nx+\frac{n(n-1)x^2}{2!}+\ldots\quad(ii)$$

Comparing (i) & (ii), R.H.S term by term, we get

$$nx=\frac{1}{4}\ \_\_\_\_(iii)\ \ and\ \ \frac{n(n-1)}{2!}x^2=\frac{3}{32}\Rightarrow n(n-1)x^2=\frac{3}{16}\ \_\_\_\_(iv)$$

By squaring (iii)

$$n^2x^2=\frac{1}{16}\ \_\_\_\_(v)$$

Divide (iv) *by* (v)

$$\frac{n(n-1)x^2}{n^2x^2}=\frac{3/116}{1/116}\Rightarrow\frac{n-1}{n}=3$$

$$\Rightarrow n-1=3n\Rightarrow -1=2n\Rightarrow n=-1/2$$

Put in (iii)

$$\left(-\frac{1}{2}\right)x=-\frac{1}{4}\ \Rightarrow x=-\frac{1}{2}$$

Now, $(1+x)^n=\left(1-\frac{1}{2}\right)^{-1/2}=\left(\frac{1}{2}\right)^{-1/2}=(2)^{1/2}=\sqrt{2}$

Hence proved.

8. If $y=\frac{1}{3}+\frac{1.3}{2!}\left(\frac{1}{3}\right)^2+\frac{1.3.5}{3!}\left(\frac{1}{3}\right)^3+\ldots$ prove that $y^2+2y-2=0$.

Sol; Given: $y=\frac{1}{3}+\frac{1.3}{2!}\left(\frac{1}{3}\right)^2+\frac{1.3.5}{3!}\left(\frac{1}{3}\right)^3+\ldots$

Adding 1 on both sides, we get

$$1+y=1+\frac{1}{3}+\frac{1.3}{2!}\left(\frac{1}{3}\right)^2+\frac{1.3.5}{3!}\left(\frac{1}{3}\right)^3+\dots \qquad (i)$$

$$\therefore (1+x)^n=1+nx+\frac{n(n-1)x^2}{2!}+\dots \qquad (ii)$$

Comparing (i) & (ii), R.H.S term by term, we get

$$nx=\frac{1}{3} \qquad (iii) \text{ and } \frac{n(n-1)x^2}{2!}=\frac{1.3}{2!}.\left(\frac{1}{3}\right)^2 \Rightarrow n(n-1)x^2=\frac{3}{9} \Rightarrow n(n-1)x^2=\frac{1}{3} \qquad (iv)$$

Squaring eq. (iii)

$$n^2x^2=\frac{1}{9} \qquad (v)$$

Dividing (iv) by (v), we get

$$\frac{n(n-1)x^2}{n^2x^2}=\frac{\frac{1}{3}}{\frac{1}{9}} \Rightarrow \frac{n-1}{n}=\frac{1}{3}\times\frac{9}{1} \Rightarrow \frac{n-1}{n}=3$$

$$n-1=3n \Rightarrow 2n=-1 \Rightarrow n=-\frac{1}{2}$$

Put value of n in (iii)

$$x\left(\frac{-1}{2}\right)=\frac{1}{3} \Rightarrow x=\frac{1}{3}\times\frac{2}{-1}=-\frac{2}{3} \Rightarrow x=\frac{-2}{3}$$

Now comparing L.H.S of (i) & (ii), we get $1+y=(1+x)^n$ put the of n & x

$$1+y=\left(1+\left(\frac{-2}{3}\right)\right)^{-\frac{1}{2}}=\left(1-\frac{2}{3}\right)^{-\frac{1}{2}}=\left(\frac{1}{3}\right)^{-\frac{1}{2}} \Rightarrow 1+y=(3)^{\frac{1}{2}}$$

Taking square on both sides, we get

$$(1+y)^2=(3)^{\frac{1}{2}\times 2}$$

$$1+y^2+2y=3$$

$$\Rightarrow \quad 1+y^2+2y-3=0$$

$$\Rightarrow \quad y^2+2y-2=0$$ Hence Proved.

9. **If $2y=\frac{1}{2^2}+\frac{1.3}{2!}.\frac{1}{2^4}+\frac{1.3.5}{3!}.\frac{1}{2^6}+\dots$, prove that $4y^2+4y-1=0$.**

Sol: Given

$$2y=\frac{1}{2^2}+\frac{1.3}{2!}.\frac{1}{2^4}+\frac{1.3.5}{3!}.\frac{1}{2^6}+\dots,$$

Adding 1 on both sides, we get

$$1+2y=1+\frac{1}{2^2}+\frac{1.3}{2!}.\frac{1}{2^4}+\frac{1.3.5}{3!}.\frac{1}{2^6}+\dots, \qquad (i)$$

$$\therefore (1+x)^n 1+nx+\frac{n(n-1)x^2}{2!}+\frac{n(n-1)(n-2)x^3}{3!}+\dots \qquad (ii)$$

Comparing (i) & (ii), R.H.S term by term, we get

$$nx=\frac{1}{2^2}=\frac{1}{4} \qquad (iii) \quad \text{and} \quad \frac{n(n-1)x^2}{2!}=\frac{1.3}{2!}.\frac{1}{2^4} \Rightarrow n(n-1)x^2=\frac{3}{16} \qquad (iv)$$

Squaring eq. (iii) $n^2x^2=\frac{1}{16} \qquad (v)$

Dividing (iv) by (v), we get

$$\frac{n(n-1)x^2}{n^2x^2}=\frac{\frac{3}{16}}{\frac{1}{16}}\Rightarrow\frac{n-1}{n}=\frac{3}{16}\times\frac{16}{1}=3$$

$$\Rightarrow n-1=3n\Rightarrow 2n=-1\Rightarrow n=\frac{-1}{2}$$

$$\Rightarrow\frac{1}{4}-\frac{3}{4}=x\Rightarrow\frac{-2}{4}=x$$

$\Rightarrow n=\frac{-1}{2}$ put in (iii), we get

$$x\left(\frac{-1}{2}\right)=\frac{1}{4}\Rightarrow x=-2\times\frac{1}{4}\Rightarrow x=\frac{-1}{2}$$

Now comparing L.H.S of (i) & (ii), we get $1+2y=(1+x)^n$ put the of n & x

$$1+2y=\left(1-\frac{1}{2}\right)^{-\frac{1}{2}}\Rightarrow 1+2y=\left(\frac{1}{2}\right)^{-\frac{1}{2}}\Rightarrow 1+2y=(2)^{\frac{1}{2}}$$

Taking square on both sides, we get

$$(1+2y)^2=(2)^{\frac{1}{2}\times 2}$$

$$1+4y^2+4y=2$$

$$\Rightarrow 1+4y^2+4y-2=0$$

$$\Rightarrow 4y^2+4y-1=0$$ Hence Proved.

Q.10. Show that the coefficient of x^r in $\frac{x}{(1-px)(1-qx)}$ is $\frac{p^r-q^r}{p-q}$.

Sol: Let $\frac{x}{(1-px)(1-qx)}=\frac{A}{1-px}+\frac{B}{1-qx}$ ______ (i)

$$\Rightarrow x=A(1-qx)+B(1-px)\text{ ______ (ii)}$$

Put $1-px=0\Rightarrow px=1\Rightarrow x=\frac{1}{p}$ in eq. (ii)

$$\frac{1}{p}=A\left(1-q\left(\frac{1}{p}\right)\right)+0\Rightarrow\frac{1}{p}=A\left(1-\frac{q}{p}\right)=A\left(\frac{p-q}{p}\right)\Rightarrow A=\frac{1}{p-q}$$

Put $1-qx=0\Rightarrow qx=1\Rightarrow x=\frac{1}{q}$ in eq. (ii)

$$\frac{1}{q}=0+B\left(1-p\left(\frac{1}{q}\right)\right)+0\Rightarrow\frac{1}{q}=B\left(1-\frac{p}{q}\right)=B\left(\frac{q-p}{q}\right)\Rightarrow B=\frac{1}{q-p}\Rightarrow B=\frac{-1}{p-q}$$

Put values in (i)

$$\frac{x}{(1-px)(1-qx)}=\frac{1}{p-q}\left(\frac{1}{1-px}-\frac{1}{1-qx}\right)=\frac{1}{p-q}\left(\sum_{n=0}^{\infty}p^nx^n-\sum_{n=0}^{\infty}p^nx^n\right)$$

Coefficient of x^r in $\sum_{n=0}^{\infty}q^nx^n$ is q^r

$$\text{Coefficient of } x^r=\frac{1}{p-q}(p^r-q^r)=\frac{p^r-q^r}{p-q}$$