1. using binomial theorem find the value of the following to three places of decimals:

i) $\sqrt{99}$

Sol: $\sqrt{99}=(99)^{\frac{1}{2}}=(100-1)^{\frac{1}{2}}=100^{\frac{1}{2}}\left(1-\frac{1}{100}\right)^{\frac{1}{2}}$

$$=10\left[1+\frac{1}{2}(-0.01)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(-0.01)^2+\ldots\right]$$

$$=10\left[1+0.005+\frac{0.5(-0.5)}{2}(-0.0001)+\ldots\right]$$

$$=10[1+0.005+0.000125+\ldots]=10(0.9499)=9.499\,approx.$$

ii) $(1.03)^{\frac{1}{3}}$

Sol: $(1.03)^{\frac{1}{3}}=(1+0.03)^{\frac{1}{3}}=1+\frac{1}{3}(0.03)+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}(0.03)^2+\ldots$

$$=1+0.01+\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\left(\frac{-2}{3}\right)(0.0009)+\ldots$$

$$=1+0.01-0.0001+\ldots=1.010(approx)$$

iii) $\frac{1}{\sqrt[5]{252}}$

Sol: $\frac{1}{\sqrt[5]{252}}=\frac{1}{(252)^{1/5}}=(252)^{-1/5}=(243+9)^{-1/5}$

$$=(243)^{-1/5}\left(1+\frac{9}{243}\right)^{-1/5}=\frac{1}{(243)^{1/5}}(1+0.037)^{-1/5}$$

$$=\frac{1}{3^{5\times\frac{1}{5}}}\left[1+\left(-\frac{1}{5}\right)(0.037)+\frac{-\frac{1}{5}\left(-\frac{1}{5}-1\right)}{2!}(0.037)^2+\ldots\right]$$

$$=\frac{1}{3}(1-0.0074+0.000167+\ldots)=0.331(approx)$$

iv) $\frac{\sqrt{7}}{\sqrt{8}}$

Sol: $\frac{\sqrt{7}}{\sqrt{8}}=\left(\frac{7}{8}\right)^{\frac{1}{2}}=\left(1-\frac{1}{8}\right)^{\frac{1}{2}}+\ldots=1+\frac{1}{2}\left(\frac{-1}{8}\right)-\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!}\left(\frac{-1}{8}\right)^2+\ldots$

$$=1-0.0625+\frac{1}{2}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{64}\right)+\ldots$$

$$=1-0.0625-0.001953=0.935(Approx)$$

2. Find the remainder when 8^{100} is divided by 7.

Sol: $8 \equiv 1(\text{mod} 7)$

$8^{100} \equiv 1^{100}(\text{mod} 7)$

$8^{100} \equiv 1(\text{mod} 7)$

Remainder is 1.

2nd Method

$8^{100} = (7+1)^{100}$

$= \binom{100}{0} 7^{100}\, 1^0 + \binom{100}{1} 7^{99}\, 1^1 + \binom{100}{2} 7^{98}\, 1^2 + \ldots\ldots + \binom{100}{99} 7^1\, 1^{99} + \binom{100}{100} 7^0\, 1^{100}$

$= 7^{100} + \binom{100}{1} 7^{99} + \binom{100}{2} 7^{98} + \ldots. + \binom{100}{99} 7 + 1$

$= 7k + 1$ (Where k is an integer resulting from the terms with a factor of 7)

Hence remainder is 1.

3. Find the remainder when 2^{100} is divided by 3.

Sol: $2 \equiv -1(\text{mod} 3)$

$2^{100} \equiv -1(\text{mod} 3)$

Remainder is 1

2nd Method

$2^{100} = (3-1)^{100}$

$= \binom{100}{0} 3^{100} (-1)^0 + \binom{100}{1} 3^{99} (-1)^1 + \binom{100}{2} 3^{98} (-1)^2 + \ldots + \binom{100}{99} 3^1 (-1)^{99} + \binom{100}{100} 3^0 (-1)^{100}$

$= 3^{100} - \binom{100}{1} 3^{99} + \binom{100}{2} 3^{98} + \ldots. - \binom{100}{99} 3 + 1$

$= 3m + 1$ (where m is an integer resulting from the terms with factor 3)

Hence remainder is 1.

4. Using the binomial theorem, find which number is larger:

i) $19^{10} + 20^{10}$ *or* 21^{10}

ii) $29^{15} + 30^{15}$ *or* 31^{15}

Sol

i) $19^{10} + 20^{10}$ *or* 21^{10}

Consider $\left(\frac{19}{21}\right)^{10} + \left(\frac{20}{21}\right)^{10} > 1$

$\left(1 - \frac{2}{21}\right)^{10} + \left(1 - \frac{1}{21}\right)^{10} > 1$

Use binomial expansion $(1-x)^n \approx 1 - nx$ for small x:

$\left(1 - \frac{20}{21}\right) + \left(1 - \frac{10}{21}\right) > 1$

$1 + 1 \frac{-20}{21} - \frac{10}{21} > 1$

$2-\frac{30}{21}>1$

$2-1.43>1$

$0.57 \not> 1$

More precisely:

$$21^{10}-20^{10}=(20+1)^{10}-20^{10}=\sum_{k=1}^{10}\binom{10}{k}20^{10-k}$$

$$19^{10}=(20-1)^{10}=20^{10}+\sum_{k=1}^{10}\binom{10}{k}20^{10-k}(-1)^k$$

$$21^{10}-19^{10}=2\left(\binom{10}{1}20^9+\binom{10}{3}20^7+\ldots+\binom{10}{9}20^1\right)$$

$$21^{10}-20^{10}>19^{10}$$

$$21^{10}>19^{10}+20^{10}$$

2nd Method Q. 4 (i)

Sol: We want to compare 20^{10} *and* $21^{10}-19^{10}$

Now, $21^{10}-19^{10}=(20+1)^{10}-(20-1)^{10}$

$$=\binom{10}{0}20^{10}\,1^0+\binom{10}{1}20^9\,1^1+\binom{10}{2}20^8\,1^2+\ldots+\binom{10}{9}20^1\,1^9+\binom{10}{10}20^0\,1^{10}$$

$$-\left[\binom{10}{0}20^{10}(-1)^0+\binom{10}{1}20^9(-1)^1+\binom{10}{2}20^8(-1)^2+\ldots+\binom{10}{9}20^1(-1)^9+\binom{10}{10}20^0(-1)^{10}\right]$$

$$=\cancel{20^{10}}+\binom{10}{1}20^9+\cancel{\binom{10}{2}20^8}+\ldots+\binom{10}{9}20+\cancel{1}-\cancel{20^{10}}+\binom{10}{1}20^9-\cancel{\binom{10}{2}20^8}+\binom{10}{3}20^7+\ldots+\binom{10}{9}20-\cancel{1}$$

$$=2\left[\binom{10}{1}20^9+\binom{10}{3}20^7+\binom{10}{5}20^5+\binom{10}{7}20^3+\binom{10}{9}20\right]$$

$$(20)20^9+(240)20^7+(504)20^5+(240)20^3+(20)20=20^{10}+\textit{positive term}$$

$$\Rightarrow\ 21^{10}-19^{10}>20^{10}\ \textit{or}\ 21^{10}>19^{10}+20^{10}$$

Hence 21^{10} is larger.

ii) $\mathbf{29^{15}+30^{15}}$ ***or*** $\mathbf{31^{15}}$

Consider $\left(\frac{29}{31}\right)^{15}+\left(\frac{30}{31}\right)^{15}>1$

$$\left(1-\frac{2}{31}\right)^{15}+\left(1-\frac{1}{31}\right)^{15}>1$$

Using approximation:

$$\left(1-\frac{30}{31}\right)+\left(1-\frac{15}{31}\right)^{10}>1$$

$2-\frac{45}{31}>1$

$2-1.45>1$

$0.55 \not> 1$

Similarly,

$$31^{15}-30^{15}=\sum_{k=1}^{15}\binom{15}{k}30^{15-k}$$

$$29^{15}=30^{15}+\sum_{k=1}^{15}\binom{15}{k}30^{15-k}(-1)^k$$

$$31^{15}-29^{15}=2\left(\binom{15}{1}30^{14}+\binom{15}{3}30^{12}+\ldots+\binom{15}{15}30^{0}\right)$$

$$31^{15}-30^{15}>29^{15}$$

$$31^{15}>29^{15}+30^{15}$$

5. Using the binomial theorem, show that:

i) 5^7+7^5 is divisible by 36

Sol: $5^7=(6-1)^7=\sum_{k=0}^{7}\binom{7}{k}6^k(-1)^{7-k}$

$$5^7=\binom{7}{0}6^0(-1)^7+\binom{7}{1}6^1(-1)^6+\binom{7}{2}6^2(-1)^5+\ldots.$$

$$5^7=-1+7.6-21.36+\textit{multiplies of } 6^3$$

$$5^7=-1+42-21.36+216.(\ldots)$$

$$5^7=41-21.36+36.(6.(\ldots))$$

$$5^7=41+36(-21+6.(\ldots))$$

$$5^7=41(\text{mod}\,36)$$

$$5^7=5(\text{mod}\,36)$$

$$7^5=(6+1)^5=\sum_{k=0}^{5}\binom{5}{k}6^k(1)^{5-k}$$

$$7^5=\binom{5}{0}6^0(1)^5+\binom{5}{1}6^1(1)^4+\binom{5}{2}6^2(1)^3+\ldots.$$

$$7^5=1+5.6+10.36+\textit{multiplies of } 6^3$$

$$7^5=1+30+10.36+216.(\ldots)$$

$$7^5=31+36(10+6.(\ldots))$$

$$7^5=31(\text{mod}\,36)$$

$$7^5=-5(\text{mod}\,36)$$

$$5^7+7^5\equiv 5+(-5)(\text{mod}\,36)$$

$$5^7+7^5\equiv 0(\text{mod}\,36)$$

Thus, 5^7+7^5 is divisible by 36.

2nd Method Q. 5 (i)

5^7+7^5 is divisible by 36

$$5^7=(6-1)^7$$

$$=\binom{7}{0}6^7(-1)^0+\binom{7}{1}6^6(-1)^1+\binom{7}{2}6^5(-1)^2+\ldots+\binom{7}{6}6^1(-1)^6+\binom{7}{7}6^0(-1)^7$$

$$6^2\left[6^5+\binom{7}{1}6^4(-1)^1+\binom{7}{2}6^3(-1)^2+\binom{7}{3}6^2(-1)^3+\binom{7}{4}6(-1)^4+\binom{7}{5}(-1)^5\right]7\times6-1$$
$$=36k_1+42-1=36k_1+41$$
$$=36k_1+36+5=36(k_1+1)+5 \qquad (i)$$
Again $7^5=(6+1)^5=\binom{5}{0}6^5 1^0+\binom{5}{1}6^4 1^1+\binom{5}{2}6^3 1^2+\ldots+\binom{5}{4}6^1 1^6+\binom{5}{7}6^0 1^7$
$$6^2\left[6^3+\binom{5}{1}6^2+\binom{5}{2}6^1+\binom{5}{3}\right]+5\times6+1=36k_2+31$$
$$=36k_2+36-5=36(k_2+1)-5 \qquad (ii)$$
Now (i) + (ii)
$$7^5+5^7=36(k_1+1)+\not{5}+36(k_2+1)-\not{5}=36(k_1+1+k_2+1)$$
Hence 7^5+5^7 *is divisible by* 36

ii) $(17)^{15}+(13)^7$ **is divisible by 6**

Sol: $17\equiv-1(\text{mod}\,6)$

$17^{15}\equiv(-1)^{15}(\text{mod}\,6)$

$17^{15}\equiv-1^{15}(\text{mod}\,6)$

$13\equiv1(\text{mod}\,6)$

$13^7\equiv(1)^7(\text{mod}\,6)$

$13^7\equiv1^7(\text{mod}\,6)$

$(17)^{15}+(13)^7\equiv-1+1(\text{mod}\,6)$

$(17)^{15}+(13)^7\equiv-1+1(\text{mod}\,6)$

Thus, $(17)^{15}+(13)^7$ is divisible by 6.

iii) $(21)^9+(19)^{11}$ **is divisible by 20**

Sol: $21=20+1$

$$21^9=(20+1)^9=\sum_{k=0}^{9}\binom{9}{k}20^k(1)^{9-k}=1+9.20+\textit{multiplies of } 20^2$$

$21^9\equiv1+180(\text{mod}\,400)$

$21^9\equiv1+0(\text{mod}\,20)$

$21^9\equiv1(\text{mod}\,20)$

$19=20-1$

$$19^{11}=(20+1)^{11}=\sum_{k=0}^{11}\binom{11}{k}20^k(-1)^{11-k}=-1+11.20+\textit{multiplies of } 20^2$$

$19^{11}\equiv-1+220(\text{mod}\,400)$

$19^{11}\equiv-1+0(\text{mod}\,20)$

$19^{11}\equiv-1(\text{mod}\,20)$

$(21)^9+(19)^{11}\equiv1+(-1)(\text{mod}\,20)$

$(21)^9+(19)^{11}\equiv0(\text{mod}\,20)$

Thus, $(21)^9+(19)^{11}$ is divisible by 20.

iv) $(31)^4+(29)^6$ **is divisible by 30**

Sol: $31=30+1$

$31^4=(30+1)^4=\sum_{k=0}^{4}\binom{4}{k}30^k(1)^{4-k}=1+4.30+multiples\ of\ 30^2$

$31^4\equiv 1+120 \pmod{900}$

$31^4\equiv 1+0 \pmod{900}$

$31^4\equiv 1 \pmod{900}$

$29=30-1$

$29^6=6.30(-1)+multiples\ of\ 30^2$

$29^6=1-180+multiples\ of\ 900$

$29^6=1-0 \pmod{30}$

$29^6=1 \pmod{30}$

v) $(101)^5+(99)^7$ **is divisible by 100**

$101=100+1$

$101^5=(100+1)^5=\sum_{k=0}^{5}\binom{5}{k}100^k(1)^{5-k}=1+5.100+multiples\ of\ 100^2$

$101^5=1+500 \pmod{10000}$

$101^5=1+0 \pmod{100}$

$101^5=1 \pmod{100}$

$99=100-1$

$99^7=(100-1)^7$

$\sum_{k=1}^{7}\binom{7}{k}100^k(-1)^{7-k}=-1+7$

$100(-1)^6+multiplies\ of\ 100^2$

$99^7=-1+700+multiples\ of\ 10000$

$99^7=-1+0 \pmod{100}$

$99^7=-1 \pmod{100}$

$(105)^5+(99)^7\equiv 1+(-1) \pmod{100}$

$(105)^5+(99)^7\equiv 0 \pmod{100}$

Thus, $(105)^5+(99)^7$ is divisible by 100

6. Using the binomial theorem, find the remainder when 3^{101} is divided by 8.

Sol: $3^{101}=3.(3^2)^{50}=3.9^{50}=3.(1+8)^{50}$

$=3\sum_{k=0}^{50}\binom{50}{k}1^{50-k}8^k=3\left(\binom{50}{0}8^0+\binom{50}{1}8^1+\binom{50}{2}8^2+\ldots\right)=3\left(1+50.8+64.\binom{50}{2}+\ldots\right)$

$3^{101}=3(1+0+0+\ldots) \pmod{8}$

$3^{101}=3 \pmod{8}$

2nd Method

Sol: $3^{101} = 3^{100}.3^1 = \left(3^2\right)^{50}.3$

$= 9^{50}.3 = (8+1)^{50}.3$

$= \left[\binom{50}{0}8^{50}1^0 + \binom{50}{1}8^{49}1^1 + \binom{50}{2}8^{48}1^2 + \dots + \binom{50}{49}8^1\ 1^{49} + \binom{50}{50}8^0 1^{50}\right].3$

$= \left[8^{50} + \binom{50}{1}8^{49} + \binom{50}{2}8^{48} + \dots + \binom{50}{49}8 + 1\right].3 = (8k+1).3$

$24k + 3 = 8(3k) + 3$

$\Rightarrow 3^{101} = 8(3k) + 3$

When 3^{101} is divided by 8 then remainder is 3

7. Using the binomial theorem, find the last digit of the number $(32)^{32}$.

Sol: $(32)^{32} = (30+2)^{32} = \sum_{k=0}^{32}\binom{32}{k}(30)^k(2)^{32-k}$

Modulo 10:

$(32)^{32} = \binom{32}{0}(30)^0(2)^{32-k}(\text{mod}10)$

$\sum_{k=0}^{32}\binom{32}{k}(30)^k(2)^{32-k}(\text{mod}10)$

$(32)^{32} = 1.1.2^{32} + 0(\text{mod}10)$

$(32)^{32} = 2^{32} + (\text{mod}10)$

$2^{32} = \left(2^4\right)^8 = 16^8 = (10+6)^8 = \sum_{j=0}^{8}\binom{8}{j}(10)^j(6)^{8-j}$

Modulo 10:

$16^8 = \binom{8}{0}(10)^0(6)^8 + \sum_{j=1}^{8}\binom{8}{j}(10)^j(6)^{8-j}$

$16^8 = 1.1.6^8 + 0(\text{mod}10)$

$16^8 = 6^8 + (\text{mod}10)$

$6^1 \equiv 6(\text{mod}10)$

$6^n \equiv 6(\text{mod}10)$

$6^8 \equiv 6(\text{mod}10)$

Last digit is 6

8. Using the binomial theorem, show that $7^n - 6n$ leaves remainder 1 when divided by 7 for all positive integers n

Sol: $7^n = (1+6)^n = \sum_{k=0}^{n}\binom{n}{k}1^{n-k}6^k = \binom{n}{0}6^0 + \binom{n}{1}6^1 + \binom{n}{k}6^k$

$7^n = 1 + 6n + 6^2\binom{n}{2} + 6^3\binom{n}{3} + \dots$

$7^n - 6n = 1 + 6^2\binom{n}{2} + 6^3\binom{n}{3} + \dots$

$7^n - 6n = 1 + 6\left(6\binom{n}{2} + 6^2\binom{n}{3} + \dots\right)$

$7^n - 6n = 1(\text{mod}6)$

9. **By using binomial theorem show that for each $n \in N, 5^n - 1$ is divisible by 4.**

Sol: we want to show $4 | (5^n - 1)$ for $n \in N$

Using $(4+1)^n = \sum_{k=0}^{n} \binom{n}{k} 4^{n-k} 1^k$:

$5^n = 4^n + n.4^{n-1} + ... + 4n + 1$

$5^n - 1 = 4^n + n.4^{n-1} + ... + 4n$

$5^n - 1 = 4(4^n + n.4^{n-1} + ... + n)$

Since the term in the parenthesis is an integer $4 | (5^n - 1)$.

10. **By using binomial theorem show that for each $n \in N, 5^n - 2^n$ is divisible by 3.**

Sol: We want to show $3 | (5^n - 2^n)$ for $n \in \mathbb{N}$.

Express 5^n using the binomial theorem with $5 = (3+2)$:

$(3+2)^n = \sum_{k=0}^{n} \binom{n}{k} 3^{n-k} = 3^n + \binom{3}{1} 3^{n-1} 2^1 + ... + \binom{n}{n-1} 3^1 2^{n-1} + 2^n$

$5^n = 3^n + n.3^{n-1}.2 + ... + 3n.2^{n-1} + 2^n$.

Now consider $5^n - 2^n$:

$5^n - 2^n = (3^n + n.3^{n-1}.2 + ... 3^n.2^{n-1} + 2^n) - 2^n$

$5^n - 2^n = 3^n + n.3^{n-1}.2 + ... + 3^n.2^{n-1}$

$5^n - 2^n = 3(3^{n-1} + n.3^{n-2}.2 + ... + n.2^{n-1})$

Since the term in the parenthesis is an integer $5^n - 2^n$ is a multiple of 3.

Therefore, $5^n - 2^n$ is divisible by 3 for all $n \in \mathbb{N}$.

11. **Show that $a^2 + (a+2)^2 + (a+4)^2 + 1$ is divisible by 12, whenever "a" is an odd integer.**

Sol: $E = a^2 + (a+2)^2 + (a+4)^2 + 1$

$E = a^2 + (a^2 + 4a + 4) + (a^2 + 8a + 16) + 1$

$E = 3a^2 + 12a + 21$

Given a is an odd integer, let $a = 2k + 1$ for some integer k.

$E = 3(2k+1)^2 + 12(2k+1) + 21$

$E = 3(4k^2 + 4k + 1) + 24k + 12 + 21$

$E = 12k^2 + 12k + 3 + 24k + 33$

$E = 12k^2 + 36k + 36$

$E = 12(k^2 + 3k + 3)$

Since k is an integer, $k^2 + 3k + 3$ is an integer.

Therefore, E is divisible by 12

12. **A company expects its manual revenue to grow at a fixed rate of 6% per year. The revenue in year 1 is R=Rs. 10,000,000. Estimate the company's revenue after 4 years using the binomial theorem for small growth rates.**

Sol: **Company Revenue Growth**

$R_n = R_0(1+r)^n$

$R_0 = 10,000,000$

$r = 0.06$

$n = 4$

$R_4 = 10,000,000(1+0.06)^4$

$(1+0.06)^4 \approx 1 + 4(0.06) + \frac{4.3}{2}(0.06)^2 + \frac{4.3.2}{6}(0.06)^3$

$$(1+0.06)^4 \approx 1+0.24+6(0.0036)+4(0.000216)$$
$$(1+0.06)^4 \approx 1+0.24+0.0216+0.000864$$
$$(1+0.06)^4 \approx 1.262464$$
$$R_4 = 10,000,000 \times 1.262464$$
$$R_4 = 12,624,640$$

13. A bank offers a compound interest rate of 10% per year. Zafar invests Rs. 2,000,000 for 4 years. How much will his investment be worth at the end of 4 years?

Sol: **Zafar's compound interest investment**

$$A = P(1+r)^n$$
$$P = 2,000,000$$
$$r = 0.10$$
$$n = 4$$
$$A = 2,000,000(1+0.10)^4$$
$$A = 2,000,000(1.10)^4$$
$$(1.10)^4 = 1.4641$$
$$A = 2,000,000 \times 1.4641$$
$$A = 2,928,200$$

14. Zaid is organizing a sports competition with 8 teams. Every team plays against every other team exactly once. How many matches will be played in total? Use Pascal's triangle to solve this.

Sol: To find the number of matches, locate the 8th row of Pascal's triangle and the 3rd number in that row (Corresponding to choosing 2 teams).

1
11
121
1331
14641
15101051
1615201561
172135352171
18285670562881

The number of matches is **28**.