Exercise 10.2

Q1: **Without using table find the values**

i $\sin 15° = \sin(45° - 30°)$

Sol.
$$= \sin 45° \cos 30° - \cos 45° \sin 30°$$

$$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

ii. $\cos 15° = \cos(45° - 30°)$

Sol. $= \cos 45° \cos 30° + \sin 45° \sin 30°$

$$= \left(\frac{1}{\sqrt{2}}\right)\cdot\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

iii. tan15°= tan (45° − 30°)

Sol. $= \dfrac{\tan 45° - \tan 30°}{1 + \tan 45° \tan 30°} = \dfrac{1 - \dfrac{1}{\sqrt{3}}}{1 + 1.\dfrac{1}{\sqrt{3}}} = \dfrac{\dfrac{\sqrt{3}-1}{\sqrt{3}}}{\dfrac{\sqrt{3}+1}{\sqrt{3}}} = \dfrac{\sqrt{3}-1}{\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{3}+1} = \dfrac{\sqrt{3}-1}{\sqrt{3}+1}$

iv. Sin105°= Sin (60° + 45°)

Sol. = Sin60° Cos45° + Cos60° Sin45°

$= (\dfrac{1}{\sqrt{2}})(\dfrac{\sqrt{3}}{2}) + (\dfrac{1}{\sqrt{2}})(\dfrac{1}{2}) = \dfrac{\sqrt{3}}{2\sqrt{2}} + \dfrac{1}{2\sqrt{2}} = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$

v. Cos105°= Cos (60° + 45°)

Sol. $= \text{Cos60° Cos45°} - \text{Sin60° Sin45°} = \left(\dfrac{1}{2}\right)\left(\dfrac{1}{\sqrt{2}}\right) - \left(\dfrac{\sqrt{3}}{2}\right)\left(\dfrac{1}{\sqrt{2}}\right) = \dfrac{1}{2\sqrt{2}} - \dfrac{\sqrt{3}}{2\sqrt{2}} = \dfrac{1-\sqrt{3}}{2\sqrt{2}}$

vi. tan105°= tan (60° + 45°)

Sol. $= \dfrac{\tan 60° + \tan 45°}{1 - \tan 60° \tan 45°} = \dfrac{\sqrt{3}+1}{1 - \sqrt{3}.1} = \dfrac{1+\sqrt{3}}{1-\sqrt{3}}$

Q2. Prove that

i. Sin $(45° + \alpha) = \dfrac{1}{\sqrt{2}}$ (Sin α + Cos α)

Sol. L.H.S = Sin $(45° + \alpha)$ = Sin 45° Cos α + Cos 45° Sin α = $\dfrac{1}{\sqrt{2}}$ Cosα + $\dfrac{1}{\sqrt{2}}$ Sinα = $\dfrac{1}{\sqrt{2}}$ (Sinα + Cosα)

ii. Cos $(\alpha + 45°) = \dfrac{1}{\sqrt{2}}$ (Cos α − Sinα)

Sol. L.H.S = Cos $(\alpha + 45°)$ = Cos α Cos 45° − Sin α Sin45° = Cos $\alpha \dfrac{1}{\sqrt{2}}$ − Sin$\alpha \dfrac{1}{\sqrt{2}}$ = $\dfrac{1}{\sqrt{2}}$ (Cos α − Sin α)= R.H.S

Q3. Prove that

i tan (45° + A) tan (45° − A) = 1

Sol. L.H.S = tan (45° + A). tan (45° − A)

$= \left(\dfrac{\tan 45° + \tan A}{1 - \tan 45° \tan A}\right) \cdot \left(\dfrac{\tan 45° - \tan A}{1 + \tan 45° \tan A}\right) = \left(\dfrac{1 + \tan A}{1 - 1.\tan A}\right)\left(\dfrac{1 - \tan A}{1 + 1.\tan A}\right)$

$= \left(\dfrac{1 + \tan A}{1 - \tan A}\right)\left(\dfrac{1 - \tan A}{1 + \tan A}\right) = 1 = R.H.S$

ii. $\tan\left(\dfrac{\pi}{4} - \theta\right) + \tan\left(\dfrac{3\pi}{4} + \theta\right) = 0$

Sol. L.H.S = $\tan\left(\dfrac{\pi}{4} - \theta\right) + \tan\left(\dfrac{3\pi}{4} + \theta\right) = \dfrac{\tan\dfrac{\pi}{4} - \tan\theta}{1 + \tan\dfrac{\pi}{4}\tan\theta} + \dfrac{\tan\dfrac{3\pi}{4} + \tan\theta}{1 - \tan\dfrac{3\pi}{4}\tan\theta} = \dfrac{1 - \tan\theta}{1 + 1.\tan\theta} + \dfrac{-1 + \tan\theta}{1 - (-1)\tan\theta}$

$= \dfrac{1 - \tan\theta}{1 + \tan\theta} + \dfrac{-1 + \tan\theta}{1 + \tan\theta} = \dfrac{1 - \tan\theta - 1 + \tan\theta}{1 + \tan\theta} = \dfrac{0}{1 + \tan\theta} = 0 = R.H.S$

iii. $\sin\left(\theta+\dfrac{\pi}{6}\right)+\cos\left(\theta+\dfrac{\pi}{3}\right)=\cos\theta$

Sol. L.H.S $=\sin\left(\theta+\dfrac{\pi}{6}\right)+\cos\left(\theta+\dfrac{\pi}{3}\right)=\sin\theta\,\cos\dfrac{\pi}{6}+\cos\theta\,\sin\dfrac{\pi}{6}+\cos\theta\,\cos\dfrac{\pi}{3}-\sin\theta\,\sin\dfrac{\pi}{3}$

$=\sin\theta\,\dfrac{\sqrt{3}}{2}+\cos\theta\,\dfrac{1}{2}+\cos\theta\,\dfrac{1}{2}-\sin\theta\,\dfrac{\sqrt{3}}{2}=\cos\theta\left(\dfrac{1}{2}+\dfrac{1}{2}\right)=\cos\theta\,(1)=\cos\theta$ = R.H.S

iv. $\dfrac{\sin\theta-\cos\theta.\tan\theta/2}{\cos\theta+\sin\theta.\tan\theta/2}=\tan\theta/2$

Sol. L.H.S $=\dfrac{\sin\theta-\cos\theta.\tan\theta/2}{\cos\theta+\sin\theta.\tan\theta/2}$

$=\dfrac{\sin\theta-\cos\theta.\dfrac{\sin\theta/2}{\cos\theta/2}}{\cos\theta+\sin\theta.\dfrac{\sin\theta/2}{\cos\theta/2}}=\dfrac{\dfrac{\sin\theta\cos\theta/2-\cos\theta\sin\theta/2}{\cos\theta/2}}{\dfrac{\cos\theta\cos\theta+\sin\theta\sin\theta/2}{\cos\theta/2}}$

$=\dfrac{\dfrac{\sin(\theta-\theta/2)}{\cos\theta/2}}{\dfrac{\cos(\theta-\theta/2)}{\cos\theta/2}}=\dfrac{\sin\theta/2}{\cos\theta/2}\times\dfrac{\cos\theta/2}{\cos\theta/2}=\tan\theta/2=R.H.S$

v. $\dfrac{1-\tan\theta.\tan\varphi}{1+\tan\theta.\tan\varphi}=\dfrac{\cos(\theta+\varphi)}{\cos(\theta-\varphi)}$

Sol. L.H.S $=\dfrac{1-\tan\theta.\tan\phi}{1+\tan\theta.\tan\phi}=\dfrac{1-\dfrac{\sin\theta}{\cos\theta}\dfrac{\sin\phi}{\cos\phi}}{1+\dfrac{\sin\theta}{\cos\theta}\dfrac{\sin\phi}{\sin\phi}}$

$=\dfrac{\dfrac{\cos\theta\cos\phi-\sin\theta\sin\phi}{\cos\theta\cos\phi}}{\dfrac{\cos\theta\cos\phi+\sin\theta\sin\phi}{\cos\theta\cos\phi}}=\dfrac{\dfrac{\cos(\theta+\phi)}{\cos\theta\cos\phi}}{\dfrac{\cos(\theta-\phi)}{\cos\theta\cos\phi}}=\dfrac{\cos(\theta+\varphi)}{\cos\theta\cos\varphi}\times\dfrac{\cos\theta\cos\varphi}{\cos(\theta-\varphi)}=\dfrac{\cos(\theta+\phi)}{\cos(\theta-\phi)}$

Q4. Show that $\cos(\alpha+\beta)\,\cos(\alpha-\beta)=\cos^2\alpha-\sin^2\beta=\cos^2\beta-\sin^2\alpha$

Sol. L.H.S $=\cos(\alpha+\beta).\cos(\alpha-\beta)$

$=(\cos\alpha\,\cos\beta-\sin\alpha\,\sin\beta).(\cos\alpha\,\cos\beta+\sin\alpha\,\sin\beta)$

$=(\cos\alpha\,\cos\beta)^2-(\sin\alpha\,\sin\beta)^2=\cos^2\alpha\,\cos^2\beta-\sin^2\alpha\,\sin^2\beta$ _____ I

$=\cos^2\alpha\,(1-\sin^2\beta)-(1-\cos^2\alpha)\,\sin^2\beta$

$=\cos^2\alpha-\cos^2\alpha\sin^2\beta-\sin^2\beta+\cos^2\alpha\sin^2\beta$

$=\cos^2\alpha-\sin^2\beta$ **Result I**

$=(1-\sin^2\alpha)-(1-\cos^2\beta)$

$=1-\sin^2\alpha-1+\cos^2\beta$

$=\cos^2\beta-\sin^2\alpha$ **Result II**

5. Show that $\dfrac{Sin(\alpha+\beta)+Sin(\alpha-\beta)}{Cos(\alpha+\beta)+Cos(\alpha-\beta)} = \tan\alpha$

Sol.

$$L.H.S = \frac{Sin(\alpha+\beta)+Sin(\alpha-\beta)}{Cos(\alpha+\beta)+Cos(\alpha-\beta)} = \frac{Sin\alpha\,Cos\beta + Cos\alpha\,Sin\beta + Sin\alpha\,Cos\beta - Cos\alpha\,Sin\beta}{Cos\alpha\,Cos\beta - Sin\alpha\,Sin\beta + Cos\alpha\,Cos\beta + Sin\alpha\,Sin\beta}$$

$$= \frac{2\,Sin\alpha\,Cos\beta}{2\,Cos\alpha\,Cos\beta} = \tan\alpha = R.H.S$$

Q6(i). Show that $\sin^2\left(\alpha+\dfrac{\beta}{2}\right) - \sin^2\left(\alpha-\dfrac{\beta}{2}\right) = \sin 2\alpha.\sin\beta$

Sol:

$$L.H.S = \left(\sin\alpha\cos\frac{\beta}{2} + \cos\alpha\sin\frac{\beta}{2}\right)^2 - \left(\sin\alpha\cos\frac{\beta}{2} - \cos\alpha\sin\frac{\beta}{2}\right)^2$$

$$= \left(\sin^2\alpha\cos^2\frac{\beta}{2} + \cos^2\alpha\sin^2\frac{\beta}{2} + 2\sin\alpha\cos\frac{\beta}{2}\cos\alpha\sin\frac{\beta}{2}\right) - \left(\sin^2\alpha\cos^2\frac{\beta}{2} + \cos^2\alpha\sin^2\frac{\beta}{2} - 2\sin\alpha\cos\frac{\beta}{2}\cos\alpha\sin\frac{\beta}{2}\right)$$

$$= \sin^2\alpha\cos^2\frac{\beta}{2} + \cos^2\alpha\sin^2\frac{\beta}{2} + 2\sin\alpha\cos\frac{\beta}{2}\cos\alpha\sin\frac{\beta}{2} - \sin^2\alpha\cos^2\frac{\beta}{2} - \cos^2\alpha\sin^2\frac{\beta}{2} + 2\sin\alpha\cos\frac{\beta}{2}\cos\alpha\sin\frac{\beta}{2}$$

$$= 4\sin\alpha\cos\frac{\beta}{2}\cos\alpha\sin\frac{\beta}{2} = 2\sin\alpha\cos\alpha.2\sin\frac{\beta}{2}\cos\frac{\beta}{2} = \sin 2\alpha\sin\beta = R.H.S$$

Q6(ii). Show that: $\sin^2\alpha + \sin^2\beta + \cos^2(\alpha+\beta) + 2\sin\alpha.\sin\beta.\cos(\alpha+\beta) = 1$

Sol:

$$L.H.S = \sin^2\alpha + \sin^2\beta + \cos^2(\alpha+\beta) + 2\sin\alpha.\sin\beta.\cos(\alpha+\beta)$$

$$= \sin^2\alpha + \sin^2\beta + \cos(\alpha+\beta)\left[\cos(\alpha+\beta) + 2\sin\alpha\sin\beta\right]$$

$$= \sin^2\alpha + \sin^2\beta + \cos(\alpha+\beta)\left[\cos\alpha\cos\beta - \sin\alpha\sin\beta + 2\sin\alpha\sin\beta\right]$$

$$= \sin^2\alpha + \sin^2\beta + \cos(\alpha+\beta)\left[\cos\alpha\cos\beta + \sin\alpha\sin\beta\right]$$

$$= \sin^2\alpha + \sin^2\beta + \cos(\alpha+\beta)\cos(\alpha-\beta)$$

$$= \sin^2\alpha + \sin^2\beta + (\cos\alpha\cos\beta - \sin\alpha\sin\beta)(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$$

$$= \sin^2\alpha + \sin^2\beta + \cos^2\alpha\cos^2\beta - \sin^2\alpha\sin^2\beta$$

$$= \sin^2\alpha + \sin^2\beta + \cos^2\alpha(1-\sin^2\beta) - (1-\cos^2\alpha)\sin^2\beta$$

$$= \sin^2\alpha + \sin^2\beta + \cos^2\alpha - \cos^2\alpha\sin^2\beta - \sin^2\beta + \cos^2\alpha\sin^2\beta$$

$$= \sin^2\alpha + \sin^2\beta + \cos^2\alpha - \sin^2\beta$$

$$= \sin^2\alpha + \cos^2\alpha + \sin^2\beta - \sin^2\beta$$

$$= 1 + 0 = 1$$

Q7. Show that:

i. $\cos(\alpha-\beta) = \dfrac{1+\tan\alpha\tan\beta}{\sec\alpha\sec\beta}$

Sol:

$$R.H.S = \frac{1+\tan\alpha\tan\beta}{\sec\alpha\sec\beta}$$

$$= \frac{1+\dfrac{\sin\alpha}{\cos\alpha}.\dfrac{\sin\beta}{\cos\beta}}{\dfrac{1}{\cos\alpha}.\dfrac{1}{\cos\beta}} = \frac{\dfrac{\cos\alpha\cos\beta + \sin\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{1}{\cos\alpha\cos\beta}} = \frac{\cos(\alpha-\beta)}{\cos\alpha\cos\beta} \times \cos\alpha\cos\beta = \cos(\alpha-\beta) = L.H.S$$

ii. $\sin(\alpha + \beta) = \dfrac{1 + \cot\alpha\tan\beta}{\csc\alpha\sec\beta}$

Sol: R.H.S $= \dfrac{1 + \cot\alpha\tan\beta}{\csc\alpha\sec\beta}$

$$= \dfrac{1 + \dfrac{\cos\alpha}{\sin\alpha}\cdot\dfrac{\sin\beta}{\cos\beta}}{\dfrac{1}{\sin\alpha}\cdot\dfrac{1}{\cos\beta}} = \dfrac{\dfrac{\sin\alpha\cos\beta + cs\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{1}{\sin\alpha\cos\beta}}$$

$$= \dfrac{\sin(\alpha + \beta)}{\sin\alpha\cos\beta} \times \sin\alpha\cos\beta = \sin(\alpha + \beta) = L.H.S$$

iii. $\cot(\alpha - \beta) = \dfrac{\cot\alpha\cot\beta + 1}{\cot\beta - \cot\alpha}$

Sol: Let R.H.S $= \dfrac{\cot\alpha\cot\beta + 1}{\cot\beta - \cot\alpha}$

$$= \dfrac{\dfrac{1}{\tan\alpha\tan\beta} + 1}{\dfrac{1}{\tan\beta} - \dfrac{1}{\tan\alpha}} = \dfrac{\dfrac{1 + \tan\alpha\tan\beta}{\tan\alpha\tan\beta}}{\dfrac{\tan\alpha - \tan\beta}{\tan\alpha\tan\beta}} = \dfrac{1 + \tan\alpha\tan\beta}{\tan\alpha\tan\beta} \times \dfrac{\tan\alpha\tan\beta}{\tan\alpha - \tan\beta}$$

$$= \dfrac{1 + \tan\alpha\tan\beta}{\tan\alpha - \tan\beta} = \dfrac{1}{\dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}} = \dfrac{1}{\tan(\alpha - \beta)} = \cot(\alpha - \beta) = L.H.S$$

iv. $\dfrac{\tan\alpha + \tan\beta}{\tan\alpha - \tan\beta} = \dfrac{\sin(\alpha + \beta)}{\sin(\alpha - \beta)}$

Sol: L.H.S $= \dfrac{\tan\alpha + \tan\beta}{\tan\alpha - \tan\beta} = \dfrac{\dfrac{\sin\alpha}{\cos\alpha} + \dfrac{\sin\beta}{\cos\beta}}{\dfrac{\sin\alpha}{\cos\alpha} - \dfrac{\sin\beta}{\cos\beta}} = \dfrac{\dfrac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\cos\alpha\cos\beta}}$

$$= \dfrac{\sin(\alpha + \beta)}{\cos\alpha\cos\beta} \times \dfrac{\cos\alpha\cos\beta}{\sin(\alpha - \beta)} = \dfrac{\sin(\alpha + \beta)}{\sin(\alpha - \beta)} = R.H.S$$

v. $\cot(\alpha + \beta) = \dfrac{\cot\alpha\cot\beta - 1}{\cot\alpha + \cot\beta}$

Sol: R.H.S $= \dfrac{\cot\alpha\cot\beta - 1}{\cot\alpha + \cot\beta} = \dfrac{\dfrac{1}{\tan\alpha\tan\beta} - 1}{\dfrac{1}{\tan\alpha} + \dfrac{1}{\tan\beta}} = \dfrac{\dfrac{1 - \tan\alpha\tan\beta}{\tan\alpha\tan\beta}}{\dfrac{\tan\alpha + \tan\beta}{\tan\alpha\tan\beta}}$

$$= \dfrac{1 - \tan\alpha\tan\beta}{\tan\alpha\tan\beta} \times \dfrac{\tan\alpha\tan\beta}{\tan\alpha + \tan\beta} = \dfrac{1 - \tan\alpha\tan\beta}{\tan\alpha + \tan\beta} = \dfrac{1}{\dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}}$$

$$= \dfrac{1}{\tan(\alpha + \beta)} = \cot(\alpha + \beta) = L.H.S$$

Q8. if $\sin\alpha = \dfrac{24}{25}$ and $\cos\beta = \dfrac{20}{29}$, where $0 < \alpha < \dfrac{\pi}{2}$

and $0 < \beta < \dfrac{\pi}{2}$ show that $\sin(\alpha - \beta) = \dfrac{333}{725}$.

Sol: $\cos^2\alpha = 1 - \sin^2\alpha$

$$= 1 - \left(\frac{24}{25}\right)^2 = 1 - \frac{576}{625} = \frac{625 - 576}{625}$$

$$\cos^2\alpha = \frac{49}{625} \Rightarrow \cos\alpha = \pm\frac{7}{25} = \frac{7}{25}(\alpha \text{ in I quad})$$

$$\sin^2\beta = 1 - \cos^2\beta$$

$$= 1 - \left(\frac{20}{29}\right)^2 = 1 - \frac{400}{841} = \frac{841 - 400}{841}$$

$$\sin^2\beta = \frac{441}{841} \Rightarrow \sin\beta = \pm\frac{21}{29} = \frac{21}{29}(\beta \text{ in I quad})$$

Using formula $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$$\sin(\alpha - \beta) = \left(\frac{24}{25}\right)\left(\frac{20}{29}\right) - \left(\frac{7}{25}\right)\left(\frac{21}{29}\right)$$

$$= \frac{480}{725} - \frac{147}{725} = \frac{480 - 147}{725} = \frac{333}{725}$$

Q9. if $\sin\alpha = \dfrac{-8}{17}$ and $\cos\beta = -\dfrac{4}{5}$, where $\dfrac{3\pi}{2} < \alpha < 2\pi$ and $\pi < \beta < \dfrac{3\pi}{2}$ Find the following.

i. $\sin(\alpha + \beta)$ ii. $\cos(\alpha + \beta)$ iii. $\tan(\alpha + \beta)$

iv. $\sin(\alpha - \beta)$ v. $\cos(\alpha - \beta)$ vi. $\tan(\alpha - \beta)$

In which quadrants do the terminal sides of the angles of measures $(\alpha + \beta)$ and $(\alpha - \beta)$ lie?

Sol: Given that α is in IV-Quad and β is in III-Quad

$$\cos^2\alpha = 1 - \sin^2\alpha$$

$$= 1 - \left(\frac{-8}{17}\right)^2 = 1 - \frac{64}{289} = \frac{289 - 64}{289}$$

$$\cos^2\alpha = \frac{225}{289} \Rightarrow \cos\alpha = \pm\frac{15}{17}$$

$$\cos\alpha = \frac{15}{17} \qquad (\alpha \text{ in IV quad})$$

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{-8/\cancel{17}}{15/\cancel{17}} = -\frac{8}{15}$$

$$\sin^2\beta = 1 - \cos^2\beta$$

$$= 1 - \left(\frac{-4}{5}\right)^2 = 1 - \frac{16}{25} = \frac{9}{25}$$

$$\sin \beta = \pm \frac{3}{5} = \frac{-3}{5} \qquad (\beta \text{ in III quad})$$

$$\Rightarrow \sin \beta = \frac{-3}{5}$$

$$\tan \beta = \frac{\sin \beta}{\cos \beta} = \frac{-3/5}{-4/5} = \frac{3}{4}$$

i. $\sin(\alpha + \beta)$

Sol: $\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$

$$= \left(\frac{-8}{17}\right)\left(-\frac{4}{5}\right) + \left(\frac{15}{17}\right)\left(\frac{-3}{5}\right) = \frac{32}{85} - \frac{45}{85} = \frac{-13}{85}$$

ii. $\cos(\alpha + \beta)$

Sol: $\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$

$$= \left(\frac{15}{17}\right)\left(\frac{-4}{5}\right) - \left(\frac{-8}{17}\right)\left(\frac{-3}{5}\right) = \frac{-60}{85} - \frac{24}{85} = \frac{-84}{85}$$

iii. $\tan(\alpha + \beta)$

Sol: $\tan(\alpha + \beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha \tan\beta} = \dfrac{\frac{-8}{15} + \frac{3}{4}}{1 - \left(\frac{-8}{15}\right)\left(\frac{3}{4}\right)} = \dfrac{\frac{-8}{15} + \frac{3}{4}}{1 + \frac{24}{60}} = \dfrac{\frac{-32+45}{60}}{\frac{60+24}{60}} = \dfrac{13}{84}$

iv. $\sin(\alpha - \beta)$

Sol: $\sin(\alpha - \beta) = \sin\alpha \cos\beta - \cos\alpha \sin\beta$

$$= \left(\frac{-8}{17}\right)\left(-\frac{4}{5}\right) - \left(\frac{15}{17}\right)\left(\frac{-3}{5}\right) = \frac{32}{85} + \frac{45}{85} = \frac{77}{85}$$

v. $\cos(\alpha - \beta)$

Sol: $\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$

$$= \left(\frac{15}{17}\right)\left(\frac{-4}{5}\right) + \left(\frac{-8}{17}\right)\left(\frac{-3}{5}\right) = \frac{-60}{85} + \frac{24}{85} = \frac{-36}{85}$$

vi. $\tan(\alpha - \beta)$

Sol: $\tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha \tan\beta} = \dfrac{\frac{-8}{15} - \frac{3}{4}}{1 + \left(\frac{-8}{15}\right)\left(\frac{3}{4}\right)} = \dfrac{\frac{-32-45}{60}}{\frac{60-24}{60}} = \dfrac{-77}{36}$

$\sin(\alpha + \beta) = -ve$, $\cos(\alpha + \beta) = -ve$

but $\tan(\alpha + \beta) = +ve$ so $\alpha + \beta$ is in III quad

Similarly,

$\sin(\alpha - \beta) = +ve$, $\cos(\alpha - \beta) = -ve$, $\tan(\alpha - \beta) = -ve$

So $\alpha - \beta$ is in II quad

10.(i) Find $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$ given that $\tan\alpha = \dfrac{3}{4}$, $\cos\beta = \dfrac{5}{13}$ and neither the terminal side of the angle of measure α nor that β is in the I quadrant.

Given $\tan\alpha = +ve$ but not in I so α is in III quad

Similarly, $\cos\beta = +ve$ but β not is in I so β is in IV quad

Sol. $1 + \tan^2\alpha = \sec^2\alpha \Rightarrow 1 + \dfrac{9}{16} = \sec^2\alpha = \dfrac{25}{16} \Rightarrow \sec\alpha = \pm\dfrac{5}{4}$

$\Rightarrow \cos\alpha = \pm\dfrac{4}{5} \Rightarrow \cos\alpha = \dfrac{-4}{5}$ (Because α in III)

$\sin^2\alpha = 1 - \cos^2\alpha = 1 - \left(\dfrac{-4}{5}\right)^2 = 1 - \dfrac{16}{25} = \dfrac{9}{25} \Rightarrow \sin\alpha = \pm\dfrac{3}{5}$

$\sin\alpha = -\dfrac{3}{5}$ (Because α is in III)

$\sin^2\beta = 1 - \cos^2\beta = 1 - \left(\dfrac{5}{13}\right)^2 = 1 - \dfrac{25}{169} = \dfrac{169-25}{169} = \dfrac{144}{169} = \sin\beta = \pm\dfrac{12}{13}$

$\sin\beta = \dfrac{-12}{13}$ (Because β is in IV)

a. $\sin(\alpha+\beta) = \sin\alpha\,\cos\beta + \cos\alpha\,\sin\beta$

Sol. $= \left(\dfrac{-3}{5}\right)\left(\dfrac{5}{13}\right) + \left(\dfrac{-4}{5}\right)\left(\dfrac{-12}{13}\right) = \dfrac{-15}{65} + \dfrac{48}{65} = \dfrac{-15+48}{65} = \dfrac{33}{65}$

b. $\cos(\alpha+\beta) = \cos\alpha\,\cos\beta - \sin\alpha\,\sin\beta$

Sol. $= \left(\dfrac{-4}{5}\right)\left(\dfrac{5}{13}\right) - \left(\dfrac{-3}{5}\right)\left(\dfrac{-12}{13}\right) = \dfrac{-20}{65} - \dfrac{36}{65} = \dfrac{-20-36}{65} = \dfrac{-56}{65}$

10 (ii) $\tan\alpha = -\dfrac{15}{8}$, $\sin\beta = -\dfrac{7}{25}$ and neither the terminal side of the angle of measure α nor that β is in the IV quadrant.

Sol. $1 + \tan^2\alpha = \sec^2\alpha \Rightarrow 1 + \left(\dfrac{-15}{8}\right)^2 = \sec^2\alpha$

> *Given α not in IV and $\tan\alpha = -ve$ so α in II Similarly β in III*

$1 + \dfrac{225}{16} = \sec^2\alpha \Rightarrow \dfrac{289}{64} = \sec^2\alpha \Rightarrow \sec\alpha = \pm\dfrac{17}{8} \Rightarrow \cos\alpha = \pm\dfrac{8}{17}$

$\boxed{\cos\alpha = -8/17}$ (Because α is in II)

$\sin^2\alpha = 1 - \cos^2\alpha = 1 - \left(\dfrac{-8}{17}\right)^2 = 1 - \dfrac{64}{289} = \dfrac{225}{289} \Rightarrow \sin\alpha = \pm\dfrac{15}{17}$

$\boxed{\sin\alpha = \dfrac{15}{17}}$ (Because α is in II)

$\cos^2\beta = 1 - \sin^2\beta = 1 - \left(\dfrac{-7}{25}\right)^2 = 1 - \dfrac{49}{625} = \dfrac{625-64}{625} = \dfrac{576}{625}$

$\cos\beta = \pm\dfrac{24}{25} \Rightarrow \boxed{\cos\beta = -\dfrac{24}{25}}$ (Because β is in III)

Now

a. $\mathrm{Sin}\,(\alpha+\beta)=\mathrm{Sin}\,\alpha\,\mathrm{Cos}\,\beta+\mathrm{Cos}\,\alpha\,\mathrm{Sin}\,\beta$

Sol. $=\left(\dfrac{15}{17}\right)\left(\dfrac{-24}{25}\right)+\left(\dfrac{-8}{17}\right)\left(\dfrac{-7}{25}\right)=\dfrac{-360}{425}+\dfrac{56}{425}=\dfrac{-360+56}{425}=\dfrac{-304}{425}$

b. $\mathrm{Cos}(\alpha+\beta)=\mathrm{Cos}\,\alpha\,\mathrm{Cos}\,\beta-\mathrm{Sin}\,\alpha\,\mathrm{Sin}\,\beta=\left(\dfrac{-8}{17}\right)\left(\dfrac{-24}{25}\right)-\left(\dfrac{15}{17}\right)\left(\dfrac{-7}{25}\right)$

Sol. $=\dfrac{192}{425}+\dfrac{105}{425}=\dfrac{192+105}{425}=\dfrac{297}{425}$

Q11. Prove that: $\dfrac{\cos 19^0+\sin 19^0}{\cos 19^0-\sin 19^0}=\tan 64^0$

Sol: $\text{R.H.S}=\tan\left(64^0\right)=\tan\left(45^0+19^0\right)$

$=\dfrac{\tan 45^0+\tan 19^0}{1-\tan 45^0\tan 19^0}=\dfrac{1+\tan 19^0}{1-\tan 19^0}=\dfrac{1+\dfrac{\sin 19^0}{\cos 19^0}}{1-\dfrac{\sin 19^0}{\cos 19^0}}=\dfrac{\dfrac{\cos 19^0+\sin 19^0}{\cos 19^0}}{\dfrac{\cos 19^0-\sin 19^0}{\cos 19^0}}=\dfrac{\cos 19^0+\sin 19^0}{\cos 19^0-\sin 19^0}=L.H.S$

Q12. Prove that $\cos\left(60^0+\theta\right)\cos\left(60^0-\theta\right)+\sin\left(60^0+\theta\right)\sin\left(60^0-\theta\right)=\cos 2\theta$

Sol: $\text{L.H.S}=\cos\left(60^0+\theta\right)\cos\left(60^0-\theta\right)+\sin\left(60^0+\theta\right)\sin\left(60^0-\theta\right)$

Using formula: $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

$=\cos\left[\left(60^0+\theta\right)-\left(60^0-\theta\right)\right]=\cos\left(60^0+\theta-60^0+\theta\right)=\cos 2\theta=R.H.S$

Q13. If α,β,γ are angles of triangle ABC, Show that $\mathrm{Cot}\,\alpha/2+\mathrm{Cot}\,\beta/2+\mathrm{Cot}\,\gamma/2=\mathrm{Cot}\,\alpha/2\,\mathrm{Cot}\,\beta/2\,\mathrm{Cot}\,\gamma/2$

Sol. We know that $\alpha+\beta+\gamma=180^\circ\Rightarrow\alpha+\beta=180^0-\gamma$

Divide both side by '2' $\dfrac{\alpha+\beta}{2}=\dfrac{180^0-\gamma}{2}\Rightarrow\dfrac{\alpha+\beta}{2}=90^0-\dfrac{\gamma}{2}\Rightarrow\dfrac{\alpha}{2}+\dfrac{\beta}{2}=90^0-\dfrac{\gamma}{2}$

$\tan(\alpha/2+\beta/2)=\tan(90^0-\gamma/2)\Rightarrow\dfrac{\tan\alpha/2+\tan\beta/2}{1-\tan\alpha/2\tan\beta/2}=\mathrm{Cot}\,\gamma/2$

$\dfrac{\dfrac{1}{\mathrm{Cot}\,\alpha/2}+\dfrac{1}{\mathrm{Cot}\,\beta/2}}{1-\dfrac{1}{\mathrm{Cot}\,\alpha/2\,\mathrm{Cot}\,\beta/2}}=\mathrm{Cot}\,\gamma/2\Rightarrow\dfrac{\dfrac{\mathrm{Cot}\,\dfrac{\alpha}{2}+\mathrm{Cot}\,\dfrac{\beta}{2}}{\mathrm{Cot}\,\dfrac{\alpha}{2}\,\mathrm{Cot}\,\dfrac{\beta}{2}}}{\dfrac{\mathrm{Cot}\,\dfrac{\alpha}{2}\,\mathrm{Cot}\,\dfrac{\beta}{2}-1}{\mathrm{Cot}\,\dfrac{\alpha}{2}\,\mathrm{Cot}\,\dfrac{\beta}{2}}}=\mathrm{Cot}\,\dfrac{\gamma}{2}$

$\dfrac{\mathrm{Cot}\,\alpha/2+\mathrm{Cot}\,\beta/2}{\mathrm{Cot}\,\alpha/2\,\mathrm{Cot}\,\beta/2}\times\dfrac{\mathrm{Cot}\,\alpha/2\,\mathrm{Cot}\,\beta/2}{\mathrm{Cot}\,\alpha/2\,\mathrm{Cot}\,\beta/2-1}=\mathrm{Cot}\,\gamma/2$

$\mathrm{Cot}\,\alpha/2+\mathrm{Cot}\,\beta/2=\mathrm{Cot}\,\gamma/2\,(\mathrm{Cot}\,\alpha/2\,\mathrm{Cot}\,\beta/2-1)$

$\mathrm{Cot}\,\alpha/2+\mathrm{Cot}\,\beta/2=\mathrm{Cot}\,\alpha/2\,\mathrm{Cot}\,\beta/2\,\mathrm{Cot}\,\gamma/2-\mathrm{Cot}\,\gamma/2$

$\mathrm{Cot}\,\alpha/2+\mathrm{Cot}\,\beta/2+\mathrm{Cot}\,\gamma/2=\mathrm{Cot}\,\alpha/2\,\mathrm{Cot}\,\beta/2\,\mathrm{Cot}\,\gamma/2$

Hence proved.

14. If $\alpha + \beta + \gamma = 180^\circ$, Show that $\cot\alpha \cot\beta + \cot\beta \cot\gamma + \cot\gamma \cot\alpha = 1$

Sol. Given $\alpha + \beta + \gamma = 180^\circ \Rightarrow \alpha + \beta = 180^\circ - \gamma \Rightarrow \tan(\alpha+\beta) = \tan(180^\circ - \gamma)$

$$\frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = -\tan\gamma \Rightarrow \frac{\dfrac{1}{\cot\alpha} + \dfrac{1}{\cot\beta}}{1 - \dfrac{1}{\cot\alpha\cot\beta}} = -\frac{1}{\cot\gamma}$$

$$\frac{\dfrac{\cot\alpha + \cot\beta}{\cot\alpha\cot\beta}}{\dfrac{\cot\alpha\cot\beta - 1}{\cot\alpha\cot\beta}} = -\frac{1}{\cot\gamma} \Rightarrow \frac{\cot\alpha + \cot\beta}{\cot\alpha\cot\beta} \times \frac{\cot\alpha\cot\beta}{\cot\alpha\cot\beta - 1} = -\frac{1}{\cot\gamma}$$

$(\cot\alpha + \cot\beta)(\cot\gamma) = -(\cot\alpha\cot\beta - 1)$

$\cot\alpha\cot\gamma + \cot\beta\cot\gamma = -\cot\alpha\cot\beta + 1$

$\cot\alpha\cot\beta + \cot\beta\cot\gamma + \cot\gamma\cot\alpha = 1$

Hence Proved.

15. Express the following in the form $r\sin(\theta + \phi)$ or $r\sin(\theta - \phi)$ where terminal sides of the angles of measures θ and ϕ are in the first quadrant:

i. $24\sin\theta + 7\cos\theta$

Sol: Take

$24 = r\cos\phi$ ____ I & $7 = r\sin\phi$ ____ II

Squaring and Adding (I) and (II)

As $r^2\cos^2\phi + r^2\sin^2\phi = (24)^2 + (7)^2$

$r^2(\cos^2\phi + \sin^2\phi) = 576 + 49$

$r^2 = 625 \Rightarrow r = 25$

Dividing (II) By (I)

$\phi = \dfrac{r\sin\theta}{r\cos\phi} = \dfrac{7}{24} \Rightarrow \tan\phi = \dfrac{7}{24}$

Then $24\sin\theta + 7\cos\theta = r\cos\phi\sin\theta + r\sin\phi\cos\theta$

$= r(\sin\theta\cos\phi + \cos\theta\sin\phi) = r\sin(\theta+\phi) = 25\sin(\theta+\phi)$

Where $\tan\phi = \dfrac{7}{24}$

ii. $12\sin\theta - 5\cos\theta$

Sol: Take $12 = r\cos\phi$ ____ (1)

and $5 = r\sin\phi$ ____ (2)

Squaring and adding (1) and (2)

$(12)^2 + (5)^2 = r^2\cos\phi + r^2\sin^2\phi c$

$144 + 25 = r^2(\cos^2\phi + \sin^2\phi)$

$169 = r^2(1) \Rightarrow r = 13$

Divide (2) by (1)

$\dfrac{5}{12} = \dfrac{r\sin\phi}{r\cos\phi}$

$\tan\phi = \dfrac{5}{12} \Rightarrow \phi = \tan^{-1}\left(\dfrac{5}{12}\right)$

Using (1) and (2) in (A)

$\Rightarrow r\cos\phi\sin\theta - r\sin\phi\cos\theta$

$= r(\sin\theta\cos\phi - \cos\theta\sin\phi) = r\sin(\theta - \phi)$

$= 13\sin(\theta - \phi)$

Where $\tan\phi = \dfrac{5}{12}$

iii. $\sin\theta - \cos\theta$

Sol: Take $1 = r\cos\phi$ ____ I & $1 = r\sin\phi$ ____ II

By squaring and Adding I & II

$(1)^2 + (1)^2 = r^2\cos^2\phi + r^2\sin^2\phi$

$1 + 1 = r^2(\cos^2\phi + \sin^2\phi) \Rightarrow 2 = r^2 \Rightarrow r = \sqrt{2}$

By dividing II by I

$\dfrac{r\sin\phi}{r\cos\phi} = \dfrac{1}{1} \Rightarrow \tan\phi = 1$

$\sin\theta - \cos\theta = r\cos\phi\sin\theta - r\sin\phi\cos\theta$

$= r(\sin\theta\cos\phi - \sin\phi\cos\theta)$

$= r\sin(\theta - \phi) = \sqrt{2}\sin(\theta - \phi)$

Where $r = \sqrt{2}$ and $\tan\phi = 1$

iv. $8\sin\theta - 6\cos\theta$

Sol: Put $8 = r\cos\phi$ ____ I & $6 = r\sin\phi$ ____ II

Squaring and adding I & II

As $r^2\cos^2\phi + r^2\sin^2\phi = (8)^2 + (6)^2$

$$r^2\left(\cos^2\phi+\sin^2\phi\right)=64+36$$
$$r^2=100\Rightarrow r=10$$
By dividing II by I
$$\frac{r\sin\phi}{r\cos\phi}=\frac{6}{8}\Rightarrow\tan\phi=\frac{3}{4}$$

Then
$$8\sin\theta-6\cos\theta=r\cos\phi\sin\theta-r\sin\phi\cos\theta$$
$$=r\left(\sin\theta\cos\phi-\cos\theta\sin\phi\right)$$
$$=r\sin\left(\theta-\phi\right)=10\sin\left(\theta-\phi\right)$$

Where $\tan\phi=\dfrac{3}{4}$

v. $\dfrac{1}{2}\sin\theta+\dfrac{\sqrt{3}}{2}\cos\theta$

Sol: Put $\dfrac{1}{2}=r\cos\phi$ ____(1)

and $\dfrac{\sqrt{3}}{2}=r\sin\phi$ ____(2)

Divide 2 by 1

$$\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}=\frac{r\sin\phi}{r\cos\phi}$$

$$\tan\phi=\sqrt{3}$$

Squaring and adding 1 & 2

$$\left(\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2=r^2\cos^2\phi+r^2\sin^2\phi$$

$$\frac{1}{4}+\frac{3}{4}=r^2\left(\cos^2\phi+\sin^2\phi\right)$$
$$\frac{4}{4}=r^2\Rightarrow r=1$$
$$\frac{1}{2}\sin\theta+\frac{\sqrt{3}}{2}=r\cos\phi\sin\theta+r\sin\phi\cos\theta$$
$$=r\left(\sin\theta\cos\phi+\cos\theta\sin\phi\right)$$
$$=r\sin\left(\theta+\phi\right)=\sin\left(\theta+\phi\right)$$

Where $\tan\phi=\sqrt{3}$

vi. $13\sin\theta-84\cos\theta$

Sol: Put
$13=r\cos\phi$ ____ I, $84=r\sin\phi$ ____ II

Squaring and Adding I & II
$$r^2\cos\phi+r^2\sin^2\phi=\left(13\right)^2+\left(84\right)^2$$
$$r^2\left(\cos\phi+\sin^2\phi\right)=169+7056$$
$$r^2=7225\Rightarrow r=85$$

By dividing II by I
$$\frac{r\sin\phi}{r\cos\phi}=\frac{84}{13}\Rightarrow\tan\phi=\frac{84}{13}$$

Then
$$13\sin\theta-84\cos\theta=r\cos\phi\sin\theta-r\sin\phi\cos\theta$$
$$=r\left(\sin\theta\cos\phi-\cos\theta\sin\phi\right)$$
$$=r\sin\left(\theta-\phi\right)=85\sin\left(\theta-\phi\right)$$

Where $\tan\phi=\dfrac{84}{13}$