Exercise 12.1

Limit of a function:

A function f(x) be defined in an open internal near the number "a" and x approaches "a" from both left and right then f(x) approaches a specific number "L" then L is called limit of f(x) written as $\lim_{x\to a} f(x) = L$

Q1. Find the limit of the following sequences if exists

(i) $a_n = \dfrac{2n+3}{n+1}$

Divide by n:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\frac{2n}{n}+\frac{3}{n}}{\frac{n}{n}+\frac{1}{n}} = \lim_{n\to\infty} \frac{2+\frac{3}{n}}{1+\frac{1}{n}}$$

As $n \to \infty, \frac{3}{n} \to 0$ and $\frac{1}{n} \to 0$:

$$\frac{2+0}{1+0} = 2$$

(ii) $b_n = \dfrac{2n+3}{n^2+1}$

Divide by n^2:

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\frac{2n}{n^2}+\frac{3}{n^2}}{\frac{n^2}{n^2}+\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\frac{2}{n}+\frac{3}{n^2}}{1+\frac{1}{n^2}}$$

As $n \to \infty, \frac{2}{n} \to 0, \frac{3}{n^2} \to 0$ and $\frac{1}{n^2} \to 0$:

$$\frac{0+0}{1+0} = 0$$

(iii) $c_n = \dfrac{5n^2}{2n+3}$

Divide by n:

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \frac{\frac{5n^2}{n}}{\frac{2n}{n}+\frac{3}{n}} = \lim_{n\to\infty} \frac{5n}{2+\frac{3}{n}}$$

As $n \to \infty, \frac{3}{n} \to 0$:

$$\frac{5(\infty)}{2+0} = \frac{\infty}{2} = \infty$$

The limit does not exist (diverges).

(iv) $d_n = \dfrac{n^2-3n+1}{2n^2+n+4}$

Divide by n^2:

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} \frac{\frac{n^2}{n^2}-\frac{3n}{n^2}+\frac{1}{n^2}}{\frac{2n^2}{n^2}+\frac{n}{n^2}+\frac{4}{n^2}} = \lim_{n\to\infty} \frac{1-\frac{3}{n}+\frac{1}{n^2}}{2+\frac{1}{n}+\frac{4}{n^2}}$$

As $n \to \infty, \frac{3}{n} \to 0, \frac{1}{n^2} \to 0,$ and $\frac{4}{n^2} \to 0$:

$$\frac{1-0+0}{2+0+0} = \frac{1}{2}$$

Q2. Evaluate each limit by using theorems of limits:

(i) $\lim_{x\to 3}(2x+4)$

Sol: $\lim_{x\to 3}(2x+4) = \lim_{x\to 3}(2x) + \lim_{x\to 3}(4) = 2\lim_{x\to 3}(x) + 4$

$2(3)+4 = 6+4 = 10$

(ii) $\lim_{x\to 1}(3x^2-2x+4)$

Sol: $\lim_{x\to 1}(3x^2-2x+4) = 3\lim_{x\to 1}(x^2) - 2\lim_{x\to 1}(x) + 4$

$3(1)^2 - 2(1) + 4 = 3-2+4 = 5$

(iii) $\lim_{x\to 3}\sqrt{x^2+x+4}$

Sol: $\lim_{x\to 3}\sqrt{x^2+x+4} = \sqrt{\lim_{x\to 3}x^2 + \lim_{x\to 3}x + \lim_{x\to 3}4}$

$= \sqrt{(3)^2+3+4} = \sqrt{9+3+4} = \sqrt{16} = 4$

(iv) $\lim_{x\to 2}\sqrt{x^2+4}$

Sol: $\lim_{x\to 2}\sqrt{x^2+4} = \sqrt{\lim_{x\to 2}x^2 + \lim_{x\to 2}4}$

$= \sqrt{(2)^2+4} = \sqrt{4+4} = \sqrt{8}$

(v) $\lim_{x\to 2}\left(\sqrt{x^3+1} - \sqrt{x^2+5}\right)$

Sol: $\lim_{x\to 2}\left(\sqrt{x^3+1} - \sqrt{x^2+5}\right) = \lim_{x\to 2}\sqrt{x^3+1} - \lim_{x\to 2}\sqrt{x^2+5}$

$= \sqrt{(2)^3+1} - \sqrt{(2)^2+5} = \sqrt{8+1} - \sqrt{4+5}$

$= \sqrt{9} - \sqrt{9} = \cancel{3} - \cancel{3} = 0$

(vi) $\lim\limits_{x\to -2}\dfrac{2x^3+5x}{3x-2}$

Sol: $\lim\limits_{x\to -2}\dfrac{2x^3+5x}{3x-2}=\dfrac{\lim\limits_{x\to -2}\left(2x^3+5x\right)}{\lim\limits_{x\to -2}(3x-2)}$

$$=\frac{2\lim\limits_{x\to -2}x^3+5\lim\limits_{x\to -2}x}{3\lim\limits_{x\to -2}(x)-2}=\frac{2(-2)^3+5(-2)}{3(-2)-2}$$

$$=\frac{2(-8)+5(-2)}{-6-2}=\frac{-16-10}{-6-2}=\frac{-26}{-8}=\frac{13}{4}$$

Q3. Evaluate each limit by using algebraic techniques.

> **نوٹ:**
> اگر limit put کرنے سے denominator میں zero بنے
> تو limit put نہیں کرنی اسے آگے حل کرنا ہے اور اگر zero
> نہ بنے تو وہیں پر limit put کریں۔

(i) $\lim\limits_{x\to -1}\dfrac{x^3-x}{x+1}$

$$=\lim_{x\to -1}\frac{x(x^2-1)}{x+1}=\lim_{x\to -1}\frac{x(x-1)\cancel{(x+1)}}{\cancel{(x+1)}}$$

$$=\lim_{x\to -1}x(x-1)=-1(-1-1)=-1(-2)=2$$

(ii) $\lim\limits_{x\to 3}\dfrac{x^2-5x+6}{x^2-2x-3}$

$$=\lim_{x\to 3}\frac{(x-2)\cancel{(x-3)}}{\cancel{(x-3)}(x+1)}=\lim_{x\to 3}\left(\frac{x-2}{x+1}\right)=\frac{3-2}{3+1}=\frac{1}{4}$$

(iii) $\lim\limits_{x\to 2}\dfrac{x^3-8}{x^2-5x+6}$

$$=\lim_{x\to 2}\frac{x^3-(2)^3}{x^2-3x-2x+6}=\lim_{x\to 2}\frac{(x-2)(x^2+2x+4)}{x(x-3)-2(x-3)}$$

$$=\lim_{x\to 2}\frac{\cancel{(x-2)}(x^2+2x+4)}{\cancel{(x-2)}(x-3)}=\lim_{x\to 2}\frac{x^2+2x+4}{x-3}$$

$$=\frac{(2)^2+2(2)+4}{2-3}=\frac{4+4+4}{-1}=-12$$

(iv) $\lim\limits_{x\to 1}\dfrac{x^3-3x^2+3x-1}{x^3-x}$

$$=\lim_{x\to 1}\frac{x^3-1-3x^2+3x}{x(x^2-1)}$$

$$=\lim_{x\to 1}\frac{(x-1)(x^2+x+1)-3x(x-1)}{x(x-1)(x+1)}$$

$$=\lim_{x\to 1}\frac{\cancel{(x-1)}(x^2+x+1-3x)}{x\cancel{(x-1)}(x+1)}$$

$$=\lim_{x\to 1}\frac{x^2+1-2x}{x(x+1)}=\frac{1+1-2}{1(1+1)}=\frac{2-2}{2}=\frac{0}{2}=0$$

(v) $\lim\limits_{x\to 2}\dfrac{x^3-6x^2+12x-8}{x^3-4x}$

$$=\lim_{x\to 2}\frac{(x-2)^3}{x(x^2-4)}=\lim_{x\to 2}\frac{(x-2)^3}{x(x-2)(x+2)}$$

$$=\lim_{x\to 2}\frac{(x-2)^2}{x(x+2)}=\frac{(2-2)^2}{2(2+2)}=\frac{(0)^2}{2(4)}=\frac{0}{8}=0$$

(vi) $\lim\limits_{x\to 1}\left(\dfrac{x^4-1}{x^2-3x+2}\right)$

$$=\lim_{x\to 1}\left(\frac{x^4-1}{x^2-3x+2}\right)=\lim_{x\to 1}\frac{(x^2-1)(x^2+1)}{x^2-x-2x+2}$$

$$=\lim_{x\to 1}\frac{\cancel{(x-1)}(x+1)(x^2+1)}{\cancel{(x-1)}(x-2)}=\lim_{x\to 1}\frac{(x+1)(x^2+1)}{(x-2)}$$

$$=\frac{(1+1)\left((1)^2+1\right)}{1-2}=\frac{(2)(2)}{-1}=\frac{4}{-1}=-4$$

(vii) $\lim\limits_{x\to 2}\dfrac{x-2}{\sqrt{x+2}-\sqrt{6-x}}$

$$=\lim_{x\to 2}\frac{x-2}{\sqrt{x+2}-\sqrt{6-x}}$$

$$=\lim_{x\to 2}\frac{x-2}{\sqrt{x+2}-\sqrt{6-x}}\times\frac{\sqrt{x+2}+\sqrt{6-x}}{\sqrt{x+2}+\sqrt{6-x}}$$

$$=\lim_{x\to 2}\frac{(x-2)\left(\sqrt{x+2}+\sqrt{6-x}\right)}{\left(\sqrt{x+2}\right)^2-\left(\sqrt{6-x}\right)^2}$$

$$=\lim_{x\to 2}\frac{(x-2)\left(\sqrt{x+2}+\sqrt{6-x}\right)}{x+2-6+x}$$

$$=\lim_{x\to 2}\frac{(x-2)\left(\sqrt{x+2}+\sqrt{6-x}\right)}{2x-4}$$

$$=\lim_{x\to 2}\frac{(x-2)\left(\sqrt{x+2}+\sqrt{6-x}\right)}{2(x-2)}$$

$$=\lim_{x\to 2}\frac{\cancel{(x-2)}\left(\sqrt{x+2}+\sqrt{6-x}\right)}{2\cancel{(x-2)}}$$

$$=\frac{\left(\sqrt{2+2}+\sqrt{6-2}\right)}{2}=\frac{\left(\sqrt{4}+\sqrt{4}\right)}{2}=\frac{(2+2)}{2}=\frac{4}{2}=2$$

(viii) $\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$

$= \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\times\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$

$= \lim_{h\to 0}\frac{(\sqrt{x+h})^2-(\sqrt{x})^2}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0}\frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}$

$= \lim_{h\to 0}\frac{h}{h(\sqrt{x+h}+\sqrt{x})} = \frac{1}{\sqrt{x+0}+\sqrt{x}}$

$= \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$

(ix) $\lim_{x\to a}\frac{x^n-a^n}{x^m-a^m}$

(divide Numerator and denominator by (x - a))

$= \frac{\lim_{x\to a}\frac{x^n-a^n}{x-a}}{\lim_{x\to a}\frac{x^m-a^m}{x-a}}$

$= \frac{na^{n-1}}{ma^{m-1}}$ (Use Theorem)

$= \frac{n}{m}a^{n-1-m+1} = \frac{n}{m}a^{n-m}$

Q4. **Evaluate the following limits.**

(i) $\lim_{x\to 0}\frac{\sin 5x}{x}$

Sol: $\lim_{x\to 0}\frac{\sin 5x}{x}$

$= \lim_{x\to 0}\frac{\sin 5x}{x}\times 5$ ('×'&'÷' by 5)

$= (1)\times 5 = 5$

(ii) $\lim_{x\to 0}\frac{\sin x^\circ}{x}$

Note:

$x^\circ = x\times 1^\circ = x\times\frac{\pi}{180} = \frac{x\pi}{180}$

Sol: $\lim_{x\to 0}\frac{\sin x^\circ}{x} = \lim_{x\to 0}\frac{\sin\frac{\pi x}{180}}{x}$

$= \lim_{x\to 0}\frac{\sin x\times\frac{\pi}{180}}{x\times\frac{\pi}{180}}\times\frac{\pi}{180}$ $\left('\times'\&'\div'\ by\ \frac{\pi}{180}\right)$

$= \lim_{x\to 0}(1)\times\frac{\pi}{180} = \frac{\pi}{180}$

(iii) $\lim_{\theta\to 0}\frac{1-\cos\theta}{\sin\theta}$

$= \lim_{\theta\to 0}\frac{2\sin^2\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)} = \frac{\lim_{\theta\to 0}\sin\left(\frac{\theta}{2}\right)}{\lim_{\theta\to 0}\cos\left(\frac{\theta}{2}\right)}$

$= \frac{\sin 0}{\cos 0} = \frac{0}{1} = 0$

(iv) $\lim_{x\to\frac{\pi}{4}}\frac{\sin x-\cos x}{x-\frac{\pi}{4}}$

By L. Hopital rule

$\lim_{x\to\frac{\pi}{4}}\frac{\sin x-\cos x}{x-\frac{\pi}{4}} = \lim_{x\to\frac{\pi}{4}}\frac{\cos x+\sin x}{1-0}$

$= \cos\left(\frac{\pi}{4}\right)+\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}$

$= \frac{2}{\sqrt{2}} = \frac{\sqrt{2}.\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}.\sqrt{2}}{\sqrt{2}} = \sqrt{2}$

(v) $\lim_{x\to 0}\frac{\cos ax-\cos bx}{x^2}$

$\lim_{x\to 0}\frac{\cos ax-\cos bx}{x^2}$ $\left(\frac{0}{0}\right)$

By L. Hopital rule

$= \lim_{x\to 0}\frac{-a\sin ax+b\sin bx}{2x}$

$= \lim_{x\to 0}\frac{-a^2\cos ax+b^2\cos bx}{2}$

$= \frac{-a^2\cos 0+b^2\cos(b.0)}{2}$

$= \frac{-a^2+b^2}{2} = \frac{b^2-a^2}{2}$

(vi) $\lim_{x\to\frac{\pi}{4}}\frac{\tan x-1}{x-\frac{\pi}{4}}$

$\lim_{x\to\frac{\pi}{4}}\frac{\tan x-1}{x-\frac{\pi}{4}}$ $\left(\frac{0}{0}\right)$

$= \lim_{x\to\frac{\pi}{4}}\frac{\sec^2 x}{1}$ By L. Hopital rule

$$= \lim_{x \to \frac{\pi}{4}} \left(\sec^2 x\right)$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{1}{\cos^2 x} = \frac{1}{\left(\cos\left(\frac{\pi}{4}\right)\right)^2} = \frac{1}{\left(\frac{1}{\sqrt{2}}\right)^2} = \frac{1}{\frac{1}{2}} = 2$$

(vii) $\lim_{x \to 0} \frac{1 - \cos 2x}{x^2}$

$$= \lim_{x \to 0} \frac{2\sin^2 x}{x^2} = 2\left(\lim_{x \to 0} \frac{\sin x}{x}\right)^2 = 2(1)^2 = 2$$

(viii) $\lim_{x \to 0} \frac{\cos ax - \cos bx}{\cos cx - \cos dx}$

$$\lim_{x \to 0} \frac{\cos ax - \cos bx}{\cos cx - \cos dx} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{-a \sin ax + b \sin bx}{-c \sin cx + d \sin dx}$$ By L. Hopital rule

$$= \lim_{x \to 0} \frac{-a^2 \cos ax + b^2 \cos bx}{-c^2 \cos cx + d^2 \cos dx}$$

$$= \frac{-a^2 \cos(0) + b^2 \cos(0)}{-c^2 \cos(0) + d^2 \cos(0)}$$

$$= \frac{-a^2(1) + b^2(1)}{-c^2(1) + d^2(1)} = \frac{b^2 - a^2}{d^2 - c^2}$$

(ix) $\lim_{x \to 1} \frac{x^3 - 1}{x^2 - 1}$

$$= \lim_{x \to 1} \frac{(x-1)(x^2 + x + 1)}{(x-1)(x+1)} = \frac{(1)^2 + (1) + 1}{1+1} = \frac{3}{2}$$

(x) $\lim_{x \to 3} \frac{x^2 - x\log x + 3\log x - 9}{x - 3}$

$$\lim_{x \to 3} \frac{x^2 - x\log x + 3\log x - 9}{x - 3} \quad \left(\frac{0}{0}\right)$$

By L. Hopital rule

$$= \lim_{x \to 3} \frac{2x - \left(x\left(\frac{1}{x}\right) + 1.\log x\right) + \frac{3}{x} - 0}{1}$$

$$= \lim_{x \to 3} \left(2x - 1 - \ln x + \frac{3}{x}\right) = 2(3) - \ln(3) - 1 + \frac{3}{3}$$

$$= 6 - \ln 3$$

(xi) $\lim_{x \to 0} \frac{x(2^x - 1)}{1 - \cos x}$

$$\lim_{x \to 0} \frac{x(2^x - 1)}{1 - \cos x} \quad \left(\frac{0}{0}\right)$$

By L. Hopital rule

$$= \lim_{x \to 0} \frac{2^x - 1 + x2^x \ln 2}{\sin x} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{2^x \ln 2 - 0 + 2^x \ln 2 + x \ln 2.2^x \ln 2}{\cos x}$$

$$= \lim_{x \to 0} \frac{2.2^x \ln 2 + x.2^x (\ln 2)^2}{\cos x}$$

$$= \frac{2.2^0 \ln 2 + 0.2^0 (\ln 2)^2}{\cos(0)} = \frac{2\ln 2}{1} = 2\ln 2$$

Q5. Express each limit in terms of e:

(i) $\lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^{2n}$

$$= \left[\lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^n\right]^2 = e^2$$

(ii) $\lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^{n/2}$

$$= \left[\lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^n\right]^{\frac{1}{2}} = e^{\frac{1}{2}} = \sqrt{e}$$

> **نوٹ:**
> کچھ سوال L. Hopital rule سے حل کیے ہیں اگر $\left(\frac{0}{0}\right)$ فارم بن جائے تو Nomerator کا ایک Derivative لینا ہے۔ اور Denomerator کا ایک Derivative لینا ہے۔ پھر اس میں Put limit کرنی ہے۔

(iii) $\lim_{n \to +\infty} \left(1 - \frac{1}{n}\right)^n = \lim_{n \to +\infty} \left[1 + \left(-\frac{1}{n}\right)\right]^n$

$$= \left[\lim_{n \to +\infty} \left(1 + \left(\frac{-1}{n}\right)\right)^{-n}\right]^{-1} = e^{-1} = \frac{1}{e}$$

(iv) $\lim_{n \to +\infty} \left(1 + \frac{1}{3n}\right)^n$

$$= \left[\lim_{n \to +\infty} \left(1 + \frac{1}{3n}\right)^{3n}\right]^{\frac{1}{3}} = e^{\frac{1}{3}}$$

(v) $\lim_{n\to+\infty}\left(1+\frac{4}{n}\right)^{n}$

$$= \lim_{n\to+\infty}\left(1+\frac{4}{n}\right)^{4n/4} = \left[\lim_{n\to+\infty}\left(1+\frac{4}{n}\right)^{n/4}\right]^4 = e^4$$

(vi) $\lim_{x\to 0}(1+3x)^{2/x}$

$$= \lim_{x\to 0}(1+3x)^{\frac{6}{3x}} = \left[\lim_{x\to 0}(1+3x)^{1/3x}\right]^6 = e^6$$

(vii) $\lim_{x\to 0}(1+2x^2)^{\frac{1}{x^2}}$

$$= \lim_{x\to 0}\left(1+2x^2\right)^{\frac{2}{2x^2}} = \left[\lim_{x\to 0}\left(1+2x^2\right)^{1/2x^2}\right]^2 = e^2$$

(viii) $\lim_{x\to 0}\frac{e^{ax}-e^{bx}}{abx}$

By L. Hopital rule

$$= \lim_{x\to 0}\frac{ae^{ax}-be^{bx}}{ab} = \frac{ae^0-be^0}{ab} = \frac{a-b}{ab} = \frac{1}{b}-\frac{1}{a}$$

(ix) $\lim_{x\to\infty}\left(\frac{x}{1+x}\right)^{x} = \lim_{x\to\infty}\left(\frac{1+x}{x}\right)^{-x}$

$$= \lim_{x\to\infty}\left(\frac{1}{x}+\frac{x}{x}\right)^{-x}$$

$$= \lim_{x\to\infty}\left(\frac{1}{x}+1\right)^{-x} = \left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\right]^{-1} = e^{-1} = \frac{1}{e}$$

(x) $\lim_{x\to 0}\frac{e^{1/x}-1}{e^{1/x}+1}, x<0,$

$\Rightarrow$ *Put* $x=-y$ *when* $x\to 0$ *then* $y\to 0$

$$= \lim_{y\to 0}\frac{e^{1/-y}-1}{e^{1/-y}+1} = \lim_{y\to 0}\frac{\frac{1}{e^{1/y}}-1}{\frac{1}{e^{1/y}}+1} = \frac{\frac{1}{\infty}-1}{\frac{1}{\infty}+1} = \frac{0-1}{0+1} = -1$$

(xi) $\lim_{x\to 0}\frac{e^{1/x}-1}{e^{1/x}+1}, x>0$

$$= \lim_{x\to 0}\frac{\cancel{e^{1/x}}\left(1-\frac{1}{e^{1/x}}\right)}{\cancel{e^{1/x}}\left(1+\frac{1}{e^{1/x}}\right)} = \frac{1-\frac{1}{\infty}}{1+\frac{1}{\infty}} = \frac{1-0}{1+0} = 1$$

(xii) $\lim_{x\to 2}\frac{e^{x}-e^{2}}{x-2} \quad \left(\frac{0}{0}\right)$

By L. Hopital rule

$$\lim_{x\to 2}\frac{e^{x}}{1} = e^2$$