Exercise 13.3

Q.1. **A car's position at time t is given by $s(t)=5t^3-3t^2+t$. Find the velocity by differentiating the position function with respect to time.**

Sol: $s(t)=5t^3-3t^2+t$

Take derivative w.r.t. t

$v(t)=s'(t)=5(3t^2)-3(2t)+1=15t^2-6t+1$

The velocity function is $v(t)=15t^2+6t+1$

Q.2. **Structural stress on a bridge is modeled by the function $S(x)=100-5x^2$, where x is the distance from the center of the bridge. Calculate the rate of change of stress at that point.**

Sol: Given the stress function: $S(x)=100-5x^2$

1. **Find the first derivative** $\frac{dS}{dx}=-10x$

2. **Set the first derivative to zero to find critical points:** $-10x=0\Rightarrow x=0$

3. **Find the second derivative:** $\frac{d^2S}{dx^2}=-10$

Since the second derivative is negative, $x=0$ to a maximum.

4. **Calculate the rate of change of stress at $x=0$ (Which is the value of the first derivative at that point):**

$\frac{dS}{dx}|_{x=0}=-10(0)=0$

Therefore, the stress is maximum at $x=0$, and the rate of change of stress at that point is 0.

Q.3. **A Company's revenue function is given by $R(x)=1000x-10x^2$, where is the number of units produced. The cost function is $C(x)=300x+2000$.**

i. **Find the profit function $P(x)$**

ii. **Determine the marginal profit when $x=15$**

Sol: Calculate the profit function $P(x)$

$P(x)=R(x)-C(x)$

Substitute the given revenue and cost functions:

$P(x)=(1000x-10x^2)-(300x+2000)$

Simplify the expression $P(x)=1000x-10x^2-300x-2000$

$P(x)=-10x^2+700x-2000$

Determine the marginal profit function $P'(x)$

The marginal profit function is the derivative of the profit function:

$$P'(x)=\frac{d}{dx}\left(-10x^2+700x-2000\right)$$

Apply the power rule: $P'(x)=-20x+700$

Calculate the marginal profit at $x=15$

Substitute $x=15$ into the marginal profit function:

$$P'(15)=-20(15)+700=-300+700=400$$

Q.4. An investment grows according to the function $A(t)=10000(1+0.05t)^3$, where $A(t)$ is the value of the investment and t is the time in years.

i. Find the rate of change of the investment after 8 years.

ii. What is the investment value after 8 years?

Sol: **a)** To find the rate of change, we need to find the derivative of the function $A(t)$ with respect to time t.

$$A(t)=10000(1+0.05t)^3$$

Take derivative w.r.t. t

$$\frac{dA}{dt}=10000.3(1+0.05t)^{3-1}.\frac{d}{dt}(1+0.05t)$$

$$\frac{dA}{dt}=30000(1+0.05t)^2.(0.05)=1500(1+0.05t)^2$$

Now, to find the rate of change after 8 years, we substitute $t=8$ into the derivative:

$$\frac{dA}{dt}|_{t=8}=1500(1+(0.05)(8))^2=1500(1+0.4)^2$$

$$\frac{dA}{dt}|_{t=8}=1500(1.4)^2=1500(1.96)=2940$$

So, the rate of change of the investment after 8 years is 2940 per year.

b) What is the investment value after 8 years?

Sol: To find the investment value after 8 years, we substitute $t=8$ into the original function $A(t)$:

$$A(8)=10000(1+0.05.8)^3$$

$$A(8)=10000(1+0.4)^3=10000(1.4)^3=10000(2.744)=27440$$

Therefore, the investment value after 8 years is 27440.

Q.5. The position of a particle moving along a line is given by $S(t)=5t^3-12t^2+8t$ where $S(t)$ is the position in meters and t is the time in seconds.

i. Determine the velocity of the particle at $t=4$ seconds

ii. Find the acceleration at $t=4$ seconds

iii. When is the particle at rest?

a) Velocity at $t=4$ seconds

Sol: 1. Position function: $s(t)=5t^3-12t^2+8t$

2. Velocity function (first derivative):

$$v(t)=\frac{ds}{dt}=15t^2-24t+8$$

3. at $t=4$:

$$v(4)=15(4)^2-24(4)+8$$

$$v(4)=15(16)-96+8=240-96+8=152 \quad m/s$$

b) Acceleration at $t = 4$ seconds:

Sol: 1. Velocity function: $v(t) = 15t^2 - 24t + 8$

2. Acceleration function (first derivative of velocity, or second derivative of position):

$$a(t) = \frac{dv}{dt} = 30t - 24$$

3. Evaluate at $t = 4$:

$$a(4) = 30(4) - 24$$

$$a(4) = 120 - 24$$

$$a(4) = 96 \quad m/s^2$$

c) When is the particle at rest?

Sol: $v(t) = 15t^2 - 24t + 8 = 0$

When particle is at rest put $v(t) = 0$

$$\Rightarrow 15t^2 - 24t + 8 = 0$$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Where $a = 15, b = -24, c = 8$

$$t = \frac{-(-24) \pm \sqrt{(-24)^2 - 4(15)(8)}}{2(15)} = \frac{24 \pm \sqrt{576 - 480}}{30} = \frac{24 \pm \sqrt{96}}{30}$$

$$t = \frac{24 \pm \sqrt{96}}{30} = \frac{24 \pm 4\sqrt{6}}{30} = \frac{12 \pm 2\sqrt{6}}{15}$$

The particle is at rest at $t = \frac{12 + 2\sqrt{6}}{15}$ seconds and $t = \frac{12 - 2\sqrt{6}}{15}$ seconds.

Q.6. The Position of a car traveling along a straight highway is given by $x(t) = 30t^2 - 4t^3$, where $x(t)$ is the distance traveled in kilometers and t is the time in hours.

a. Find the car's velocity at $t = 3$ hours. **b. Determine the car's acceleration at $t = 3$ hours.**

a) Velocity at $t = 3$ hours.

Sol: 1. Position function: $x(t) = 30t^2 - 4t^3$

2. Velocity function (first derivative):

$$v(t) = \frac{dx}{dt} = 60t - 12t^2$$

3. Evaluate at $t = 3$:

$$v(3) = 60(3) - 12(3)^2 = 180 - 108 = 72 \quad km/h$$

b) Acceleration at $t = 3$ hours.

Sol: 1. Velocity function $v(t) = 60t - 12t^2$

2. Acceleration function (first derivative of velocity):

$$a(t) = \frac{dv}{dt} = 60 - 24t$$

3. Evaluate at $t = 3$:

$$a(3) = 60 - 24(3)$$

$$a(3) = 60 - 72 = -12 \quad km/h^2$$

Q.7. **The stress on a beam under a varying load is given by $S(x) = 400x - x^3$ where $S(x)$ is the stress in pascals (Pa) and x is the distance from the fixed end in metres. Calculate the rate of change of stress of at 6 meters.**

Sol: 1. Stress function: $S(x) = 400x - x^3$

2. First derivative (rate of change):

$$\frac{dS}{dx} = 400 - 3x^2$$

3. Evaluate at $x = 6$:

$$\frac{dS}{dx}\Big|_{x=6} = 400 - 3(6)^2$$

$$\frac{dS}{dx}\Big|_{x=6} = 400 - 108$$

$$\frac{dS}{dx}\Big|_{x=6} = 292 \;\; Pa/m$$

Q.8. **The cost $C(r)$ to construct a cylindrical tank depends on the radius of the base, and is given by $C(r) = 8000\pi r^2 + \frac{150000}{r}$, where the first term represents the cost of the base and the second term represents the cost of the walls. Determine the rate of change of the cost at $r = 4$ metres.**

Sol: Find the derivative of $C(r)$

$$C(r) = 8000\pi r^2 + \frac{150000}{r}$$

$$C'(r) = 16000\pi r - \frac{150000}{r^2}$$

$$C'(4) = 16000\pi(4) - \frac{150000}{4^2} = 64000\pi - \frac{150000}{16}$$

$$C'(4) = 64000\pi - 9375 \approx \$191540.79$$