Exercise 14.1

Q.1. **Let $\underline{u}=3\underline{i}+2\underline{j}-5\underline{k}$, $\underline{v}=\underline{i}-5\underline{j}-\underline{k}$ and $\underline{w}=-4\underline{i}-\underline{j}+7\underline{k}$. Find the following:**

i. $\underline{u}+2\underline{v}+\underline{w}$

Sol: By using the value of $\underline{u},\underline{v}$ and $\underline{w}$

$$\underline{u}+2\underline{v}+\underline{w}=3\underline{i}+2\underline{j}-5\underline{k}+2\left(\underline{i}-5\underline{j}-\underline{k}\right)+\left(-4\underline{i}-\underline{j}+7\underline{k}\right)$$

$$=3\underline{i}+2\underline{j}-5\underline{k}+2\underline{i}-10\underline{j}-2\underline{k}-4\underline{i}-\underline{j}+7\underline{k}=\underline{i}-9\underline{j}+0\underline{k}=\underline{i}-9\underline{j}$$

ii. $\underline{v}-3\underline{w}$

Sol: By using the value of $\underline{v}$ and $\underline{w}$

$$\underline{v}-3\underline{w}=\underline{i}-5\underline{j}-\underline{k}-3\left(-4\underline{i}-\underline{j}+7\underline{k}\right)=\underline{i}-5\underline{j}-\underline{k}+12\underline{i}+3\underline{j}-21\underline{k}=13\underline{i}-2\underline{j}-22\underline{k}$$

iii. $|3\underline{v}+\underline{w}|$

Sol: $3\underline{v}+\underline{w}=3\left(\underline{i}-5\underline{j}-\underline{k}\right)+\left(-4\underline{i}-\underline{j}+7\underline{k}\right)=3\underline{i}-15\underline{j}-3\underline{k}-4\underline{i}-\underline{j}+7\underline{k}=-\underline{i}-16\underline{j}+4\underline{k}$

$$|3\underline{v}+\underline{w}|=\sqrt{(-1)^2+(-16)^2+(4)^2}=\sqrt{1+256+16}=\sqrt{273}$$

Q.2. **Find the magnitude of the vector $\underline{v}$ and write the direction cosines of $\underline{v}$.**

i. $\underline{v}=3\underline{i}-2\underline{j}+6\underline{k}$

Sol: $|\underline{v}|=\sqrt{(3)^2+(-2)^2+(6)^2}$

$$|\underline{v}|=\sqrt{9+4+36}=\sqrt{49}=7$$

$$\cos\alpha=\frac{a}{|\underline{v}|},\quad \cos\beta=\frac{b}{|\underline{v}|},\quad \cos\gamma=\frac{c}{|\underline{v}|}$$

$$\cos\alpha=\frac{3}{7},\quad \cos\beta=\frac{-2}{7},\quad \cos\gamma=\frac{6}{7}$$

> نوٹ:
> Direction cosine معلوم کرنے کے لیے $\underline{i},\underline{j}$ اور $\underline{k}$ کے
> Co-efficients کو magnitude سے تقسیم کریں الگ الگ

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ii. $\underline{v} = -4\underline{i} + 4\underline{j} + 2\underline{k}$

Sol: $|\underline{v}| = \sqrt{(-4)^2 + (4)^2 + (2)^2} = \sqrt{16+16+4} = \sqrt{36} = 6$

$\cos\alpha = \frac{-4}{6} = \frac{-2}{3}$, $\cos\beta = \frac{4}{6} = \frac{2}{3}$, $\cos\gamma = \frac{2}{6} = \frac{1}{3}$

iii. $\underline{v} = -6\underline{i} + 8\underline{j}$

Sol: $|\underline{v}| = \sqrt{(-6)^2 + (8)^2}$

$|\underline{v}| = \sqrt{36+64} = \sqrt{100} = 10$

$\cos\alpha = \frac{-6}{10}$, $\cos\beta = \frac{8}{7}$

Q.3. Find t, so that $|2\underline{i} + (t-1)\underline{j} + t\underline{k}| = \sqrt{13}$

Sol: $\sqrt{(2)^2 + (t-1)^2 + (t)^2} = \sqrt{13}$

$\sqrt{4 + t^2 + 1 - 2t + t^2} = \sqrt{13}$

$\sqrt{2t^2 - 2t + 5} = \sqrt{13}$

Squaring both sides

$2t^2 - 2t + 5 = 13$

$2t^2 - 2t + 5 - 13 = 0$

$2t^2 - 2t - 8 = 0$

$t^2 - t - 4 = 0$

$a = 1,\ b = -1,\ c = -4$

By using Quadratic formula

$t = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(-4)}}{2(1)} \Rightarrow t = \frac{1 \pm \sqrt{1+16}}{2} = \frac{1 \pm \sqrt{17}}{2}$

$\Rightarrow t = \frac{1 \pm \sqrt{17}}{2}$

Q.4. Find a unit vector in the direction of $\underline{v} = -\underline{i} + 4\underline{j} - 8\underline{k}$

Sol: $|\underline{v}| = \sqrt{(-1)^2 + (4)^2 + (-8)^2}$

$|\underline{v}| = \sqrt{1+16+64} = \sqrt{81} = 9$

$\hat{v} = \frac{\underline{v}}{|\underline{v}|} = -\frac{1}{9}\hat{i} + \frac{4}{9}\hat{j} - \frac{8}{9}\hat{k}$

Q.5. If $\underline{u} = 2\underline{i} + \underline{j} - 3\underline{k}$, $\underline{v} = -\underline{i} + 4\underline{j} + 2\underline{k}$ and $\underline{w} = 3\underline{i} - 2\underline{j} + \underline{k}$ Find a unit vector parallel to $4\underline{u} - 3\underline{v} + 2\underline{w}$.

Sol: $4\underline{u} - 3\underline{v} + 2\underline{w} = 4(2\underline{i} + \underline{j} - 3\underline{k}) - 3(-\underline{i} + 4\underline{j} + 2\underline{k}) + 2(3\underline{i} - 2\underline{j} + \underline{k})$

$= 8\underline{i} + 4\underline{j} - 12\underline{k} + 3\underline{i} - 12\underline{j} - 6\underline{k} + 6\underline{i} - 4\underline{j} + 2\underline{k}$

$= 17\underline{i} - 12\underline{j} - 16\underline{k}$

$|4\underline{u} - 3\underline{v} + 2\underline{w}| = \sqrt{(17)^2 + (-12)^2 + (-16)^2} = \sqrt{289+144+256} = \sqrt{689}$

$\hat{a} = \frac{4\underline{u} - 3\underline{v} + 2\underline{w}}{|4\underline{u} - 3\underline{v} + 2\underline{w}|} = \frac{17}{\sqrt{689}}\underline{i} - \frac{12}{\sqrt{689}}\underline{j} - \frac{16}{\sqrt{689}}\underline{k}$

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Q.6. **Find a vector whose**

i. Magnitude is 5 and is parallel to $3\underline{i}+4\underline{j}-\underline{k}$ **ii. Magnitude is 7 and is parallel to $-\underline{i}+\underline{j}+\underline{k}$**

i. $3\underline{i}+4\underline{j}-\underline{k}$

Sol: $\underline{v}=3\underline{i}+4\underline{j}-\underline{k}$

$$|\underline{v}|=\sqrt{(3)^2+(4)^2+(-1)^2}=\sqrt{9+16+1}=\sqrt{26}$$

$$\hat{v}=\frac{\underline{v}}{|\underline{v}|}=\frac{1}{\sqrt{26}}\left(3\underline{i}+4\underline{j}-\underline{k}\right)$$

Given Magnitude is 5

Required Vector $=\underline{a}=5.\hat{v}=\frac{5}{\sqrt{26}}\left(3\underline{i}+4\underline{j}-\underline{k}\right)=\frac{5}{\sqrt{26}}\left(3\underline{i}+4\underline{j}-\underline{k}\right)=\frac{1}{\sqrt{26}}\left(15\underline{i}+20\underline{j}-5\underline{k}\right)$

ii. $-\underline{i}+\underline{j}+\underline{k}$

Sol: $\underline{v}=-\underline{i}+\underline{j}+\underline{k}$

$$|\underline{v}|=\sqrt{(-1)^2+(1)^2+(1)^2}=\sqrt{1+1+1}=\sqrt{3}$$

$$\hat{v}=\frac{\underline{v}}{|\underline{v}|}=\frac{1}{\sqrt{3}}\left(-\underline{i}+\underline{j}+\underline{k}\right)$$

Given magnitude is7

Required Vector $\underline{a}=7.\hat{v}=\frac{7}{\sqrt{3}}\left(-\underline{i}+\underline{j}+\underline{k}\right)$

$$=\frac{1}{\sqrt{3}}\left(-7\underline{i}+7\underline{j}+7\underline{k}\right)$$

Q.7. **If $\underline{u}=x\underline{i}+2\underline{j}+3\underline{k}$, $\underline{v}=\underline{i}+y\underline{j}-3\underline{k}$ and $\underline{w}=-2\underline{i}-3\underline{j}$ represent the sides of a triangle Find the values of x and y**

Sol: When $\underline{u},\underline{v},\underline{w}$ are sides of triangle then

$\underline{u}+\underline{v}+\underline{w}=0$

$x\underline{i}+2\underline{j}+3\underline{k}+\underline{i}+y\underline{j}-3\underline{k}-2\underline{i}-3\underline{j}=0$

$x\underline{i}-\underline{i}+y\underline{j}-\underline{j}+0\underline{k}=0 \; or \; (x-1)\underline{i}+(y-1)\underline{j}+0\underline{k}=0\underline{i}+0\underline{j}+0\underline{k}$

$\Rightarrow x-1=0 \; and \; y-1=0 \Rightarrow x=1 \; and \; y=1$

Q.8. **The position vectors of the points A,B,C and D are $\underline{u}=\underline{i}+2\underline{j}+\underline{k}$, $\underline{v}=7\underline{i}+8\underline{j}+4\underline{k}$, $\underline{w}=-\underline{i}+\underline{k}$ and $\underline{z}=\underline{i}+2\underline{j}+2\underline{k}$ respectively. Show that $\overrightarrow{AB}$ is parallel $\overrightarrow{CD}$**

Sol: $\overrightarrow{AB}=\underline{v}-\underline{u}=7\underline{i}+8\underline{j}+4\underline{k}-\underline{i}-2\underline{j}-\underline{k}=6\underline{i}+6\underline{j}+3\underline{k}$

$\overrightarrow{CD}=\underline{z}-\underline{w}=\underline{i}+2\underline{j}+2\underline{k}+\underline{i}-\underline{k}=2\underline{i}+2\underline{j}+\underline{k}$

To check if $\overrightarrow{AB}$ is paralleled to $\overrightarrow{CD}$.

$\overrightarrow{AB}=6\underline{i}\times 6\underline{j}+3\underline{k}=3\left(2\underline{i}+2\underline{j}+\underline{k}\right)$

$\overrightarrow{AB}=3(\overrightarrow{CD})$

$\overrightarrow{AB}$ is paralleled to $\overrightarrow{CD}$.

نوٹ:

اگر دو ویکٹر آپس میں برابر ہیں یا درمیان میں کوئی constant ہے

تو Parallel ہوگا $\underline{u}=\lambda\underline{v}$ or $\underline{u}=\underline{v}$

Q.9. We say that two vectors $\underline{v}$ and $\underline{w}$ in space are parallel if there is a scalar c such that $\underline{v} = c\underline{w}$. The vectors point in the same direction if $c > 0$ and the vectors point in the opposite direction if $c < 0$

a) Find two vectors of length 2 parallel to vector $\underline{v} = 2\underline{i} - 4\underline{j} + 4\underline{k}$

b) Find the constant a so that the vectors $\underline{v} = \underline{i} - 3\underline{j} + 4\underline{k}$ and $\underline{w} = a\underline{i} + 9\underline{j} - 12\underline{k}$ are parallel.

c) Find a vector of length 5 in the direction opposite that of $\underline{v} = \underline{i} - 2\underline{j} + 3\underline{k}$.

d) Find a and b so that the vectors $3\underline{i} - \underline{j} + 4\underline{k}$ and $a\underline{i} + b\underline{j} - 2\underline{k}$ are parallel.

a) Find two vectors of length 2 parallel to vector $\underline{v} = 2\underline{i} - 4\underline{j} + 4\underline{k}$

Sol: $|\underline{v}| = \sqrt{(2)^2 + (-4)^2 + (4)^2} = \sqrt{4+16+16} = \sqrt{36} = 6$

$\hat{v} = \frac{1}{6}\left(2\underline{i} - 4\underline{j} + 4\underline{k}\right) = \frac{1}{3}\underline{i} - \frac{2}{3}\underline{j} + \frac{2}{3}\underline{k}$

Required vector $= 2\hat{v} = \frac{2}{3}\underline{i} - \frac{4}{3}\underline{j} + \frac{4}{3}\underline{k}$.

Vectors in same direction: $+2\hat{v} = \frac{2}{3}\underline{i} - \frac{4}{3}\underline{j} + \frac{4}{3}\underline{k}$

Vectors in opposite direction $-2\hat{v} = -\frac{2}{3}\underline{i} + \frac{4}{3}\underline{j} - \frac{4}{3}\underline{k}$

b) Find the constant a so that the vectors $\underline{v} = \underline{i} - 3\underline{j} + 4\underline{k}$ and $\underline{w} = a\underline{i} + 9\underline{j} - 12\underline{k}$ are parallel.

Sol: $\underline{v} = c\underline{w}$

$\underline{i} - 3\underline{j} + 4\underline{k} = c\left(a\underline{i} + 9\underline{j} - 12\underline{k}\right)$

$\underline{i} - 3\underline{j} + 4\underline{k} = ac\underline{i} + 9c\underline{j} - 12c\underline{k}$

Comparison of $\underline{j}$ component $-3 = 9c \Rightarrow c = -\frac{1}{3}$

By comparing $\underline{i}$ component $ac = 1 \Rightarrow a\left(\frac{-1}{3}\right) = 1 \Rightarrow a = -3$

c) Find a vector of length 5 in the direction opposite that of $\underline{v} = \underline{i} - 2\underline{j} + 3\underline{k}$.

Sol: $|\underline{v}| = \sqrt{(1)^2 + (-2)^2 + (3)^2} = \sqrt{1+4+9} = \sqrt{14} \Rightarrow \hat{v} = \frac{1}{\sqrt{14}}(i - 2j + 3k)$

Required vector $= -5.\hat{v} = \frac{-5}{\sqrt{14}}\left(\underline{i} - 2\underline{j} + 3\underline{k}\right) = \frac{-5}{\sqrt{14}}\underline{i} + \frac{10}{\sqrt{14}}\underline{j} - \frac{15}{\sqrt{14}}\underline{k}$

d) Find a and b so that the vectors $3\underline{i} - \underline{j} + 4\underline{k}$ and $a\underline{i} + b\underline{j} - 2\underline{k}$ are parallel.

Sol: $\underline{v} = \underline{c}.\underline{w}$

$\Rightarrow 3\underline{i} - \underline{j} + 4\underline{k} = c\left(a\underline{i} + b\underline{j} - 2\underline{k}\right)$

$\Rightarrow 3\underline{i} - \underline{j} + 4\underline{k} = ac\underline{i} + bc\underline{j} - 2c\underline{k}$

By comparison of component

$3 = ac$ ____ (I), $-1 = bc$ and $4 = -2c \Rightarrow c = -2$

Put values of c in I and II

$3 = (-2)a$ $\qquad$ $-1 = b(-2)$

$\Rightarrow a = \frac{-3}{2}$, $\qquad$ $\Rightarrow b = \frac{1}{2}$

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Q.10 **A spacecraft moves from point $(120,240,-50)$ to point $(130,210,80)$ in kilometers. What is the magnitude of the displacement vector in kilometers?**

Sol: Final point $=(130,210,80)$

Initial point $=(120,240,-50)$

Displacement $=(130-120,210-240,80-(-50))=(10,-30,130)$

$$|\underline{d}|=\sqrt{(10)^2+(-30)^2+(130)^2}=\sqrt{100+900+16900}=\sqrt{17900}$$

$$|\underline{d}|\approx 133.77\,km$$

Q.11 **Find the direction cosines for the given vector:**

i. $\underline{u}=-6\underline{i}+3\underline{j}+2\underline{k}$ ii. $\underline{v}=4\underline{i}+2\underline{j}-5\underline{k}$

iii. $\overrightarrow{PQ}$, **where** $P(9,3,13)$ **and** $Q(11,6,19)$.

i. $\underline{u}=-6\underline{i}+3\underline{j}+2\underline{k}$

Sol: $|\underline{u}|=\sqrt{(-6)^2+(3)^2+(2)^2}=\sqrt{36+9+4}=\sqrt{49}=7$

Direction cosines are

$$\cos\alpha=-\frac{6}{7},\quad \cos\beta=\frac{3}{7},\quad \cos\gamma=\frac{2}{7}$$

ii. $\underline{v}=4\underline{i}+2\underline{j}-5\underline{k}$

Sol: $|\underline{v}|=\sqrt{(4)^2+(2)^2+(-5)^2}=\sqrt{16+4+25}=\sqrt{45}=3\sqrt{5}$

Direction cosines are

$$\cos\alpha=\frac{4}{3\sqrt{5}},\quad \cos\beta=\frac{2}{3\sqrt{5}},\quad \cos\gamma=\frac{-5}{3\sqrt{5}}$$

iii. $\overrightarrow{PQ}$, **where** $P(9,3,13)$ **and** $Q(11,6,19)$.

Sol: $\overrightarrow{PQ}=(11,6,19)-(9,3,13)=(11-9,6-3,19-13)=(2,3,6)$

$$\overrightarrow{PQ}=2\underline{i}+3\underline{j}+6\underline{k}$$

$$\left|\overrightarrow{PQ}\right|=\sqrt{(2)^2+(3)^2+(6)^2}=\sqrt{4+9+36}=\sqrt{49}=7.$$

Direction cosines are $\cos\alpha=\frac{2}{7},\quad \cos\beta=\frac{3}{7},\quad \cos\gamma=\frac{6}{7}$

Q.12. **Which of the following triple can be the direction angles of a single vector?**

i. $45°,45°,60°$ ii. $30°,45°,60°$ iii. $45°,60°,60°$

i. $45°,45°,60°$

Sol: Here $\cos\alpha=\cos 45°=\frac{1}{\sqrt{2}},\cos\beta=\cos 45°=\frac{1}{\sqrt{2}},\cos\gamma=\cos 60°=\frac{1}{2}$

Now $\cos^2\alpha+\cos^2\beta+\cos^2\gamma$

$$=\cos^2 45°+\cos^2 45°+\cos^2 60°=\left(\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{2}\right)^2=\frac{1}{2}+\frac{1}{2}+\frac{1}{4}=1+\frac{1}{4}=\frac{5}{4}\neq 1$$

These angles cannot be the direction angles of a single vector.

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ii. $30^\circ, 45^\circ, 60^\circ$

Sol: Here $\cos\alpha = \cos 30^\circ = \frac{\sqrt{3}}{2} \Rightarrow \cos\beta = \cos 45^\circ = \frac{1}{\sqrt{2}} \Rightarrow \cos\gamma = \cos 60^\circ = \frac{1}{2}$

Now $\cos^2\alpha + \cos^2\beta + \cos^2\gamma$

$$= \cos^2 30^\circ + \cos^2 45^\circ + \cos^2 60^\circ = \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{3}{4} + \frac{1}{2} + \frac{1}{4} = \frac{3}{4} + \frac{2}{4} + \frac{1}{4} = \frac{3+2+1}{4} = \frac{6}{4} = \frac{3}{2} \neq 1$$

These angles cannot be the direction angles of a single vector.

iii. $45^\circ, 60^\circ, 60^\circ$

Sol: Here $\cos\alpha = \cos 45^\circ = \frac{1}{\sqrt{2}} \Rightarrow \cos\beta = \cos 60^\circ = \frac{1}{2} \Rightarrow \cos\gamma = \cos 60^\circ = \frac{1}{2}$

Now $\cos^2\alpha + \cos^2\beta + \cos^2\gamma$

$$= \cos^2 45^\circ + \cos^2 60^\circ + \cos^2 60^\circ = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = \frac{2}{4} + \frac{1}{4} + \frac{1}{4} = \frac{2+1+1}{4} = \frac{4}{4} = 1$$

Since the sum of the squares of the cosines is 1, these angles are the direction angles of a single vector.

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