



Exercise 14.3



Q1: Compute the cross product $\underline{a} \times \underline{b}$ and $\underline{b} \times \underline{a}$. Check your answer by showing that each \underline{a} and \underline{b} are perpendicular to $\underline{a} \times \underline{b}$ and $\underline{b} \times \underline{a}$.

i. $\underline{a} = 2\underline{i} + \underline{j} - \underline{k}$ and $\underline{b} = \underline{i} - \underline{j} + \underline{k}$

Sol: $\underline{a} = 2\underline{i} + \underline{j} - \underline{k}$ and $\underline{b} = \underline{i} - \underline{j} + \underline{k}$

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix} = \underline{i}(1-1) - \underline{j}(2+1) + \underline{k}(-2-1) = 0\underline{i} - 3\underline{j} - 3\underline{k}$$

$$\underline{b} \times \underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix} = \underline{i}(1-1) - \underline{j}(-1-2) + \underline{k}(1+2) = 0\underline{i} + 3\underline{j} + 3\underline{k}$$

$$\text{Now } \underline{a} \cdot (\underline{a} \times \underline{b}) = (2\underline{i} + \underline{j} - \underline{k}) \cdot (-3\underline{j} - 3\underline{k}) = (2)(0) + (1)(-3) + (-1)(-3) = -3 + 3 = 0$$

So, \underline{a} and $\underline{a} \times \underline{b}$ are perpendicular.

$$\underline{a} \cdot (\underline{b} \times \underline{a}) = (2\underline{i} + \underline{j} - \underline{k}) \cdot (3\underline{j} + 3\underline{k}) = (2)(0) + (1)(3) + (-1)(3) = 0 + 3 - 3 = 0$$

So, \underline{a} and $\underline{b} \times \underline{a}$ are perpendicular.

نوٹ:
 اگر \underline{u} اور \underline{v} دو ویکٹرز کا dot product صفر ہوگا تو دونوں ویکٹرز
 Perpendicular ہوں گے اور اگر cross product صفر ہوگا
 تو parallel ہوں گے



$$\underline{b} \cdot (\underline{a} \times \underline{b}) = (\underline{i} - \underline{j} + \underline{k}) \cdot (-3\underline{j} - 3\underline{k}) = (1)(0) + (-1)(-3) + (1)(-3) = 3 - 3 = 0$$

So, \underline{b} and $\underline{a} \times \underline{b}$ are perpendicular.

$$\underline{b} \cdot (\underline{b} \times \underline{a}) = (\underline{i} - \underline{j} + \underline{k}) \cdot (3\underline{j} + 3\underline{k}) = (1)(0) + (-1)(3) + (1)(3) = 3 - 3 = 0$$

So, \underline{b} and $\underline{b} \times \underline{a}$ are perpendicular.

ii. $\underline{a} = \underline{i} + 3\underline{j} + 2\underline{k}$ and $\underline{b} = 2\underline{i} - \underline{j} + \underline{k}$

Sol: $\underline{a} = \underline{i} + 3\underline{j} + 2\underline{k}$ and $\underline{b} = 2\underline{i} - \underline{j} + \underline{k}$

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 3 & 2 \\ 2 & -1 & 1 \end{vmatrix} = \underline{i}(3+2) - \underline{j}(1-4) + \underline{k}(-1-6) = 5\underline{i} + 3\underline{j} - 7\underline{k}$$

$$\underline{b} \times \underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -1 & 1 \\ 1 & 3 & 2 \end{vmatrix} = \underline{i}(-2-3) - \underline{j}(4-1) + \underline{k}(6+1) = -5\underline{i} - 3\underline{j} + 7\underline{k}$$

$$\text{Now } \underline{a} \cdot (\underline{a} \times \underline{b}) = (\underline{i} + 3\underline{j} + 2\underline{k}) \cdot (5\underline{i} + 3\underline{j} - 7\underline{k}) = (1)(5) + (3)(3) + (2)(-7) = 5 + 9 - 14 = 0$$

So, \underline{a} and $\underline{a} \times \underline{b}$ are perpendicular.

$$\underline{a} \cdot (\underline{b} \times \underline{a}) = (\underline{i} + 3\underline{j} + 2\underline{k}) \cdot (-5\underline{i} - 3\underline{j} + 7\underline{k}) = (1)(-5) + (3)(-3) + (2)(7) = -5 - 9 + 14 = 0$$

So, \underline{a} and $\underline{b} \times \underline{a}$ are perpendicular.

$$\underline{b} \cdot (\underline{a} \times \underline{b}) = (2\underline{i} - \underline{j} + \underline{k}) \cdot (5\underline{i} + 3\underline{j} - 7\underline{k}) = (2)(5) + (-1)(3) + (2)(-7) = 10 - 3 - 7 = 0$$

So, \underline{b} and $\underline{a} \times \underline{b}$ are perpendicular.

$$\underline{b} \cdot (\underline{b} \times \underline{a}) = (2\underline{i} - \underline{j} + \underline{k}) \cdot (-5\underline{i} - 3\underline{j} + 7\underline{k}) = (2)(-5) + (-1)(-3) + (1)(7) = -10 + 3 + 7 = 0$$

So, \underline{b} and $\underline{b} \times \underline{a}$ are perpendicular.

iii. $\underline{a} = 2\underline{i} - 2\underline{j} + \underline{k}$ and $\underline{b} = -\underline{i} + \underline{j} + 3\underline{k}$

Sol: $\underline{a} = 2\underline{i} - 2\underline{j} + \underline{k}$ and $\underline{b} = -\underline{i} + \underline{j} + 3\underline{k}$

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -2 & 1 \\ -1 & 1 & 3 \end{vmatrix} = \underline{i}(-6-1) - \underline{j}(6+1) + \underline{k}(2-2) = -7\underline{i} - 7\underline{j} + 0\underline{k}$$

$$\underline{b} \times \underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & 1 & 3 \\ 2 & -2 & 1 \end{vmatrix} = \underline{i}(1+6) - \underline{j}(-1-6) + \underline{k}(2-2) = 7\underline{i} + 7\underline{j} + 0\underline{k}$$

$$\text{Now } \underline{a} \cdot (\underline{a} \times \underline{b}) = (2\underline{i} - 2\underline{j} + \underline{k}) \cdot (-7\underline{i} - 7\underline{j} + 0\underline{k}) = (2)(-7) + (-2)(-7) + (1)(0) = -14 + 14 + 0 = 0$$

So, \underline{a} and $\underline{a} \times \underline{b}$ are perpendicular.

$$\underline{b} \cdot (\underline{a} \times \underline{b}) = (-\underline{i} + \underline{j} + 3\underline{k}) \cdot (-7\underline{i} - 7\underline{j} + 0\underline{k}) = (-1)(-7) + (1)(-7) + (3)(0) = 7 - 7 + 0 = 0$$

So, \underline{b} and $\underline{a} \times \underline{b}$ are perpendicular.

$$\underline{a} \cdot (\underline{b} \times \underline{a}) = (2\underline{i} - 2\underline{j} + \underline{k}) \cdot (7\underline{i} + 7\underline{j} + 0\underline{k}) = (2)(7) + (-2)(7) + (1)(0) = 14 - 14 + 0 = 0$$

So, \underline{a} and $\underline{b} \times \underline{a}$ are perpendicular.

$$\underline{b} \cdot (\underline{b} \times \underline{a}) = (-\underline{i} + \underline{j} + 3\underline{k}) \cdot (7\underline{i} + 7\underline{j} + 0\underline{k}) = (-1)(7) + (1)(7) + (3)(0) = 7 + 7 + 0 = 0$$

So, \underline{b} and $\underline{b} \times \underline{a}$ are perpendicular.

iv. $\underline{a} = -4\underline{i} + \underline{j} - 2\underline{k}$ and $\underline{b} = 2\underline{i} + \underline{j} + \underline{k}$

Sol: $\underline{a} = -4\underline{i} + \underline{j} - 2\underline{k}$ and $\underline{b} = 2\underline{i} + \underline{j} + \underline{k}$

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -4 & 1 & -2 \\ 2 & 1 & 1 \end{vmatrix} = \underline{i}(1+2) - \underline{j}(-4+4) + \underline{k}(-4-2) = 3\underline{i} - 0\underline{j} - 6\underline{k}$$

$$\underline{b} \times \underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 1 & 1 \\ -4 & 1 & -2 \end{vmatrix} = \underline{i}(-2-1) - \underline{j}(-4+4) + \underline{k}(2+4) = -3\underline{i} - 0\underline{j} + 6\underline{k}$$

Now $\underline{a} \cdot (\underline{a} \times \underline{b}) = (-4\underline{i} + \underline{j} - 2\underline{k}) \cdot (3\underline{i} - 0\underline{j} - 6\underline{k}) = (-4)(3) + (1)(-0) + (-2)(-6) = -12 - 0 + 12 = 0$

So, \underline{a} and $\underline{a} \times \underline{b}$ are perpendicular.

$$\underline{b} \cdot (\underline{a} \times \underline{b}) = (2\underline{i} + \underline{j} + \underline{k}) \cdot (3\underline{i} - 0\underline{j} - 6\underline{k}) = (2)(3) + (1)(-0) + (1)(-6) = 6 - 0 - 6 = 0$$

So, \underline{b} and $\underline{a} \times \underline{b}$ are perpendicular.

$$\underline{a} \cdot (\underline{b} \times \underline{a}) = (2\underline{i} + \underline{j} + \underline{k}) \cdot (-3\underline{i} - 0\underline{j} + 6\underline{k}) = (2)(-3) + (1)(-0) + (1)(6) = -6 - 0 + 6 = 0$$

So, \underline{a} and $\underline{b} \times \underline{a}$ are perpendicular.

$$\underline{b} \cdot (\underline{b} \times \underline{a}) = (2\underline{i} + \underline{j} + \underline{k}) \cdot (-3\underline{i} - 0\underline{j} + 6\underline{k}) = (2)(-3) + (1)(-0) + (1)(6) = -6 - 0 + 6 = 0$$

So, \underline{b} and $\underline{b} \times \underline{a}$ are perpendicular.

Q2. Find a unit vector perpendicular to the plane containing \underline{a} and \underline{b} . Also find sine of the angle between them:

i. $\underline{a} = \underline{i} + 6\underline{j} - 3\underline{k}$, $\underline{b} = 2\underline{i} + \underline{j} + 3\underline{k}$

Sol: $\underline{a} = \underline{i} + 6\underline{j} - 3\underline{k}$, $\underline{b} = 2\underline{i} + \underline{j} + 3\underline{k}$

Given in component form:

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 6 & -3 \\ 2 & 1 & 3 \end{vmatrix} = \underline{i}(18+3) - \underline{j}(3+6) + \underline{k}(1-12) = 21\underline{i} - 9\underline{j} - 11\underline{k}$$

$$|\underline{a} \times \underline{b}| = \sqrt{(21)^2 + (-9)^2 + (-11)^2} = \sqrt{442 + 81 + 121} = \sqrt{643}$$

Unit vector perpendicular to the plane:

$$\hat{n} = \frac{\underline{a} \times \underline{b}}{|\underline{a} \times \underline{b}|} = \frac{1}{\sqrt{643}} (21\underline{i} - 9\underline{j} - 11\underline{k})$$

Compute $|\underline{a}|$ and $|\underline{b}|$:

$$|\underline{a}| = \sqrt{(1)^2 + (6)^2 + (-3)^2} = \sqrt{1+36+9} = \sqrt{46}$$

$$|\underline{b}| = \sqrt{(2)^2 + (1)^2 + (3)^2} = \sqrt{4+1+9} = \sqrt{14}$$

Sine of the angle between \underline{a} and \underline{b} :

$$\sin \theta = \frac{|\underline{a} \times \underline{b}|}{|\underline{a}| |\underline{b}|} = \frac{\sqrt{643}}{\sqrt{46} \sqrt{14}} = \frac{\sqrt{643}}{\sqrt{644}}$$

ii. $\underline{a} = -\underline{i} - \underline{j} - \underline{k}, \underline{b} = 2\underline{i} - 3\underline{j} + 4\underline{k}$

Sol: $\underline{a} = -\underline{i} - \underline{j} - \underline{k}, \underline{b} = 2\underline{i} - 3\underline{j} + 4\underline{k}$

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & -1 & -1 \\ 2 & -3 & 4 \end{vmatrix} = \underline{i}(-4-3) - \underline{j}(-4+2) + \underline{k}(3+2) = -7\underline{i} + 2\underline{j} + 5\underline{k}$$

$$|\underline{a} \times \underline{b}| = \sqrt{(-7)^2 + (2)^2 + (5)^2} \\ = \sqrt{49+4+25} = \sqrt{78}$$

Unit vector perpendicular to the plane:

$$\hat{n} = \frac{\underline{a} \times \underline{b}}{|\underline{a} \times \underline{b}|} = \frac{1}{\sqrt{78}}(-7\underline{i} + 2\underline{j} + 5\underline{k})$$

Compute $|\underline{a}|$ and $|\underline{b}|$:

$$|\underline{a}| = \sqrt{(-1)^2 + (-1)^2 + (-1)^2} = \sqrt{1+1+1} = \sqrt{3}$$

$$|\underline{b}| = \sqrt{(2)^2 + (-3)^2 + (4)^2} = \sqrt{4+9+16} = \sqrt{29}$$

Sine of the angle between \underline{a} and \underline{b} :

$$\sin \theta = \frac{|\underline{a} \times \underline{b}|}{|\underline{a}||\underline{b}|} = \frac{\sqrt{78}}{\sqrt{3}\sqrt{29}} = \frac{\sqrt{78}}{\sqrt{3 \times 29}} = \frac{\sqrt{78}}{\sqrt{87}}$$

iii. $\underline{a} = \underline{i} + \underline{j} + \underline{k}, \underline{b} = \underline{i} - \underline{j} - \underline{k}$

Sol: $\underline{a} = \underline{i} + \underline{j} + \underline{k}, \underline{b} = \underline{i} - \underline{j} - \underline{k}$

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{vmatrix}$$

$$= \underline{i}(-1+1) - \underline{j}(-1-1) + \underline{k}(-1-1) = 0\underline{i} + 2\underline{j} - 2\underline{k}$$

$$|\underline{a} \times \underline{b}| = \sqrt{(0)^2 + (2)^2 + (-2)^2} = \sqrt{0+4+4} = \sqrt{8} = 2\sqrt{2}$$

Unit vector perpendicular to the plane:

$$\hat{n} = \frac{\underline{a} \times \underline{b}}{|\underline{a} \times \underline{b}|} = \frac{1}{2\sqrt{2}}(0\underline{i} + 2\underline{j} - 2\underline{k})$$

$$\hat{n} = \frac{1}{\sqrt{2}}(0\underline{i} + \underline{j} - \underline{k})$$

Compute $|\underline{a}|$ and $|\underline{b}|$:

$$|\underline{a}| = \sqrt{(-3)^2 + (2)^2 + (5)^2} \\ = \sqrt{9+4+25} = \sqrt{38}$$

$$|\underline{b}| = \sqrt{(1)^2 + (-1)^2 + (-1)^2} = \sqrt{1+1+1} = \sqrt{3}$$

Sine of the angle between \underline{a} and \underline{b} :

$$\sin \theta = \frac{|\underline{a} \times \underline{b}|}{|\underline{a}||\underline{b}|} = \frac{2\sqrt{2}}{\sqrt{3}\sqrt{3}} = \frac{2\sqrt{2}}{3}$$

iv. $\underline{a} = 5\underline{i} + \underline{j} - 3\underline{k}, \underline{b} = -2\underline{i} + 4\underline{j} + \underline{k}$

Sol: $\underline{a} = 5\underline{i} + \underline{j} - 3\underline{k}, \underline{b} = -2\underline{i} + 4\underline{j} + \underline{k}$

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 5 & 1 & -3 \\ -2 & 4 & 1 \end{vmatrix} = \underline{i}(1+12) - \underline{j}(5-6) + \underline{k}(20+2) = 13\underline{i} + \underline{j} + 22\underline{k}$$

$$|\underline{a} \times \underline{b}| = \sqrt{(13)^2 + (1)^2 + (22)^2} = \sqrt{169+1+484} = \sqrt{654}$$

Unit vector perpendicular to the plane:

$$\hat{n} = \frac{\underline{a} \times \underline{b}}{|\underline{a} \times \underline{b}|} = \frac{1}{\sqrt{654}}(13\underline{i} + \underline{j} + 22\underline{k})$$

Compute $|\underline{a}|$ and $|\underline{b}|$:

$$|\underline{a}| = \sqrt{(5)^2 + (1)^2 + (-3)^2} = \sqrt{25+1+9} = \sqrt{35}$$

$$|\underline{b}| = \sqrt{(-2)^2 + (4)^2 + (1)^2} = \sqrt{4+16+1} = \sqrt{21}$$

Sine of the angle between \underline{a} and \underline{b} :

$$\sin \theta = \frac{|\underline{a} \times \underline{b}|}{|\underline{a}||\underline{b}|} = \frac{\sqrt{654}}{\sqrt{35}\sqrt{21}} = \frac{\sqrt{654}}{\sqrt{35 \times 21}} = \frac{\sqrt{654}}{\sqrt{735}}$$

Q3. Find the area of the triangle, formed by the points P, Q and R .

i. $P(2,3,5); Q(1,2,0); R(4,1,2)$

Sol: First, find the vectors \overrightarrow{PQ} and \overrightarrow{PR} :

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (1-2)\underline{i} + (2-3)\underline{j} + (0-5)\underline{k} = -\underline{i} - \underline{j} - 5\underline{k}$$

$$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = (4-2)\underline{i} + (1-3)\underline{j} + (2-5)\underline{k} = 2\underline{i} - 2\underline{j} - 3\underline{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & -1 & -5 \\ 2 & -2 & -3 \end{vmatrix} = \underline{i}(3-10) - \underline{j}(3+10) + \underline{k}(2+2) = -7\underline{i} - 13\underline{j} + 4\underline{k}$$

$$|\overrightarrow{PQ} \times \overrightarrow{PR}| = \sqrt{(-7)^2 + (-13)^2 + (4)^2} = \sqrt{49+169+16} = \sqrt{234} = \sqrt{9 \times 26} = 3\sqrt{26}$$

Finally, calculate the area of the triangle:

$$\text{Area} = \frac{1}{2} |\overrightarrow{PQ} \times \overrightarrow{PR}| = \frac{1}{2} \sqrt{234} \text{ square units.}$$

ii. $P(0,0,1); Q(2,-1,2); R(-1,3,2)$

Sol: First, find the vectors \overrightarrow{PQ} and \overrightarrow{PR} :

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (2-0)\underline{i} + (-1-0)\underline{j} + (2-1)\underline{k} = 2\underline{i} - \underline{j} + \underline{k}$$

$$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = (-1-0)\underline{i} + (3-0)\underline{j} + (2-1)\underline{k} = -\underline{i} + 3\underline{j} + \underline{k}$$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -1 & 1 \\ -1 & 3 & 1 \end{vmatrix} = \underline{i}(-1-3) - \underline{j}(2+1) + \underline{k}(6-1) = -4\underline{i} - 3\underline{j} + 5\underline{k}$$

$$|\vec{PQ} \times \vec{PR}| = \sqrt{(-4)^2 + (-3)^2 + (5)^2} = \sqrt{16+9+25} = \sqrt{50}$$

$$\text{Area of triangle} = \frac{1}{2} |\vec{PQ} \times \vec{PR}| = \frac{1}{2} \sqrt{50} = \frac{1}{2} \sqrt{25 \times 2} = \frac{1}{2} \times 5\sqrt{2} = \frac{5\sqrt{2}}{2} \text{ square units.}$$

Q4. Find the area of a parallelogram, whose vertices are:

i. $A(1, 1, 1); B(4, 2, 3); C(5, 6, 7); D(2, 5, 5)$

Sol: First, calculate the vectors \vec{AB} and \vec{AD} :

$$\vec{AB} = \vec{OB} - \vec{OA} = (4-1)\underline{i} + (2-1)\underline{j} + (3-1)\underline{k} = 3\underline{i} + \underline{j} + 2\underline{k}$$

$$\vec{AD} = \vec{OD} - \vec{OA} = (2-1)\underline{i} + (5-1)\underline{j} + (5-1)\underline{k} = \underline{i} + 4\underline{j} + 4\underline{k}$$

$$\vec{AB} \times \vec{AD} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & 1 & 2 \\ 1 & 4 & 4 \end{vmatrix} = \underline{i}(4-8) - \underline{j}(12-2) + \underline{k}(12-1) = -4\underline{i} - 10\underline{j} + 11\underline{k}$$

$$|\vec{AB} \times \vec{AD}| = \sqrt{(-4)^2 + (-10)^2 + (11)^2} = \sqrt{16+100+121} = \sqrt{237}$$

The area of the parallelogram is $\sqrt{237}$ square units.

ii. $A(4, 5, 6); B(1, 3, 2); C(-2, 0, 1); D(1, 2, 5)$

Sol: First, calculate the vectors \vec{AB} and \vec{AD} :

$$\vec{AB} = \vec{OB} - \vec{OA} = (1-4)\underline{i} + (3-5)\underline{j} + (2-6)\underline{k} = -3\underline{i} - 2\underline{j} - 4\underline{k}$$

$$\vec{AD} = \vec{OD} - \vec{OA} = (1-4)\underline{i} + (2-5)\underline{j} + (5-6)\underline{k} = -3\underline{i} - 3\underline{j} - \underline{k}$$

$$\vec{AB} \times \vec{AD} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -3 & -2 & -4 \\ -3 & -3 & -1 \end{vmatrix} = \underline{i}(2-12) - \underline{j}(3-12) + \underline{k}(9-6) = -10\underline{i} - 9\underline{j} + 3\underline{k}$$

$$|\vec{AB} \times \vec{AD}| = \sqrt{(-10)^2 + (-9)^2 + (3)^2} = \sqrt{100+81+9} = \sqrt{190}$$

The area of the parallelogram for is $\sqrt{190}$ square units.

Q5. If the cross product of the vectors $\underline{u} = 7\underline{i} - 4\underline{j} + 5\underline{k}$ and $\underline{v} = a\underline{i} - b\underline{j} + 3\underline{k}$ is zero, then find values of a and b

Sol: $\underline{u} = 7\underline{i} - 4\underline{j} + 5\underline{k}, \underline{v} = a\underline{i} - b\underline{j} + 3\underline{k}$

$$\underline{u} \times \underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 7 & -4 & 5 \\ a & -b & 3 \end{vmatrix} = \underline{i}(-12+5b) - \underline{j}(21-5a) + \underline{k}(-7b+4a)$$

Given that $\underline{u} \times \underline{v} = 0$

$$(-12+5b)\underline{i} - (21-5a)\underline{j} + (-7b+4a)\underline{k} = 0 \approx 0\underline{i} + 0\underline{j} + 0\underline{k}$$

$$-12+5b=0 \quad \text{and} \quad 21-5a=0$$

$$5b=12 \quad \text{and} \quad 5a=21$$

$$b = \frac{12}{5} \quad \text{and} \quad a = \frac{21}{5}$$

Q6. Which vectors, if any, are perpendicular or parallel?

i. $\underline{u} = 5\underline{i} - \underline{j} + \underline{k}; \underline{v} = \underline{j} - 5\underline{k}; \underline{w} = -15\underline{i} + 3\underline{j} - 3\underline{k}$

Sol: $\underline{u} = 5\underline{i} - \underline{j} + \underline{k}; \underline{v} = \underline{j} - 5\underline{k}; \underline{w} = -15\underline{i} + 3\underline{j} - 3\underline{k}$

$$\underline{u} \cdot \underline{v} = (5\underline{i} - \underline{j} + \underline{k}) \cdot (\underline{j} - 5\underline{k}) = 5(0) + (-1)(1) + (1)(-5) = -1 - 5 = -6 \neq 0$$

Not perpendicular

$$\underline{v} \cdot \underline{w} = (\underline{j} - 5\underline{k}) \cdot (-15\underline{i} + 3\underline{j} - 3\underline{k}) = (0)(-15) + (1)(3) + (-5)(-3) = 3 + 15 = 18 \neq 0$$

Not perpendicular

$$\underline{u} \cdot \underline{w} = (5\underline{i} - \underline{j} + \underline{k}) \cdot (-15\underline{i} + 3\underline{j} - 3\underline{k}) = (5)(-15) + (-1)(3) + (1)(-3) = -75 - 3 - 3 = -81 \neq 0$$

Not perpendicular

$$\text{Now, } \underline{w} = -15\underline{i} + 3\underline{j} - 3\underline{k} = -3(5\underline{i} - \underline{j} + \underline{k}) = -3\underline{u}$$

So, \underline{u} & \underline{w} are parallel.

ii. $\underline{u} = \underline{i} + 2\underline{j} - \underline{k}; \underline{v} = -\underline{i} + \underline{j} + \underline{k}; \underline{w} = \frac{-\pi}{2}\underline{i} - \pi\underline{j} + \frac{\pi}{2}\underline{k}$

Sol: $\underline{u} = \underline{i} + 2\underline{j} - \underline{k}; \underline{v} = -\underline{i} + \underline{j} + \underline{k}; \underline{w} = \frac{-\pi}{2}\underline{i} - \pi\underline{j} + \frac{\pi}{2}\underline{k}$

For parallel,

$$\underline{w} = \frac{-\pi}{2}\underline{i} - \pi\underline{j} + \frac{\pi}{2}\underline{k} = \frac{-\pi\underline{i} - 2\pi\underline{j} + \pi\underline{k}}{2} = \frac{-\pi}{2}(\underline{i} + 2\underline{j} - \underline{k})$$

$$\underline{w} = \frac{-\pi}{2}\underline{u}$$

So, \underline{u} & \underline{w} are parallel.

Now for perpendicular

$$\underline{u} \cdot \underline{v} = (\underline{i} + 2\underline{j} - \underline{k}) \cdot (-\underline{i} + \underline{j} + \underline{k}) = 1(-1) + 2(1) + (-1)(1) = -1 + 2 - 1 = 0$$

For \underline{u} & \underline{w} no need to check for perpendicular because they are parallel.

Q7. Use the definition of cross product, for any vectors $\underline{u}, \underline{v}, \underline{w}$ and scalar k , prove that

i. $\underline{u} \times (-\underline{u}) = \underline{0}$

ii. $\underline{u} \times \underline{v} = -\underline{v} \times \underline{u}$

iii. $\underline{u} \times (k\underline{v}) = (k\underline{u}) \times \underline{v} = k(\underline{u} \times \underline{v})$

iv. $\underline{u} \times (\underline{v} + \underline{w}) = (\underline{u} \times \underline{v}) + (\underline{u} \times \underline{w})$

Sol: Let $\underline{u} = (u_1, u_2, u_3), \underline{v} = (v_1, v_2, v_3), \underline{w} = (w_1, w_2, w_3)$.

i. $\underline{u} \times (-\underline{u}) = \underline{0}$

Sol: $\underline{u} \times (-\underline{u}) = \underline{0}$

$$\underline{u} \times (-\underline{u}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ u_1 & u_2 & u_3 \\ -u_1 & -u_2 & -u_3 \end{vmatrix}$$

We can factor out -1 from the third row:

$$= (-1) \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \end{vmatrix} \quad (R_2 \text{ and } R_3 \text{ are identical})$$

$$= (-1) \cdot 0 = 0$$

Thus, $\underline{u} \times (-\underline{u}) = 0$.

ii.

$$\underline{u} \times \underline{v} = \underline{v} \times \underline{u}$$

Sol:

$$\underline{u} \times \underline{v} = \underline{v} \times \underline{u}$$

Now, consider $\underline{v} \times \underline{u}$:

$$\underline{v} \times \underline{u} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{vmatrix}$$

$$\underline{v} \times \underline{u} = - \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \quad (\text{interchanging } R_2 \text{ and } R_3)$$

$$\underline{v} \times \underline{u} = -\underline{u} \times \underline{v}$$

Therefore, $\underline{u} \times \underline{v} = -\underline{v} \times \underline{u}$.

iii. $\underline{u} \times (k\underline{v}) = (k\underline{u}) \times \underline{v} = k(\underline{u} \times \underline{v})$

Sol: $\underline{u} \times (k\underline{v}) = (k\underline{u}) \times \underline{v} = k(\underline{u} \times \underline{v})$

Let's start with $\underline{u} \times (k\underline{v})$:

$$\underline{u} \times (k\underline{v}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ u_1 & u_2 & u_3 \\ kv_1 & kv_2 & kv_3 \end{vmatrix}$$

A property of determinants is that if a row (or column) is multiplied by a scalar k , the determinant is multiplied by k . We can factor k out of the third row:

$$= k \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = k(\underline{u} \times \underline{v})$$

Now, let's consider $(k\underline{u}) \times \underline{v}$:

$$(k\underline{u}) \times \underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ ku_1 & ku_2 & ku_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Similarly, we can factor k out of the second row:

$$= k \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = k(\underline{u} \times \underline{v})$$

Thus, $\underline{u} \times (k\underline{v}) = (k\underline{u}) \times \underline{v} = k(\underline{u} \times \underline{v})$

iv. $\underline{u} \times (\underline{v} + \underline{w}) = (\underline{u} \times \underline{v}) + (\underline{u} \times \underline{w})$

Sol: $\underline{u} \times (\underline{v} + \underline{w}) = (\underline{u} \times \underline{v}) + (\underline{u} \times \underline{w})$

$$\begin{aligned} \underline{u} \times (\underline{v} + \underline{w}) &= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ u_1 & u_2 & u_3 \\ v_1 + w_1 & v_2 + w_2 & v_3 + w_3 \end{vmatrix} \\ &= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} + \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \quad (\text{Separate } R_3 \text{ elements}) \\ &= (\underline{u} \times \underline{v}) + (\underline{u} \times \underline{w}) \end{aligned}$$

Therefore, $\underline{u} \times (\underline{v} + \underline{w}) = (\underline{u} \times \underline{v}) + (\underline{u} \times \underline{w})$

Q8. Prove that: $\underline{a} \times (\underline{b} + \underline{c}) + \underline{b} \times (\underline{c} + \underline{a}) + \underline{c} \times (\underline{a} + \underline{b}) = \underline{0}$

Sol: L.H.S = $\underline{a} \times (\underline{b} + \underline{c}) + \underline{b} \times (\underline{c} + \underline{a}) + \underline{c} \times (\underline{a} + \underline{b})$

$$= \underline{a} \times \underline{b} + \underline{a} \times \underline{c} + \underline{b} \times \underline{c} + \underline{b} \times \underline{a} + \underline{c} \times \underline{a} + \underline{c} \times \underline{b}$$

$$= \underline{a} \times \underline{b} + \underline{a} \times \underline{c} + \underline{b} \times \underline{c} - \underline{a} \times \underline{b} - \underline{a} \times \underline{c} - \underline{b} \times \underline{c} = 0$$

$$\underline{a} \times \underline{a} = 0 \text{ and } \underline{b} \times \underline{a} = -\underline{a} \times \underline{b}$$

Q9. If $\underline{a} + \underline{b} + \underline{c} = \underline{0}$, then prove that $\underline{a} \times \underline{b} = \underline{b} \times \underline{c} = \underline{c} \times \underline{a}$

Sol: Take cross product with \underline{a}

$$\underline{a} \times (\underline{a} + \underline{b} + \underline{c}) = \underline{a} \times \underline{0}$$

$$\underline{a} \times \underline{a} + \underline{a} \times \underline{b} + \underline{a} \times \underline{c} = \underline{0}$$

$$0 + \underline{a} \times \underline{b} + \underline{a} \times \underline{c} = \underline{0}$$

$$\underline{a} \times \underline{b} = -\underline{a} \times \underline{c} \Rightarrow \underline{a} \times \underline{b} = \underline{c} \times \underline{a} \quad (i)$$

Take cross product with \underline{b}

$$\underline{b} \times (\underline{a} + \underline{b} + \underline{c}) = \underline{b} \times \underline{0}$$

$$\underline{b} \times \underline{a} + \underline{b} \times \underline{b} + \underline{b} \times \underline{c} = \underline{0}$$

$$\underline{b} \times \underline{a} + \underline{0} + \underline{b} \times \underline{c} = \underline{0}$$

$$-\underline{a} \times \underline{b} + \underline{b} \times \underline{c} = \underline{0} \Rightarrow \underline{a} \times \underline{b} = \underline{b} \times \underline{c} \quad (ii)$$

Combining (i) and (ii) then $\underline{a} \times \underline{b} = \underline{b} \times \underline{c} = \underline{c} \times \underline{a}$

Q10. Prove that: $\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$

Note: Some Statement is incorrect in Book

Sol: Suppose two unit vectors \overrightarrow{OA} and \overrightarrow{OB}

$$\overrightarrow{OA} = \cos \alpha \underline{i} + \sin \alpha \underline{j}$$

$$\overrightarrow{OB} = \cos \beta \underline{i} + \sin \beta \underline{j}$$

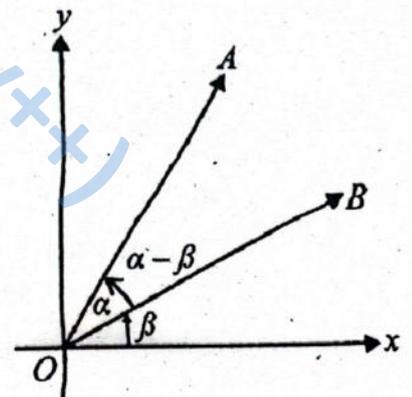
We know that $\overrightarrow{OB} \times \overrightarrow{OA} = \left| \overrightarrow{OB} \right| \left| \overrightarrow{OA} \right| \sin(\alpha - \beta) \underline{k}$

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cos \beta & \sin \beta & 0 \\ \cos \alpha & \sin \alpha & 0 \end{vmatrix} = 1 \cdot 1 \sin(\alpha - \beta) \underline{k}, \quad \left(\left| \overrightarrow{OA} \right| = \left| \overrightarrow{OB} \right| = 1 \right)$$

$$\underline{i}(0 - 0) - \underline{j}(0 - 0) + \underline{k}(\sin \alpha \cos \beta - \cos \alpha \sin \beta) = \sin(\alpha - \beta) \underline{k}$$

$$(\sin \alpha \cos \beta - \cos \alpha \sin \beta) \underline{k} = \sin(\alpha - \beta) \underline{k}$$

Hence $\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$



Q11. Show that $|\underline{a} \times \underline{b}|^2 = |\underline{a}|^2 |\underline{b}|^2 - (\underline{a} \cdot \underline{b})^2$

Sol: L.H.S = $|\underline{a} \times \underline{b}|^2 = |\underline{a}|^2 |\underline{b}|^2 \sin^2 \theta$ $\because \underline{a} \times \underline{b} = ab \sin \theta$
 $= |\underline{a}|^2 |\underline{b}|^2 (1 - \cos^2 \theta)$
 $= |\underline{a}|^2 |\underline{b}|^2 - |\underline{a}|^2 |\underline{b}|^2 \cos^2 \theta = |\underline{a}|^2 |\underline{b}|^2 - (\underline{a} \cdot \underline{b})^2 = R.H.S$ $\because \underline{a} \cdot \underline{b} = ab \cos \theta$

Q12. Use the definition of cross product, prove that for any vectors \underline{u} and \underline{v} $(\underline{u} + \underline{v}) \times (\underline{u} - \underline{v}) = -2(\underline{u} \times \underline{v})$

Sol: L.H.S = $(\underline{u} + \underline{v}) \times (\underline{u} - \underline{v}) = \underline{u} \times (\underline{u} - \underline{v}) + \underline{v} \times (\underline{u} - \underline{v})$
 $= (\underline{u} \times \underline{u}) - (\underline{u} \times \underline{v}) + (\underline{v} \times \underline{u}) - (\underline{v} \times \underline{v}) = 0 - (\underline{u} \times \underline{v}) + (\underline{v} \times \underline{u}) - 0$
 $= -(\underline{u} \times \underline{v}) + (\underline{v} \times \underline{u}) = -(\underline{u} \times \underline{v}) + (-\underline{u} \times \underline{v})$
 $= -(\underline{u} \times \underline{v}) - (\underline{u} \times \underline{v}) = -2(\underline{u} \times \underline{v}) = R.H.S$

Q13. Find the moment about the point $M(1, -3, 3)$ of the force represented by \overrightarrow{AB} , where the coordinates of points $A(4, 3, -1)$ and $B(-1, 3, 7)$ are given.

Sol: $\underline{F} = (-1-4)\underline{i} + (3-3)\underline{j} + (7-(-1))\underline{k} = -5\underline{i} + 0\underline{j} + 8\underline{k}$
 $\underline{r} = (4-1)\underline{i} + (3-(-3))\underline{j} + (-1-3)\underline{k} = 3\underline{i} + 6\underline{j} - 4\underline{k}$

Moment = $\underline{r} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & 6 & -4 \\ -5 & 0 & 8 \end{vmatrix} = \underline{i}(48-0) - \underline{j}(24-20) + \underline{k}(0-(-30)) = 48\underline{i} - 4\underline{j} + 30\underline{k}$

Q14. A force $\underline{F} = 6\underline{i} + 4\underline{j} - 4\underline{k}$ is applied at the point $A(1, -1, 2)$. Find the moment of the force about the point $B(3, -2, 3)$.

Sol: $\underline{F} = 6\underline{i} + 4\underline{j} - 4\underline{k}$
 $\underline{r} = (1-3)\underline{i} + (-1-(-2))\underline{j} + (2-3)\underline{k} = -2\underline{i} + 1\underline{j} - 1\underline{k}$

Moment = $\underline{r} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -2 & 1 & -1 \\ 6 & 4 & -4 \end{vmatrix} = \underline{i}(-4+4) - \underline{j}(8+6) + \underline{k}(-8-6) = 0\underline{i} - 14\underline{j} - 14\underline{k}$

Q15. Give a force $\underline{F} = 2\underline{i} + \underline{j} - 3\underline{k}$ acting at a point $A(1, -2, 1)$. Find the moment of \underline{F} about the point $B(2, 0, -2)$.

Sol: $\underline{F} = 2\underline{i} + \underline{j} - 3\underline{k}$
 $\underline{r} = (1-2)\underline{i} + (-2-0)\underline{j} + (1-(-2))\underline{k} = -1\underline{i} - 2\underline{j} + 3\underline{k}$

Moment = $\underline{r} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & -2 & 3 \\ 2 & 1 & -3 \end{vmatrix} = \underline{i}(6-3) - \underline{j}(3-6) + \underline{k}(-1-(-4)) = 3\underline{i} - (-3)\underline{j} + 3\underline{k}$
 $M = 3\underline{i} + 3\underline{j} + 3\underline{k}$

Q16. A force $\underline{F} = -2\underline{i} + \underline{j} - 3\underline{k}$ is applied at $P(-1, -3, 2)$. Find its moment about the point $Q(4, 2, 2)$.

Sol: $\underline{F} = -2\underline{i} + \underline{j} - 3\underline{k}$
 $\underline{r} = (-1-4)\underline{i} + (-3-2)\underline{j} + (2-2)\underline{k} = -5\underline{i} - 5\underline{j} + 0\underline{k}$

Moment = $\underline{r} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -5 & -5 & 0 \\ -2 & 1 & -3 \end{vmatrix} = \underline{i}(15-0) - \underline{j}(15-0) + \underline{k}(-5-10) = 15\underline{i} - 15\underline{j} - 15\underline{k}$