A time dilation approach to speeding up simulations in regions of extreme phase space

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1 Theoretic foundation

I think I have the formalism of the time dilation correct now. Start from an equation like this.

$$\frac{\partial f}{\partial t} + \mathscr{L}(f) = 0 \tag{1}$$

Separate f by adding zero and grouping the terms we want to drop.

$$\frac{\partial f}{\partial t} + \frac{\partial (f/\beta)}{\partial t} - \frac{\partial (f/\beta)}{\partial t} + \mathscr{L}(f) = 0 \tag{2}$$

$$\frac{\partial (f/\beta)}{\partial t} + \frac{\partial (f - f/\beta)}{\partial t} + \mathscr{L}(f) = 0 \tag{3}$$

We drop this extra time derivative. When $\beta = 1$, this term is zero. However, when $0 < \beta < 1$, this term is not negligible, but is the reason we get the CFL relaxation. We arrive at the equation

$$\frac{\partial (f/\beta)}{\partial t} + \mathscr{L}(f) = 0. \tag{4}$$

Assuming time independence of β, we can pull it out of the time derivative and put it in front of the $\mathscr{L}$.

$$\frac{\partial f}{\partial t} + \beta \mathscr{L}(f) = 0. \tag{5}$$

Equation (5) is what we can implement simply in Gkeyll. An equivalent way of expressing this is to put move the beta into the time derivative ($\partial / \partial(\beta t)$) which means that the screened cells where $\beta < 1$ effectively take smaller time steps of $\beta \Delta t$, which is why this method is described as time dilation. However, this modification is not conservative – we dropped a critical term. The conservative way to do this is to modify the Hamiltonian contours, as we do with the force softening. For instance, consider an advection equation with translation and acceleration such as

$$\frac{\partial f}{\partial t} + \nabla_z \cdot (\dot{z} f) + \nabla_v \cdot (\dot{v} f) = 0, \tag{6}$$

where $f(z,v,t)$ is the distribution function which varies along space and velocity. A conservative modification of this operator, retaining it's divergence form, would be

$$\frac{\partial f}{\partial t} + \nabla_z \cdot (\beta(z,v)\dot{z}f) + \nabla_v \cdot (\beta(z,v)\dot{v}f) = 0, \tag{7}$$

however it but fundamentally constitutes a different problem than equation (6) because it modifies the phase-space contours. In contrast, the modification made by the time dilation algorithm is

$$\frac{\partial f}{\partial t} + \beta(z)\left[\nabla_z \cdot (\dot{z}f) + \nabla_v \cdot (\dot{v}f)\right] = 0, \tag{8}$$

which does not conserve energy, particle number, or momentum but preserves the phase space contours.

To explain the breaking of conservation, equation (8) can be re-expressed in a form that makes the loss of conservation explicit. Using the product rule, both transport terms may be decomposed as

$$\beta\,\nabla_z \cdot (\dot{z}f) = \nabla_z \cdot (\beta\,\dot{z}f) - (\nabla_z\beta) \cdot (\dot{z}f), \tag{9}$$

$$\beta\nabla_v \cdot (\dot{v}f) = \nabla_v \cdot (\beta\dot{v}f) - (\nabla_v\beta) \cdot (\dot{v}f). \tag{10}$$

Substituting this identity into the evolution equation yields

$$\frac{\partial f}{\partial t} + \nabla_z \cdot (\beta\dot{z}f) + \nabla_v \cdot (\beta\dot{v}f) = (\nabla_z\beta) \cdot (\dot{z}f) + (\nabla_v\beta) \cdot (\dot{v}f), \tag{11}$$

which consists of a conservative flux term on the left-hand side and an explicit source term on the right-hand side. The source term vanishes only when β is spatially uniform, in which case the equation reduces to a conservative form. When $\nabla_z\beta, \nabla_v\beta \neq 0$, the right-hand side acts as an artificial source or sink, explaining the loss of particle, momentum, and energy conservation in the time-dilation scheme.

Although the time-dilation formulation does not preserve the exact conservation properties of the Vlasov equation, it can nevertheless be a reasonable approximation in regimes where the distribution function is small and dynamically unimportant. In regions of phase space where $f/\max(f) \ll 1$, such as the high-energy tail or sparsely populated velocity regions, the absolute contribution of conservation errors to integrated moments (density, momentum, and energy) is correspondingly small. In these regions, the dominant numerical constraint often arises from large characteristic velocities that impose restrictive CFL conditions despite carrying negligible particle weight. Introducing a local slowing factor $\beta(z) < 1$ effectively relaxes the timestep limitation while introducing errors proportional to $\dot{z}f\,\nabla_z\beta$ and $\dot{v}f\,\nabla_v\beta$, which remain subdominant when f is small. As a result, the approximation primarily alters the evolution of low-density phase-space regions that weakly influence the bulk plasma dynamics. This regime-dependent tradeoff between strict conservation and computational efficiency may be acceptable for exploratory studies or diagnostic calculations, provided that conserved quantities are carefully monitored and the approximation is not applied in regions where f significantly contributes to macroscopic moments. This situation aligns with the goals of the Gkeyll code, as it similarly trades conservation for improved robustness and computational efficiency in low-density regions.

From a theoretical perspective, placing the time-dilation factor outside the divergence operator corresponds to a local rescaling of time along characteristics. While this modification alters the governing equations, it does so in a manner that is transparent and directly tied to asymptotic scale-separation arguments, rather than implicitly modifying the Hamiltonian or the phase-space Jacobian. In practice, numerical experiments indicate that the resulting conservation errors are orders of magnitude smaller than other modeling and discretization errors for the choices of $\beta(z,v)$ considered here. Moreover, the simplicity and generality of the approach make it attractive from a software and algorithmic standpoint, providing a unified framework for multiscale time integration. For these reasons, while the method is not intended as a strictly conservative replacement for the Gyrokinetic equation, it constitutes a useful and robust tool for accelerating kinetic simulations in regimes where the time step is restricted by regions of phase space where the distribution function is small.

The time-dilation factor β is constructed to minimize screening, ensuring the effective timestep is as large as possible while respecting a user-specified minimum timestep constraint. Specifically, a global minimum timestep Δt_{min} is prescribed based on physical considerations, such as the CFL constraint neglecting regions where $f/\max(f) \ll 1$. At each update, the local CFL-limited timestep $\Delta t_{\mathrm{CFL}}(z,v)$ is computed from the characteristic phase-space velocities. The dilation factor is then defined as

$$\beta(z,v) = \min\left(1, \frac{\Delta t_{\mathrm{CFL}}(z,v)}{\Delta t_{\mathrm{min}}}\right),$$

so that time is slowed only in regions where the local CFL constraint would otherwise demand a timestep smaller than Δt_{min}. In regions where the local dynamics permit larger timesteps, $\beta = 1$ and the governing equations are unmodified. Importantly, the global timestep remains uniform across phase space and is set by Δt_{min}, while β locally rescales the effective characteristic speeds. This ensures that time dilation is applied primarily in sparsely populated regions of phase space that impose severe CFL constraints but contribute negligibly to macroscopic moments, thereby achieving maximal timestep relaxation with minimal impact on the physical solution.

2 Alternative derivation which only neglects advection

The derivation in the previous section neglects all $(1-\beta)\partial_t f$, however an alternative derivation may be considered which only dilates advection. Start with the base problem, which appears in the mirror simulations.

$$\frac{\partial f}{\partial t} + \{H, f\} + C(f) + S = 0 \tag{12}$$

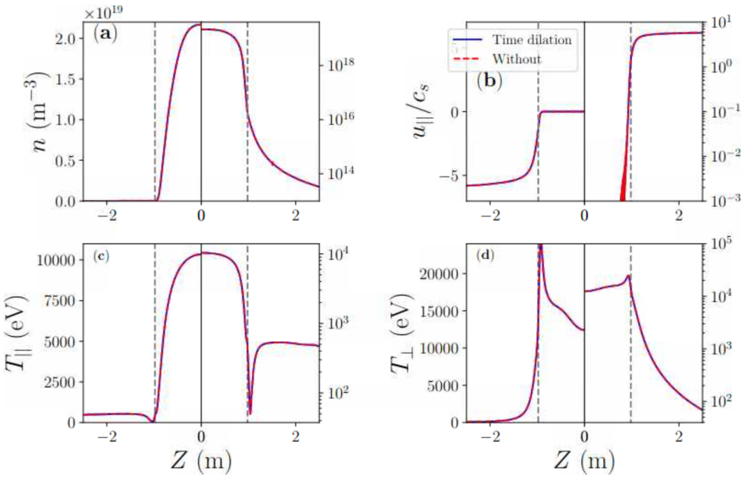

Figure 4.20: A comparison of two simulations, both with $N_z = 800$, run for 7 POA cycles. $N_\mu = 32, N_{v_\parallel} = 64$, with $\tau_{\text{OAP}} = 2\,\text{s}$ and $\tau_{\text{FDP}} = 20\,\mu\text{s}$. The timestep is determined based on the maximum CFL rate for the region where $JBf > 1 \times 10^{-16}$.

Let's add zero to the Poisson bracket

$$\frac{\partial f}{\partial t} + (1 + \beta - \beta)\{H, f\} + C(f) + S = 0. \tag{13}$$

Then drop $1 - \beta$.

$$\frac{\partial f}{\partial t} + \beta\{H, f\} + C(f) + S = 0. \tag{14}$$

This is originally what I implemented and shows the conservation errors described in the previous section without introducing modifications to the collision rate or the source. However, this is not a global time dilation modification, but only applies to the advection terms. It may be valuable and more intuitive to explain this in terms of dilating time itself in all terms.

Data from a simulation with equation (14) implemented is shown in figure 2. Minimal differences are noted between the two simulations.

3 Future work

- I haven't fully implemented equation (5) in Gkeyll. I'd like to meet to ensure this is a good idea that we want before spending considerable amounts of time on software development.
- What kinds of tests can we imagine? Conservation studies to look at the effects? Is all we care about conservation?

- What other kinds of effects does this scheme have on the simulation. Conservation issues are known and quantifiable. I could imagine additional issues which are due to extra trapping of information.