## GPEC field line tracing maths

Some definitions:

- $\eta$ = machine poloidal angle
- $\phi$ = machine toroidal angle
- $r$ = machine minor radius
- $R$ = machine major radius
- $\theta$ = magnetic poloidal angle
- $\zeta$ = magnetic toroidal angle
- subscript $N$: 'normalized'
- $bf\%f = RB_\phi = F(\psi)$ (see Ideal MHD equ 6.11)
- $\psi_0 = \psi$ on axis, $\psi_a = \psi$ at separatrix

$$
\begin{aligned}
q &= \frac{F(\psi)}{2\pi} \oint \frac{dl_p}{R^2 B_p} \\
&= \frac{1}{2\pi} \oint \frac{dl_p B_\phi}{R B_p}
\end{aligned}
$$

## y_out spline:

y_out(0) $= \eta$ machine poloidal angle ($\boldsymbol{F}_p \Leftrightarrow \boldsymbol{F}_\eta$)

y(1)

$$
\begin{aligned}
\frac{dy(1)}{d\eta} &= \frac{r}{B_z \cos(\eta) - B_R \sin(\eta)} = \frac{r}{B_\eta} \\
&\Rightarrow dy(1) = \frac{r d\eta}{B_\eta} = \frac{dl_p}{B_\eta} \\
&\Rightarrow y(1) = \int \frac{dl_\eta}{B_\eta} \\
&\Rightarrow y(1)|_{2\pi} = \oint \frac{dl_\eta}{B_\eta} =? \frac{dV}{d\psi} \frac{1}{2\pi} \text{ (see Freidberg Ideal MHD equ 6.22, } dl_p \text{ is poloidal arc length)}
\end{aligned}
$$

y(2)

$$
\begin{aligned}
\frac{dr}{dl_\eta} &= \frac{B_r}{B_\eta} \\
\Rightarrow \frac{dr}{r d\eta} &= \frac{B_r}{B_\eta} \\
\Rightarrow \frac{dr}{d\eta} &= \frac{B_r}{B_\eta} r \\
\frac{dy(2)}{d\eta} &= \frac{dy(1)}{d\eta}[B_R \cos(\eta) + B_Z \sin(\eta)] = \frac{B_r}{B_\eta} r \\
&\Rightarrow y(2) = r
\end{aligned}
$$

y(3)

$$
\begin{aligned}
\frac{dy(3)}{d\eta} &= \frac{dy(1)/d\eta}{R^2} = \frac{r}{R^2 B_\eta} \\
&\Rightarrow y(3) = \int \frac{dl_\eta}{R^2 B_\eta} \\
&\Rightarrow y(3)|_{2\pi}? = 2\pi \frac{q(\psi)}{F(\psi)} \text{ (see Ideal MHD equ 6.35)}
\end{aligned}
$$

y(4) Should be unnormalised magnetic poloidal angle $\theta$

$$
\begin{aligned}
\frac{dy(4)}{d\eta} &= \frac{dy(1)}{d\eta}\Big[\frac{B_p^{\alpha_p}(|B|)^{\alpha_B}}{R^{\alpha_R}}\Big] \\
&= \frac{r}{B_\eta}\Big[\frac{B_p^{\alpha_p}(|B|)^{\alpha_B}}{R^{\alpha_R}}\Big] \\
&\propto \frac{r}{B_\eta}(B \cdot \nabla\theta) \\
&= \frac{r}{B_\eta}(B_\theta) \\
\Rightarrow \frac{dy(4)}{r d\eta} &= \frac{dy(4)}{dl_\eta} = \frac{B_\theta}{B_\eta} \\
&\Rightarrow y(4) = \theta
\end{aligned}
$$

## ff spline:

ff%xs Normalised magnetic poloidal angle $\theta_N$

$$
\text{ff\%xs} = \frac{y(4)}{y(4)|_{\eta=2\pi}} = \frac{\theta(\eta)}{\theta(\eta = 2\pi)}
$$

ff%f(1) = $r^2$

ff%f(2) = normalised $\eta$ - normalised $\theta$

$= \frac{\eta}{2\pi} - \frac{y(4)|_\eta}{y(4)|_{\eta=2\pi}}$

$= \frac{\eta}{2\pi} - \frac{\theta(\eta)}{\theta(\eta=2\pi)}$

ff%f(3) = $\phi(\eta) - \zeta(\eta)$

$$
\begin{aligned}
ff\%f(3) &= F(\int^\eta \frac{r}{R^2 B_\eta} d\eta - \theta_N(\eta) \oint \frac{r}{R^2 B_\eta} d\eta) \\
&= F(\int^\eta \frac{r}{R^2 B_\eta} d\eta - \theta_N(\eta) \oint \frac{r}{R^2 B_\eta} d\eta) \\
\\
&= F \int^\eta \frac{r}{R^2 B_\eta} d\eta - \theta_N(\eta) F \oint \frac{r}{R^2 B_\eta} d\eta \\
&= F \int^\eta \frac{r}{R^2 B_\eta} d\eta - \theta_N(\eta)[F \oint^\eta \frac{r}{R^2 B_\eta} d\eta] \\
&= \int^\eta \frac{r}{R} \frac{B_\phi}{B_\eta} d\eta - \theta_N(\eta)[\oint \frac{r}{R} \frac{B_\phi}{B_\eta} d\eta] \\
&= \int^{l_\eta} \frac{1}{R} \frac{B_\phi}{B_\eta} dl_\eta - \theta_N(\eta)[\oint \frac{1}{R} \frac{B_\phi}{B_\eta} dl_\eta] \\
&= \int^{l_\eta} \frac{1}{R} \frac{dl_\phi}{dl_\eta} dl_\eta - \theta_N(\eta)[\oint \frac{1}{R} \frac{dl_\phi}{dl_\eta} dl_\eta] \\
&= \int^{l_\phi(\eta)} \frac{1}{R} dl_\phi - \theta_N(\eta)[\oint \frac{1}{R} dl_\phi] \\
&= \int^{\phi(\eta)} \frac{1}{R} R d\phi - \theta_N(\eta)[\oint \frac{1}{R} R d\phi] \\
&= \int^{\phi(\eta)} d\phi - \theta_N(\eta)[\oint d\phi] \\
&= \phi(\eta) - \theta_N(\eta)\phi(\eta = 2\pi) \\
&= \phi(\eta) - \zeta(\eta)
\end{aligned}
$$

Note $\theta_N(\eta)$ varies from 0 to 1 as $\eta$ increases. $\phi(\eta = 2\pi)$ is to ensure normalisation.

ff%f(4) = normalised $\frac{dV}{d\psi}$ - $\theta_N(\eta)$

$= \frac{dV}{d\psi}(\eta)/[\frac{dV}{d\psi}(\eta = 2\pi)] - \theta_N(\eta)$

## rzphi spline:

rzphi(i) = ff%f(i) for i = 1,2,3.

rzphi(4) = 'jacobian'
[in this section, $J$ is the jacobian, not current]

$$
\begin{aligned}
d\boldsymbol{r} &= RdRdZd\phi \\
&= Jd\psi d\theta d\zeta \\
V &= \int d\boldsymbol{r} \\
&= \int Jd\psi d\theta d\zeta \\
&= \oint d\zeta \int_{\psi_o}^{\psi} d\psi \oint J(\theta,\psi) \\
&= \int_0^1 d\zeta \int_{\psi_o}^{\psi} d\psi \oint d\theta J(\theta,\psi) \\
&= \int_{\psi_o}^{\psi} d\psi \oint d\theta J(\theta,\psi) \\
\frac{dV}{d\psi} &= \oint d\theta J(\theta,\psi) \\
\Rightarrow J(\theta,\psi) &= \frac{d}{d\theta}\frac{dV}{d\psi}
\end{aligned}
$$

$$
\begin{aligned}
\text{rzphi(4)}(\theta) &= \Big(1 + \frac{d}{d\theta_N} ff\%f(4)\Big) \times y(1)|_{\theta_N=1} \times 2\pi \times psio \\
&= \Big(1 + \frac{d}{d\theta_N}\Big[\frac{\frac{dV}{d\psi}(\theta_N)}{\frac{dV}{d\psi}(\theta_N=1)} - \theta_N\Big]\Big) \times \frac{dV}{d\psi}(\theta_N=1) \times (\psi_a - \psi_0) \\
&= \Big(1 + \frac{1}{\frac{dV}{d\psi}(\theta_N=1)}\frac{d}{d\theta_N}\frac{dV}{d\psi}(\theta_N) - \frac{d}{d\theta_N}\theta_N\Big) \times \frac{dV}{d\psi}(\theta_N=1) \times (\psi_a - \psi_0) \\
&= \Big(1 + \frac{1}{\frac{dV}{d\psi}(\theta_N=1)}\frac{d}{d\theta_N}\frac{dV}{d\psi}(\theta_N) - 1\Big) \times \frac{dV}{d\psi}(\theta_N=1) \times (\psi_a - \psi_0) \\
&= \Big(\frac{d}{d\theta_N}\frac{dV}{d\psi}(\theta_N)\Big) \times (\psi_a - \psi_0) \\
&= J \times (\psi_a - \psi_0) \text{ (this is the Jacobian with real units of } \psi.)
\end{aligned}
$$