Portfolio Construction - the Mean Variance Framework

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Outline

- **Mean Variance optimization and the Efficient Frontier**
- **Equivalence with the quadratic utility function maximization**
- **The Capital Allocation Line and the Tangency Portfolio**
- **Sign constrained optimization**

Mean Variance Optimization and the Efficient Frontier

Generalization with n assets

We now create a portfolio of n assets and introduce the following notations:

$$\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} \; \boldsymbol{\iota} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \; \widetilde{\mathbf{r}} = \begin{pmatrix} \widetilde{r}_1 \\ \widetilde{r}_2 \\ \vdots \\ \widetilde{r}_n \end{pmatrix} \; \boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{pmatrix} \; \boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_{1,2} & \cdots & \sigma_{1,n} \\ \sigma_{2,1} & \sigma_2^2 & \cdots & \sigma_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n,1} & \sigma_{n,2} & \cdots & \sigma_n^2 \end{pmatrix}$$

$\mathbf{w}$: $(n \times 1)$ vector of the relative weights (%) of assets in the portfolio

$\mathbf{r}$: $(n \times 1)$ vector of assets' returns, with T realizations

$\boldsymbol{\iota}$: $(n \times 1)$ unitary vector

$\boldsymbol{\mu}$: $(n \times 1)$ vector of assets' expected returns: $\boldsymbol{\mu} = \mathbb{E}[\widetilde{\mathbf{r}}]$

$\boldsymbol{\Sigma}$: $(n \times n)$ covariance matrix of assets' returns: $\boldsymbol{\Sigma} = \mathbb{E}[(\widetilde{\mathbf{r}} - \boldsymbol{\mu}).(\widetilde{\mathbf{r}} - \boldsymbol{\mu})^{'}]$. It is assumed to be semi-definite positive and invertible.

Portfolio Characteristics

The return, expected return and variance on the portfolio write:

$$r_P = \mathbf{w}^{'}.\mathbf{r}$$
$$\mathbb{E}[r_P] = \mu_P = \mathbb{E}[\mathbf{w}^{'}.\mathbf{r}] = \mathbf{w}^{'}.\mathbb{E}[\mathbf{r}] = \mathbf{w}^{'}\boldsymbol{\mu}$$

$$\begin{aligned}
V(r_P) &= V[\mathbf{w}^{'}.\mathbf{r}] && (1)\\
&= \mathbb{E}[(r_P - \mathbb{E}[r_P])^2] = \mathbb{E}[(\mathbf{w}^{'}.\mathbf{r} - \mathbf{w}^{'}.\boldsymbol{\mu}).(\mathbf{w}^{'}.\mathbf{r} - \mathbf{w}^{'}.\boldsymbol{\mu})^{'}] && (2)\\
&= \mathbb{E}[\mathbf{w}^{'}.(\mathbf{r} - \boldsymbol{\mu}).(\mathbf{w}^{'}.(\mathbf{r} - \boldsymbol{\mu}))^{'}] && (3)\\
&= \mathbb{E}[\mathbf{w}^{'}.(\mathbf{r} - \boldsymbol{\mu}).(\mathbf{r} - \boldsymbol{\mu})^{'}.\mathbf{w}] && (4)\\
&= \mathbf{w}^{'}.\mathbb{E}[(\mathbf{r} - \boldsymbol{\mu}).(\mathbf{r} - \boldsymbol{\mu})^{'}].\mathbf{w} = \mathbf{w}^{'}.\boldsymbol{\Sigma}.\mathbf{w} && (5)
\end{aligned}$$

"Feasible set" with 6 Eurozone stocks

- In what follows, we use the daily timeseries of returns on 6 Eurozone stocks (L'Oréal, Bouygues, LVMH, Shell, Air Liquide, Hermès) over the 2019-2022 period. We estimate their expected returns and their covariances using **"historical estimates"**, that have been annualized:

$$\boldsymbol{\mu} = \begin{pmatrix} 14.7\% \\ -11.3\% \\ 24.5\% \\ -18.6\% \\ 12.1\% \\ 25.2\% \end{pmatrix} \boldsymbol{\Sigma} = \begin{pmatrix} 8.9\% & 5.4\% & 7.3\% & 5.4\% & 5.3\% & 6.3\% \\ 5.4\% & 16.5\% & 8.1\% & 11.9\% & 6.1\% & 5.4\% \\ 7.3\% & 8.1\% & 13.0\% & 8.6\% & 6.1\% & 9.2\% \\ 5.4\% & 11.9\% & 8.6\% & 22.4\% & 6.3\% & 4.9\% \\ 5.3\% & 6.1\% & 6.1\% & 6.3\% & 7.7\% & 5.1\% \\ 6.3\% & 5.4\% & 9.2\% & 4.9\% & 5.1\% & 11.1\% \end{pmatrix}$$

- We then create several portfolios based on those 6 assets. Assets' weights have to be positive, sum to 1, and are allowed to vary by steps of 0.1%, which yields around $2,800$ possible combinations. We then plot in the mean/standard deviation space this **"feasible set"** of portfolios.

Feasible set with 6 Eurozone stocks

Figure 1: Feasible set

Optimal portfolio, formalization of the problem

A portfolio w^* is efficient in the sense of Markowitz if, for any portfolio z:

$$\begin{cases} \sigma_z \leq \sigma_{w^*} \implies \mu_z \leq \mu_{w^*} \\ \sigma_z = \sigma_{w^*} \implies \mu_z \leq \mu_{w^*} \end{cases}$$

An efficient portfolio is thus a portfolio with the lowest variance among all portfolios of expected return μ. It is found by solving the following program:

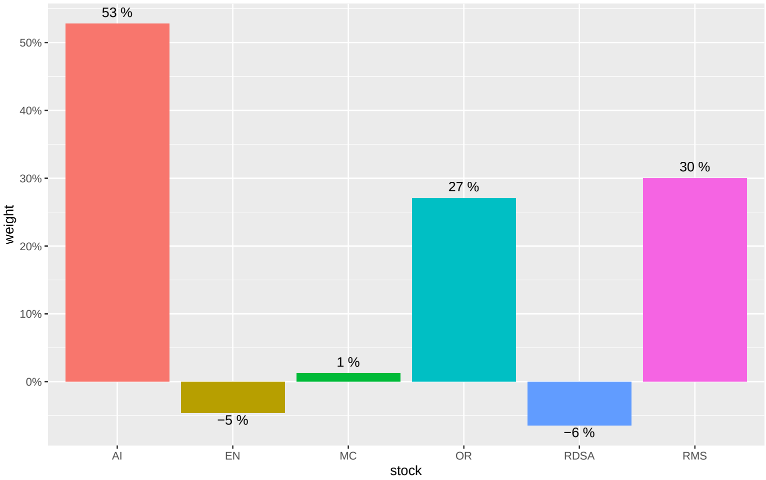
$$\min_{\mathbf{w}} \ \mathbf{w}^{'}.\boldsymbol{\Sigma}\mathbf{w} \quad \text{s.t.} \begin{cases} \mathbf{w}^{'}\boldsymbol{\mu} & = \mu \\ \mathbf{w}^{'}\boldsymbol{\iota} & = 1 \end{cases}$$

The budget constraint

- The condition $\mathbf{w}'\boldsymbol{\iota} = 1$ is called the budget (or financing) constraint.
- In that formulation, there is only a restriction on the total weights (the portfolio has to be fully invested)
- Individual weights are not constrained. In particular, they can be negative! This means short selling is authorized

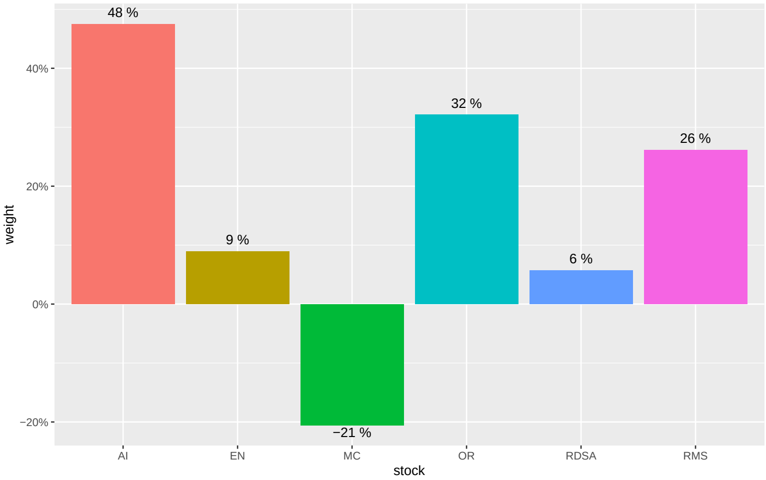
Weights of the efficient portfolio with a 20% target return

Figure 2: Weights of the efficient portfolio with a 20% target return



Weights of the efficient portfolio with a 10% target return

Figure 3: Weights of the efficient portfolio with a 10% target return



Minimum Variance Frontier

The set of minimum variance optimal portfolios is obtained by repeating the previous optimization program for all attainable values of expected returns. The sets of optimal weights form portfolios which constitute the **Minimum Variance Frontier**. The optimization program has a closed form solution.

A Lagrangian to solve the optimization program

If we have to minimize a function f under constraints $g_i(x) = c$, for $i = 1, 2, ..., m$, we can internalize the constraints into the initial function using Lagrangian multipliers λ_i, $i \in [1; m]$, to create a Lagrangian, which writes:

$$L(x, \lambda_1, ..., \lambda_m) = f(x) + \sum_{i=1}^{m} \lambda_i.(c - g_i(x)) \tag{6}$$

It is solved for values of x which satisfy the following First Order Conditions (FOCs):

$$\begin{cases} \dfrac{\delta L(x)}{\delta x} = \dfrac{\delta f(x)}{\delta x} - \displaystyle\sum_{i=1}^{m} \lambda_i.\dfrac{\delta g_i(x)}{\delta x} = 0 \\ \dfrac{\delta L(x)}{\delta \lambda_i} = c - g_i(x) = 0, \quad 1 \leq i \leq m \end{cases}$$

In the present situation, the Lagrangian writes:

$$L = \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} - \lambda_1.(\mathbf{w}^{'}\boldsymbol{\mu} - \mu) - \lambda_2.(\mathbf{w}'\boldsymbol{\iota} - 1) \tag{7}$$

First Order Conditions of the optimization program

FOCs write: $\begin{cases} \dfrac{\delta L}{\delta \mathbf{w}} = 2.\boldsymbol{\Sigma}\mathbf{w} - \lambda_1.\boldsymbol{\mu} - \lambda_2.\boldsymbol{\iota} & = 0 \\ \mathbf{w}^{'}\boldsymbol{\mu} & = \mu \\ \mathbf{w}^{'}\boldsymbol{\iota} & = 1 \end{cases} \implies$

$$\begin{cases} \dfrac{\lambda_1}{2}.\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + \dfrac{\lambda_2}{2}.\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota} & = \mathbf{w} \\ \boldsymbol{\mu}'\mathbf{w} & = \mu \\ \boldsymbol{\iota}'\mathbf{w} & = 1 \end{cases} \implies \begin{cases} \dfrac{\lambda_1}{2}.\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + \dfrac{\lambda_2}{2}.\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota} & = \mathbf{w} \\ \dfrac{\lambda_1}{2}.\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + \dfrac{\lambda_2}{2}.\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota} & = \mu \\ \dfrac{\lambda_1}{2}.\boldsymbol{\iota}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + \dfrac{\lambda_2}{2}.\boldsymbol{\iota}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota} & = 1 \end{cases}$$

by multiplying the first equation successively by $\boldsymbol{\mu}'$ and $\boldsymbol{\iota}'$.

Then we pose that
$\mathrm{A} = \boldsymbol{\iota}^{'}\boldsymbol{\Sigma}^{-1}.\boldsymbol{\mu} \quad B = \boldsymbol{\mu}^{'}\boldsymbol{\Sigma}^{-1}.\boldsymbol{\mu} \quad C = \boldsymbol{\iota}^{'}\boldsymbol{\Sigma}^{-1}.\boldsymbol{\iota} \quad D = B.C - A^2$

Expression of the Lagrangian multipliers

The system can be partially solved, starting with the values of λ_1 and λ_2, by focusing on the last two equations, that we rewrite as a product of matrices:

$$\begin{cases} \frac{\lambda_1}{2}.B + \frac{\lambda_2}{2}.A & = \mu \\ \\ \frac{\lambda_1}{2}.A + \frac{\lambda_2}{2}.C & = 1 \end{cases} \iff \frac{1}{2}\begin{pmatrix} A & C \\ B & A \end{pmatrix} \cdot \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \mu \end{pmatrix} \implies$$

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = 2\begin{pmatrix} A & C \\ B & A \end{pmatrix}^{-1} . \begin{pmatrix} 1 \\ \mu \end{pmatrix} \implies \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \frac{2}{A^2 - B.C}\begin{pmatrix} A & -C \\ -B & A \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \mu \end{pmatrix}$$

$$\implies \begin{cases} \lambda_1(\mu) & = \frac{2}{D}.(C.\mu - A) \\ \\ \lambda_2(\mu) & = \frac{2}{D}.(B - A.\mu) \end{cases}$$

Solution of the optimization program: optimal weights

The system rewrites:

$$\begin{cases} \mathbf{w} & = \dfrac{\lambda_1}{2}.\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + \dfrac{\lambda_2}{2}.\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota} \\ \\ \lambda_1 & = \dfrac{2}{D}.(C.\mu - A) \\ \\ \lambda_2 & = \dfrac{2}{D}.(B - A.\mu) \end{cases}$$

We then inject the expressions of λ_1 and λ_2 into the first equation:

$$\begin{aligned} \boldsymbol{w} &= \frac{1}{D}.(C.\mu - A).\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + \frac{1}{D}.(B - A.\mu).\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota} \\ &= \frac{1}{D}.(B.\boldsymbol{\Sigma}^{-1}.\boldsymbol{\iota} - A.\boldsymbol{\Sigma}^{-1}.\boldsymbol{\mu}) + \frac{1}{D}.(C.\boldsymbol{\Sigma}^{-1}.\boldsymbol{\mu} - A.\boldsymbol{\Sigma}^{-1}.\boldsymbol{\iota}).\mu \end{aligned}$$

Optimal weights of assets in investor's portfolio

At last, the optimal weights of assets in the investor's portfolio write:

$$\mathbf{w}^* = \mathbf{g} + \mathbf{h}\mu \text{ with } \begin{cases} \mathbf{g} = \dfrac{1}{D}.(B.\boldsymbol{\Sigma}^{-1}.\boldsymbol{\iota} - A.\boldsymbol{\Sigma}^{-1}.\boldsymbol{\mu}) \\ \\ \mathbf{h} = \dfrac{1}{D}.(C.\boldsymbol{\Sigma}^{-1}.\boldsymbol{\mu} - A.\boldsymbol{\Sigma}^{-1}.\boldsymbol{\iota}) \end{cases}$$

The investor chooses a portfolio according to his target expected return μ and parameters **g** and **h**, which depend on his estimates of assets's expected returns and covariances.

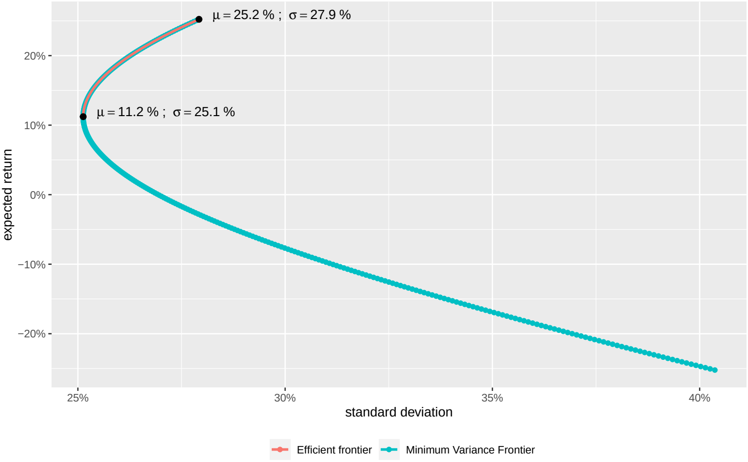
Equation of the Minimum Variance Frontier

At last, the equation of the Minimum Variance Frontier is:

$$
\begin{aligned}
\sigma =& \sqrt{\mathbf{w}^{*\,\prime}.\boldsymbol{\Sigma}.\mathbf{w}^*} = \sqrt{\mathbf{w}^{*\prime}.\boldsymbol{\Sigma}.(\frac{\lambda_1}{2}.\boldsymbol{\Sigma}^{-1}.\boldsymbol{\mu} + \frac{\lambda_2}{2}.\boldsymbol{\Sigma}^{-1}.\boldsymbol{\iota})} \\
=& \sqrt{\frac{\lambda_1}{2}.\mathbf{w}^{*\prime}.\boldsymbol{\Sigma}.\boldsymbol{\Sigma}^{-1}.\boldsymbol{\mu} + \frac{\lambda_2}{2}.\mathbf{w}^{*\prime}.\boldsymbol{\Sigma}.\boldsymbol{\Sigma}^{-1}.\boldsymbol{\iota}} = \sqrt{\frac{\lambda_1}{2}.\mu + \frac{\lambda_2}{2}} \\
=& \sqrt{\frac{1}{D}.(C.\mu^2 - 2.A.\mu + B)}
\end{aligned}
$$

Solution of the optimization program

We can plot the Minimum Variance Frontier in the standard deviation/expected return space.



Minimum Variance and Efficient Frontiers

The upper part of the frontier is called the **Efficient Frontier**, as it dominates the lower part: for any level of standard deviation, there exists a portfolio on the upper part and another one on the lower part but the portfolio on the upper part has a larger expected return, so it mean variance dominates the other portfolio, hence it is efficient.

Noting k the investor's risk target, a portfolio on the efficient frontier is found by solving:

$$\max_{\mathbf{w}} \ \mathbf{w}^{'}\boldsymbol{\mu} \quad \text{s.t.} \begin{cases} \mathbf{w}^{'}.\boldsymbol{\Sigma}\mathbf{w} & = k \\ \mathbf{w}^{'}\boldsymbol{\iota} & = 1 \end{cases}$$

Alternative solution of the optimization program

There is another way to solve the optimization problem presented above.

Starting again from the FOCs: $$\begin{cases} 2.\boldsymbol{\Sigma}\mathbf{w} - \lambda_1.\boldsymbol{\mu} - \lambda_2.\boldsymbol{\iota} & = 0 \\ \boldsymbol{\mu}'.\mathbf{w} & = \mu \\ \boldsymbol{\iota}'.\mathbf{w} & = 1 \end{cases}$$

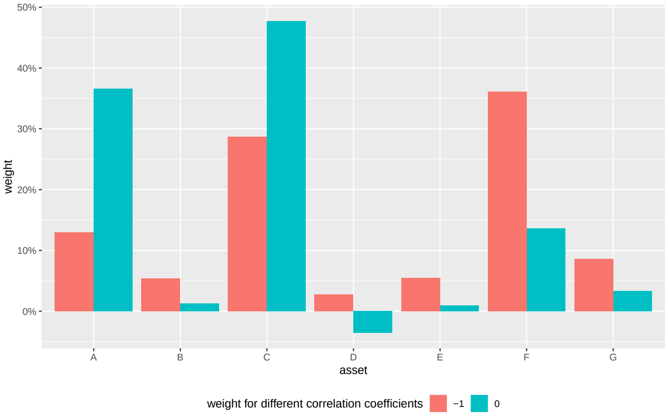
One introduces $\mathbf{M} = \begin{pmatrix} 2.\boldsymbol{\Sigma} & -\boldsymbol{\mu} & -\boldsymbol{\iota} \\ \boldsymbol{\mu}' & 0 & 0 \\ \boldsymbol{\iota}' & 0 & 0 \end{pmatrix}$. Thus $\mathbf{M}. \begin{pmatrix} \mathbf{w} \\ \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \mu \\ 1 \end{pmatrix}$

When $\mathbf{M}$ is invertible, the solution of the program writes:

$$\begin{pmatrix} \mathbf{w} \\ \lambda_1 \\ \lambda_2 \end{pmatrix} = \mathbf{M}^{-1}. \begin{pmatrix} 0 \\ \mu \\ 1 \end{pmatrix}$$

Impact of the correlation between asset classes on efficient portfolios

Figure 5: Efficient portfolio weights and correlation coefficient



For a 12% target of expected return, the efficient portfolio is way less concentrated in a few assets when asset returns are negatively correlated

The Global Minimum Variance Portfolio

The portfolio most on the left on the frontier is referred to as the Global Minimum Variance Portfolio (GMVP). It is obtained by solving the following program:

$$\min_{\mathbf{w}} \ \mathbf{w}^{'}.\mathbf{\Sigma}.\mathbf{w} \quad \text{s.t.} \ \mathbf{w}^{'}\boldsymbol{\iota} \ = 1$$

The Lagrangian writes:

$$L = \mathbf{w}'\mathbf{\Sigma}\mathbf{w} - \lambda.(\mathbf{w}'\boldsymbol{\iota} - 1) \tag{8}$$

FOCs write: $\begin{cases} \dfrac{\delta L}{\delta \mathbf{w}} = 2.\mathbf{\Sigma}\mathbf{w} - \lambda.\boldsymbol{\iota} & = 0 \\ \mathbf{w}^{'}\boldsymbol{\iota} & = 1 \end{cases}$

$$\Longrightarrow \begin{cases} \dfrac{\lambda}{2}.\mathbf{\Sigma}^{-1}\boldsymbol{\iota} & = \mathbf{w} \\ \boldsymbol{\iota}'\mathbf{w} & = 1 \end{cases} \Longrightarrow \begin{cases} \dfrac{\lambda}{2}.\mathbf{\Sigma}^{-1}\boldsymbol{\iota} & = \mathbf{w} \\ \\ \dfrac{\lambda}{2}.\boldsymbol{\iota}^{'}\mathbf{\Sigma}^{-1}\boldsymbol{\iota} & = 1 \end{cases}$$

The Global Minimum Variance Portfolio (GMVP)

The solution writes: $\mathbf{w}_{GMPV} = \dfrac{\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota}}{\boldsymbol{\iota}^{'}\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota}} = \dfrac{\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota}}{C}$

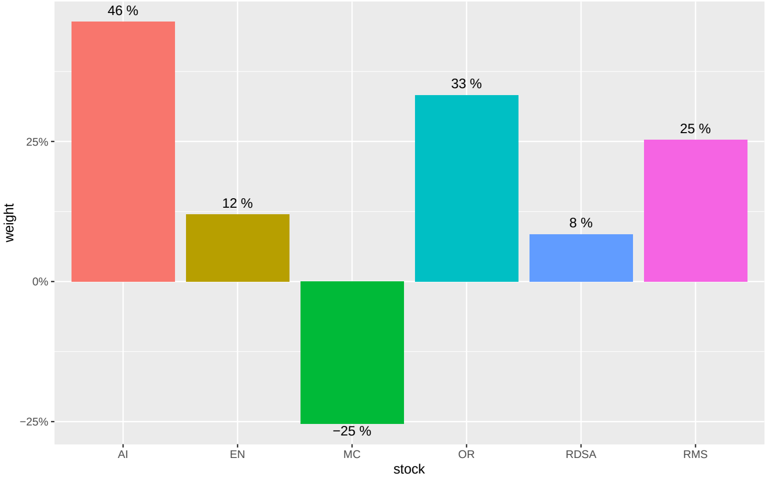
The expected return and the standard deviation on the GMVP write:

$$\begin{cases} \mu_{GMVP} & = \boldsymbol{\mu}^{'}.\mathbf{w}_{GMVP} = \dfrac{\boldsymbol{\mu}^{'}.\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota}}{\boldsymbol{\iota}^{'}\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota}} = \dfrac{A}{C} \\ \\ \sigma_{GMVP} & = \sqrt{\mathbf{w}^{'}_{GMVP}.\boldsymbol{\Sigma}.\mathbf{w}_{GMVP}} = \dfrac{1}{\sqrt{\boldsymbol{\iota}^{'}.\boldsymbol{\Sigma}^{-1}.\boldsymbol{\iota}}} = \dfrac{1}{\sqrt{C}} \end{cases}$$

Like for the equation of the minimum variance frontier, there is an alternate solution to find the weights of the GMVP.

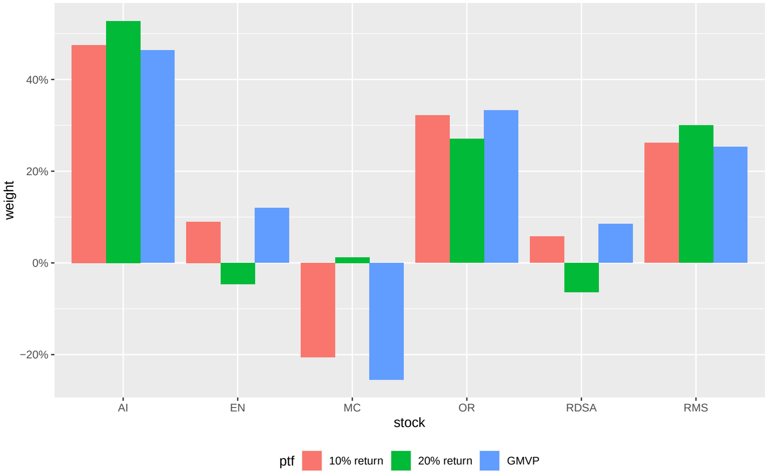
Weights of the Global Minimum Variance Portfolio

Figure 6: Weights of the unconstrained GMVP



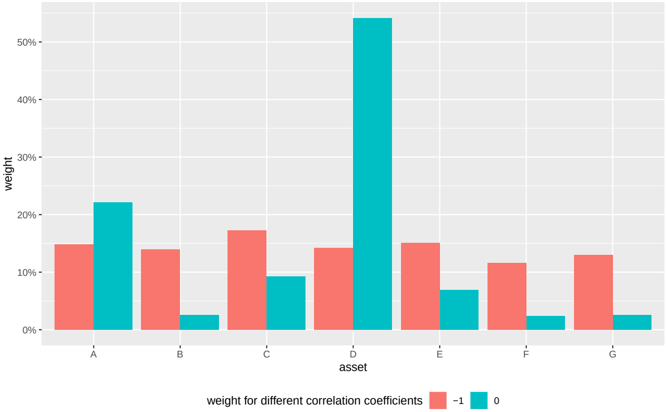
Compared weights

Figure 7: Compared weights between unconstrained GMVP and portfolios with 10% and 20% target returns



Impact of the correlation between asset classes on the GMVP

Figure 8: GMVP weights and correlation coefficient



The GMVP is way less concentrated in a few assets when asset returns are negatively correlated

The two-fund Black Theorem (1972)

According to Black's 2-fund separation theorem (1972), when there is no constraint on short sales, the asset weights of any minimum variance portfolio are a linear combination of the asset weights of any two other minimum variance portfolios. Thus the minimum variance frontier can be traced out once the weights of two minimum variance portfolios have been found out.

Equivalence of MVO with the quadratic utility function maximization

Equivalence with the quadratic utility function maximization

- All portfolios on the upper part of the frontier are mean-variance efficient, so all are relevant for an investor
- An investor selects a portfolio on the efficient frontier depending on his target expected return
  - But the result would be equivalent had he maximized a quadratic utility function!
  - Optimizing for all potential risk aversion coefficients λ is equivalent to optimizing for a series of corresponding targets of expected returns μ

Utility function and risk aversion coefficient

- One commonly used utility function in economics is the quadratic utility function:

$$U(X) = X - \frac{\lambda}{2}X^2 \tag{9}$$

with λ the risk aversion coefficient and X the investor's wealth

The higher the value of λ, the more risk averse the investor.

- The investor's expected utility writes:

$$\mathbb{E}[U(X)] = \mathbb{E}[X] - \frac{\lambda}{2}\mathbb{E}[X^2] \tag{10}$$

- X is usually the wealth generated by an asset, but the utility from an asset can be translated into a utility received from its future cash flows and thus its future return r_P:

$$\mathbb{E}[U(\widetilde{r}_P)] = \mathbb{E}[\widetilde{r}_P] - \frac{\lambda}{2}\mathbb{E}[\widetilde{r}_P^2] \tag{11}$$

Equivalence between Markowitz framework and maximization of expected utility

- To select the optimal portfolio, the investor maximizes the expected utility of the portfolio's return
- Yet, maximizing $\mathbb{E}[r_P] - \frac{\lambda}{2}\mathbb{E}[r_P^2]$ means, for a given $\mathbb{E}[r_P] = \mu$, to select a portfolio which minimizes $\mathbb{E}[r_P^2]$, thus a portfolio which minimizes $\mathbb{E}[r_P^2] - \mu^2$ as well.
- Eventually, this portfolio minimizes $\mathbb{E}[r_P^2] - \mathbb{E}[r_P]^2 = V(r_P)$. So this portfolio is actually the minimum variance portfolio among all portfolios of expected return μ.
- Thus, an agent with a quadratic utility function actually follows a mean-variance criteria. The higher the risk aversion coefficient, the more on the left of the efficient frontier the investor is positioned

Representing utility function and efficient frontier together

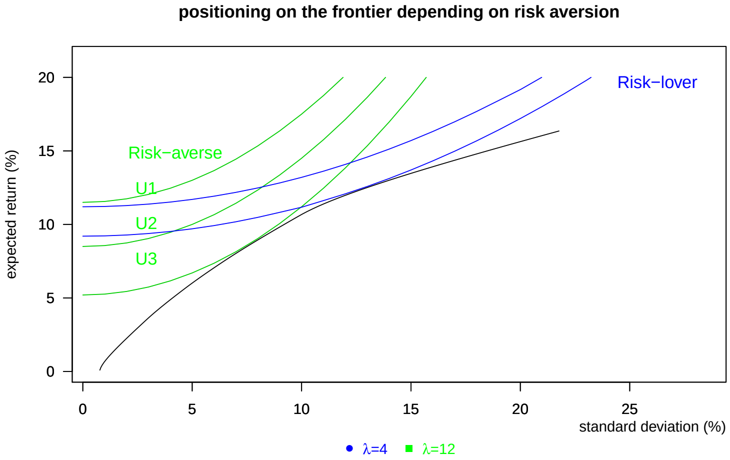
- For an investor of risk aversion coefficient λ, expected utility is maximized at r_P^* and is worth U^* ("certainty equivalent"):

$$\mathbb{E}[U(r_P^*)] = \mathbb{E}[r_P^*] - \frac{\lambda}{2}\mathbb{E}[{r_P^*}^2] = U^* \qquad (12)$$

This can be rewritten: $\mathbb{E}[r_P^*] = U^* + \frac{\lambda}{2}\mathbb{E}[{r_P^*}^2]$

- This equation can be plotted in the mean variance and in the mean standard deviation spaces, as an indifference curve. In the first instance this is a line of slope $\frac{\lambda}{2}$, in the second instance this is a hyperbola. The investor is indifferent between all the portfolios that lie on that curve. The point on this curve which is also located on the efficient frontier is the optimal portfolio for the investor, as is shown next.

Expected utility for all portfolios for different levels of risk aversion



Introduction of a riskfree asset and getting Capital Allocation Line

Characteristics of the riskfree asset and inclusion in a portfolio

A **riskfree asset** r_f yields the same return in all future states of nature:

$$\begin{cases} \mathbb{E}[r_f] = r_f \\ V(r_f) = 0 \\ Cov(r_f, r_j) = 0 \quad \text{for any risky asset } \ j \end{cases}$$

We build a portfolio i by combining a portfolio of risky assets P and the riskfree asset in proportions w_P and w_f, $w_f + w_P = 1$.
The portfolio return writes
$r_i = w_P.r_P + w_f.r_f = w_P.r_P + (1 - w_P).r_f = r_f + w_P.(r_P - r_f)$.

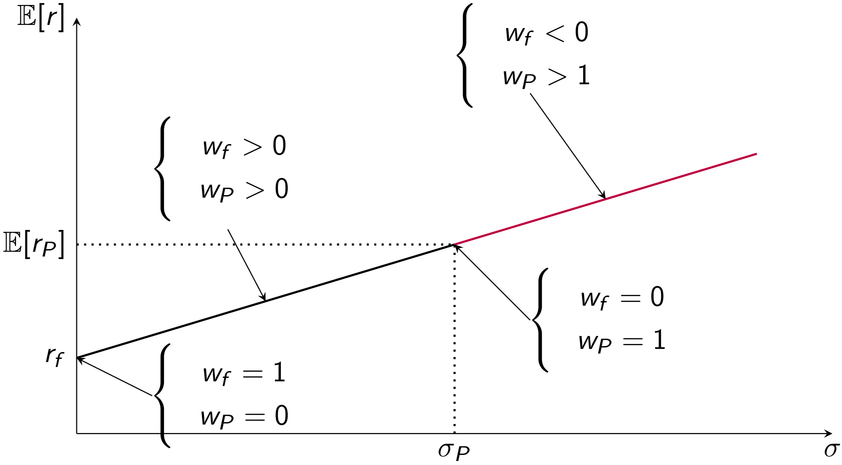
We then have:

$$\begin{cases} \mathbb{E}[r_i] = r_f + w_P.(\mathbb{E}[r_P] - r_f) \\ \sigma(r_i) = w_P.\sigma(r_P) \end{cases}$$

If $w_P \geq 100\%$, cash is borrowed at r_f to leverage on the risky asset, and $\sigma_i \geq \sigma_P$.

Portfolio including the riskfree asset: graphical representation

Since $w_P = \dfrac{\sigma(r_i(w_P))}{\sigma(r_P)}$, $\mathbb{E}[r_i] = r_f + \dfrac{\mathbb{E}[r_P] - r_f}{\sigma(r_P)}.\sigma(r_i)$.



What asset could stand for the riskfree asset?

The Markowitz framework being a 2-date model, expected returns and variances of assets are computed in the cross section at the investment end date.
This has implications for the riskfree asset identity: it should have the same payoff in all states of natures.
Cash and cash equivalents fulfill the conditions of the riskfree asset. The most relevant asset is a bond:

- issued by a default-free issuer
- with a fixed coupon rate, if any
- without reinvestment risk over the horizon of investment, thus maturing at the investment horizon, and not paying coupon during the period

What asset could stand for the riskfree asset?

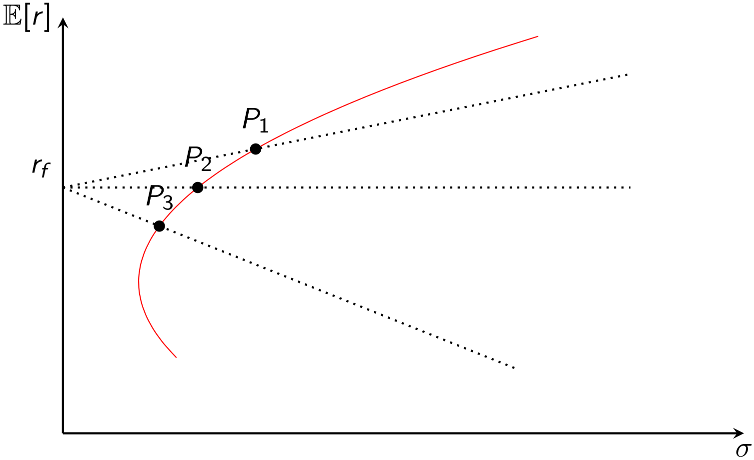
Thus a either a bill, or a bond with a residual maturity such that no coupon is paid during its residual life unless at maturity, or a Zero-coupon Bond, or a Strip of a coupon bond, issued by a sovereign entity (theoretically default free and also often in practice) maturing at the investment horizon can be used as a riskfree asset.

Any other asset incurs interest rate risk at the horizon date (either reinvestment, roll, or reset risk) or even credit risk.

NB: The riskfree asset price is not necessarily constant between t_0 and t_1

On a multiperiod framework, the timeseries of a bond's returns display variability, and thus they should be considered as risky assets and should be included in the mean variance optimizer.

Combination of the riskfree asset with different portfolios of risky assets located on the efficient frontier



Capital Allocation Line and Tangency Portfolio

One combination seems particularly interesting:

- When we combine the riskfree asset with the portfolio on the efficient frontier such that the line describing all combinations is tangent to the efficient frontier
- This line "mean variance dominates" any other line formed by combining the risk free asset with any other portfolio on the efficient frontier, but it also dominates the efficient frontier as well. This is the **Capital Allocation Line** (CAL).

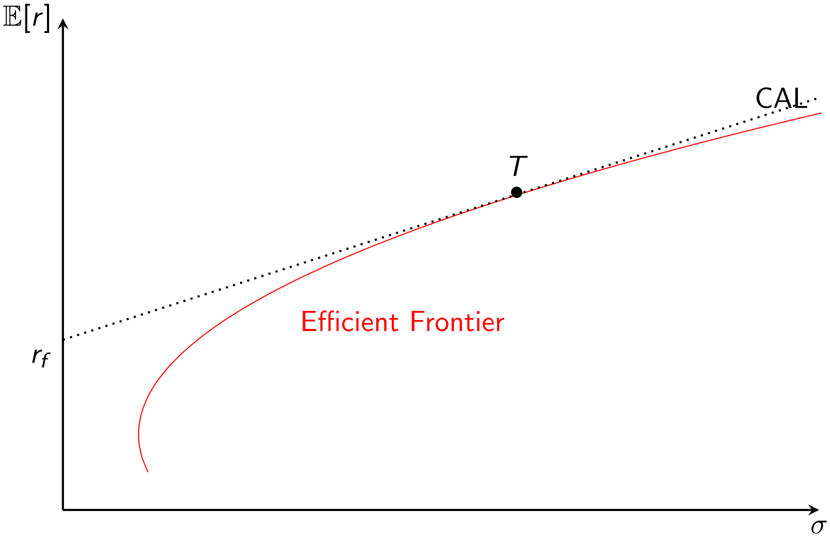
Taking the previous notations, we define the (ex ante) **Sharpe Ratio** (SR) as

$$SR = \frac{\mathbb{E}[r_P] - r_f}{\sigma_P} \tag{13}$$

The CAL has the highest Sharpe Ratio

The portfolio at the intersection of the CAL and the efficient frontier is named after the **Tangency portfolio, T**. It is the optimal portfolio among portfolios on the efficient frontier.

Capital Allocation Line and Tangency Portfolio



Optimal portfolio and Capital Allocation Line equation

The CAL goes through the riskfree asset and T, which is made of risky assets only. Portfolios on the CAL are portfolios made of those assets in varying proportions. To retrieve the equation of the CAL, let us constitute a portfolio made of the riskfree asset and the risky assets.

There are at least three methods to retrieve the optimal weights of assets in the portfolio, and at last the equation of the CAL:

- **By minimizing the variance of this portfolio, s.t. an expected return and a budget constraint**
- **By maximizing the Sharpe ratio of this portfolio, s.t. a budget constraint**.
- **By maximizing the investor's expected utility function if it is quadratic**

Optimal weights - 1st method

The investment universe is made of the same n risky assets and the riskfree asset. For an expected return target, again, the optimal portfolio is the one with the lowest variance, subject to a budget constraint:

$$\min_{\mathbf{x}} \ \mathbf{x}^{'}.\mathbf{V}.\mathbf{x} \quad \text{s.t.} \ \begin{cases} \mathbf{x}^{'}\boldsymbol{\mu}_{n+1} & = \mu \\ \mathbf{x}^{'}\boldsymbol{\iota}_{n+1} & = 1 \end{cases}$$

$\mathbf{x} = \begin{pmatrix} \mathbf{w} \\ w_f \end{pmatrix}$ is the $((n+1) \times 1)$ vector of the relative weights (%) of n risky assets and riskfree asset in the portfolio

$\boldsymbol{\mu}_{n+1} = \begin{pmatrix} \boldsymbol{\mu} \\ r_f \end{pmatrix}$ is the $((n+1) \times 1)$ vector of n risky assets and riskfree asset's expected returns

$\boldsymbol{\iota}_{n+1}$ is the $((n+1) \times 1)$ unitary vector

$\mathbf{V} = \begin{pmatrix} \boldsymbol{\Sigma} & \mathbf{0} \\ \mathbf{0}' & 0 \end{pmatrix}$ is the $((n+1) \times (n+1))$ covariances matrix of assets' returns

We also introduce $\widetilde{\mu} = \mu - r_f$ and the $(n \times 1)$ vector $\widetilde{\boldsymbol{\mu}} = \boldsymbol{\mu} - \boldsymbol{\iota}.r_f$.

Optimal weights - 1st method

Let us rewrite this optimization program as a function of $\boldsymbol{w}$ only.

Let us first observe that $\sigma_{i,f} = 0$ for any risky asset so $\mathbf{x}^{'}.\boldsymbol{V}\mathbf{x} = \mathbf{w}^{'}.\boldsymbol{\Sigma}\mathbf{w}$

We also have $\mathbf{x}^{'}\boldsymbol{\iota}_{n+1} = 1 \Leftrightarrow \mathbf{w}^{'}\boldsymbol{\iota} = 1 - w_f$

By internalizing the budget constraint, the expected return rewrites:

$$\mathbf{x}^{'}\boldsymbol{\mu}_{n+1} = \mathbf{w}^{'}\boldsymbol{\mu} + w_f.r_f = \mathbf{w}^{'}\boldsymbol{\mu} + (1 - \mathbf{w}^{'}\boldsymbol{\iota}).r_f = \mathbf{w}^{'}(\boldsymbol{\mu} - \boldsymbol{\iota}.r_f) + r_f = \mathbf{w}^{'}\widetilde{\boldsymbol{\mu}} + r_f$$

Thus $\mathbf{x}^{'}\boldsymbol{\mu}_{n+1} = \mu \Leftrightarrow \mathbf{w}^{'}\widetilde{\boldsymbol{\mu}} = \widetilde{\mu}$

The optimization program becomes:

$$\min_{\mathbf{w}} \ \mathbf{w}^{'}.\boldsymbol{\Sigma}\mathbf{w} \quad \text{s.t. } \mathbf{w}^{'}\widetilde{\boldsymbol{\mu}} = \widetilde{\mu}$$

Optimal weights - 1st method

The Lagrangian then writes:

$$L = \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} + \lambda.(\widetilde{\mu} - \mathbf{w}'\widetilde{\boldsymbol{\mu}}) \tag{14}$$

FOCs write:

$$\begin{cases} \dfrac{\delta L}{\delta \mathbf{w}} = 2.\boldsymbol{\Sigma}\mathbf{w} - \lambda.\widetilde{\boldsymbol{\mu}} & = 0 \\ \mathbf{w}'\widetilde{\boldsymbol{\mu}} & = \widetilde{\mu} \end{cases} \Longrightarrow \begin{cases} \dfrac{\lambda}{2}.\boldsymbol{\Sigma}^{-1}\widetilde{\boldsymbol{\mu}} & = \mathbf{w} \\ \widetilde{\boldsymbol{\mu}}'.\mathbf{w} & = \widetilde{\mu} \end{cases}$$

$$\Longrightarrow \begin{cases} \dfrac{\lambda}{2}.\boldsymbol{\Sigma}^{-1}\widetilde{\boldsymbol{\mu}} & = \mathbf{w} \\ \\ \dfrac{\lambda}{2}.\widetilde{\boldsymbol{\mu}}'\boldsymbol{\Sigma}^{-1}\widetilde{\boldsymbol{\mu}} & = \widetilde{\mu} \end{cases} \Longrightarrow \begin{cases} \mathbf{w} & = \dfrac{\lambda}{2}.\boldsymbol{\Sigma}^{-1}\widetilde{\boldsymbol{\mu}} \\ \\ \dfrac{\lambda}{2} & = \dfrac{\widetilde{\mu}}{\widetilde{\boldsymbol{\mu}}'\boldsymbol{\Sigma}^{-1}\widetilde{\boldsymbol{\mu}}} \end{cases} \Longrightarrow \mathbf{w} = \frac{\boldsymbol{\Sigma}^{-1}\widetilde{\boldsymbol{\mu}}}{\widetilde{\boldsymbol{\mu}}'\boldsymbol{\Sigma}^{-1}\widetilde{\boldsymbol{\mu}}}.\widetilde{\mu}$$

At last, the weight in the riskfree asset w_f is retrieved.

Optimal weights - 2nd method

The CAL is also the line in the mean-variance plan with the highest Sharpe ratio. Thus it can be found by maximizing the Sharpe Ratio of any portfolio s.t. a budget constraint:

$$\max_{\mathbf{x}} \; SR_P \quad \text{s.t. } \mathbf{x}^{'}\boldsymbol{\iota}_{n+1} \; = 1$$

We have $\mathbf{x}^{'}.\boldsymbol{\mu}_{n+1} - r_f = \mathbf{w}^{'}\widetilde{\boldsymbol{\mu}}$ and $\mathbf{x}^{'}.\mathbf{V}.\mathbf{x} = \mathbf{w}^{'}.\boldsymbol{\Sigma}.\mathbf{w}$.

At last:

$$\begin{aligned} & \max_{\mathbf{x}} \; SR_P \quad \text{s.t. } \mathbf{x}^{'}\boldsymbol{\iota}_{n+1} \; = 1 \\ \iff \quad & \max_{\mathbf{w}} \; \frac{\mathbf{w}^{'}\widetilde{\boldsymbol{\mu}}}{\sqrt{\mathbf{w}^{'}.\boldsymbol{\Sigma}\mathbf{w}}} \end{aligned}$$

Optimal weights - 2nd method

Now to find the CAL we only have to maximize the previous expression, and we get:

$$\frac{\widetilde{\mu}.\sqrt{\mathbf{w}^{'}.\mathbf{\Sigma w}} - \mathbf{w}^{'}.\widetilde{\mu}.2.\mathbf{\Sigma w}.\frac{1}{2}.(\mathbf{w}^{'}.\mathbf{\Sigma w})^{-\frac{1}{2}}}{\mathbf{w}^{'}.\mathbf{\Sigma w}} = 0 \tag{15}$$

$\Longrightarrow \widetilde{\boldsymbol{\mu}}.(\boldsymbol{w}^{'}.\mathbf{\Sigma}\mathrm{w})^{\frac{1}{2}} - \widetilde{\mu}.\mathbf{\Sigma w}.(\mathbf{w}^{'}.\mathbf{\Sigma w})^{-\frac{1}{2}} = 0$, by posing $\widetilde{\mu} = \mathbf{w}'\widetilde{\boldsymbol{\mu}}$

$\Longrightarrow \widetilde{\boldsymbol{\mu}} = \widetilde{\mu}.\frac{\mathbf{\Sigma w}}{\mathbf{w}^{'}.\mathbf{\Sigma w}} \quad \Longrightarrow \widetilde{\boldsymbol{\mu}} = \mathbf{\Sigma}\boldsymbol{w}.\frac{\widetilde{\mu}}{\sigma^2}$, by posing $\sigma = \mathbf{w}'\mathbf{\Sigma}.\mathbf{w}$

$\Longrightarrow \widetilde{\boldsymbol{\mu}} = \mathbf{\Sigma}\boldsymbol{w}.\delta$, by posing $\delta = \frac{\widetilde{\mu}}{\sigma^2}$

At last, $\boldsymbol{w}^* = \frac{\mathbf{\Sigma}^{-1}.\widetilde{\boldsymbol{\mu}}}{\delta}$, and w_f is retrieved then.

Optimal weights - 3rd method

Here we maximize the investor's expected utility, with λ his risk aversion coefficient:

$$\max_{\mathbf{x}} \ \mathbf{x}^{'}\boldsymbol{\mu}_{n+1} - \frac{\lambda}{2}\mathbf{x}'\boldsymbol{V}\mathbf{x} \quad \text{s.t. } \mathbf{x}^{'}\boldsymbol{\iota} = 1$$

By internalizing the budget constraint, this rewrites:

$$\max_{\mathbf{w}} \ \mathbf{w}^{'}\boldsymbol{\mu} + (1 - \mathbf{w}'\boldsymbol{\iota}).r_f - \frac{\lambda}{2}\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} \quad \Longleftrightarrow \quad \max_{\mathbf{w}} \quad \mathbf{w}^{'}\widetilde{\boldsymbol{\mu}} - \frac{\lambda}{2}\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}$$

Optimal weights are given by : $\widetilde{\boldsymbol{\mu}} - \lambda\boldsymbol{\Sigma}\mathbf{w} = 0$

$$\text{At last, } \boldsymbol{w}^* = \frac{\boldsymbol{\Sigma}^{-1}.\widetilde{\boldsymbol{\mu}}}{\lambda}$$

Optimal weights - summary

The results from the three methods to get optimal weights, which are equivalent, are summarized as follows:

1. $\mathbf{w}^* = \dfrac{\boldsymbol{\Sigma}^{-1}\widetilde{\boldsymbol{\mu}}}{\widetilde{\boldsymbol{\mu}}'\boldsymbol{\Sigma}^{-1}\widetilde{\boldsymbol{\mu}}}.\widetilde{\mu}$
2. $\boldsymbol{w}^* = \dfrac{\boldsymbol{\Sigma}^{-1}.\widetilde{\boldsymbol{\mu}}}{\delta}$
3. $\boldsymbol{w}^* = \dfrac{\boldsymbol{\Sigma}^{-1}.\widetilde{\boldsymbol{\mu}}}{\lambda}$

It becomes obvious that the parameter δ introduced in the Sharpe Ratio maximization is the risk aversion coefficient of the investor!

In the MVO framework, finding optimal weights for a portfolio can be done by fixing a target expected return μ, or equivalently by formulating a risk aversion coefficient λ.

Capital Allocation Line equation

Using the first expression of optimal weights, the equation of the CAL then writes:

$$\sigma = \sqrt{\mathbf{w}_P'.\mathbf{\Sigma}.\mathbf{w}_P} = \sqrt{\mathbf{w}_P'.\mathbf{\Sigma}.\frac{\mathbf{\Sigma}^{-1}\widetilde{\boldsymbol{\mu}}}{\widetilde{\boldsymbol{\mu}}'\mathbf{\Sigma}^{-1}\widetilde{\boldsymbol{\mu}}}.\widetilde{\mu}} = \sqrt{\mathbf{w}_P'.\frac{\widetilde{\boldsymbol{\mu}}}{\widetilde{\boldsymbol{\mu}}'\mathbf{\Sigma}^{-1}\widetilde{\boldsymbol{\mu}}}.\widetilde{\mu}}$$

$$= \sqrt{\frac{\widetilde{\mu}}{\widetilde{\boldsymbol{\mu}}'\mathbf{\Sigma}^{-1}\widetilde{\boldsymbol{\mu}}}.\widetilde{\mu}} = \sqrt{\frac{\widetilde{\mu}^2}{\widetilde{\boldsymbol{\mu}}'\mathbf{\Sigma}^{-1}\widetilde{\boldsymbol{\mu}}}} = \frac{|\widetilde{\mu}|}{\sqrt{\widetilde{\boldsymbol{\mu}}'\mathbf{\Sigma}^{-1}\widetilde{\boldsymbol{\mu}}}} = \frac{\widetilde{\mu}}{\sqrt{\widetilde{B}}}$$

With $\widetilde{B} = \widetilde{\boldsymbol{\mu}}'\mathbf{\Sigma}^{-1}\widetilde{\boldsymbol{\mu}}$

This is the equation of a line indeed!

It can rewrite $\mu = r_f + \sqrt{\widetilde{B}}.\sigma$.

Tobin's separation theorem (1958)

The positioning of the investor on the CAL is found in one of the three methods explained above, and depends on a target expected return or a risk aversion coefficient.

Actually, that decision can be split into two independent decisions (Tobin, 1958):

- **An investment decision**, which consists of building an efficient portfolio of risky assets only T, independently from the investor's objective. This is achieved by identifying the tangency portfolio
- **A financing decision**, which consists of finding the optimal mix between the riskfree asset and the tangency portfolio, in accordance with the investor's risk aversion λ. This is achieved by finding the optimal combination between the tangency portfolio and the riskfree asset by maximizing the investor's expected utility.

Instead of fixing $\widetilde{\mu}$ to get $\mathbf{w}^*$, one finds T (common to all investors) and one finds w_f^* and w_T^* by fixing λ.

The tangency portfolio belongs to both the EF only made of risky assets and the CAL. Instead of finding their intersection in the mean standard deviation space, let us find the weights of the different assets in the portfolio, and then let us retrieve the portfolio's coordinates.
Starting from the formula of optimal weights of a portfolio located on the CAL, it is sufficient to add the condition that $\boldsymbol{\iota}'.\mathbf{w} = 1$, as there is one portfolio on the CAL made of risky assets only:

$$\begin{cases} \mathbf{w} & = \dfrac{\boldsymbol{\Sigma}^{-1}\widetilde{\boldsymbol{\mu}}}{\widetilde{\boldsymbol{\mu}}'\boldsymbol{\Sigma}^{-1}\widetilde{\boldsymbol{\mu}}}.\widetilde{\mu} \\ \boldsymbol{\iota}'.\mathbf{w} & = 1 \end{cases} \implies \begin{cases} \mathbf{w} & = \dfrac{\boldsymbol{\Sigma}^{-1}\widetilde{\boldsymbol{\mu}}}{\widetilde{\boldsymbol{\mu}}'\boldsymbol{\Sigma}^{-1}\widetilde{\boldsymbol{\mu}}}.\widetilde{\mu} \\ 1 & = \dfrac{\boldsymbol{\iota}'.\boldsymbol{\Sigma}^{-1}\widetilde{\boldsymbol{\mu}}}{\widetilde{\boldsymbol{\mu}}'\boldsymbol{\Sigma}^{-1}\widetilde{\boldsymbol{\mu}}}.\widetilde{\mu} \end{cases}$$

$$\implies \begin{cases} \mathbf{w} & = \boldsymbol{\Sigma}^{-1}\widetilde{\boldsymbol{\mu}}.\dfrac{\widetilde{\mu}}{\widetilde{\boldsymbol{\mu}}'\boldsymbol{\Sigma}^{-1}\widetilde{\boldsymbol{\mu}}} \\ \dfrac{\widetilde{\mu}}{\widetilde{\boldsymbol{\mu}}'\boldsymbol{\Sigma}^{-1}\widetilde{\boldsymbol{\mu}}} & = \dfrac{1}{\boldsymbol{\iota}'.\boldsymbol{\Sigma}^{-1}\widetilde{\boldsymbol{\mu}}} \end{cases} \implies \boldsymbol{w}_T = \frac{\boldsymbol{\Sigma}^{-1}\widetilde{\boldsymbol{\mu}}}{\boldsymbol{\iota}'.\boldsymbol{\Sigma}^{-1}\widetilde{\boldsymbol{\mu}}}!$$

The tangency portfolio - Coordinates

We pose $\boldsymbol{\iota}'.\boldsymbol{\Sigma}^{-1}\widetilde{\boldsymbol{\mu}} = \widetilde{A}$, hence $\mathbf{w}_T = \dfrac{\boldsymbol{\Sigma}^{-1}\widetilde{\boldsymbol{\mu}}}{\widetilde{A}}$

The expected return and the standard deviation of the tangency portfolio then write:

$$\begin{cases} \mu_T = r_f + \widetilde{\mu}_T = r_f + \widetilde{\boldsymbol{\mu}}'.\mathbf{w}_T = r_f + \dfrac{\widetilde{\boldsymbol{\mu}}'\boldsymbol{\Sigma}^{-1}\widetilde{\boldsymbol{\mu}}}{\widetilde{A}} = r_f + \dfrac{\widetilde{B}}{\widetilde{A}} \\ \sigma_T = \sqrt{\mathbf{w}_T'.\boldsymbol{\Sigma}.\mathbf{w}_T} = \sqrt{\dfrac{\widetilde{\mu}_T}{\widetilde{A}}} = \sqrt{\dfrac{\mu_T - r_f}{\widetilde{A}}} = \sqrt{\dfrac{\widetilde{B}}{\widetilde{A}}/\widetilde{A}} = \dfrac{\sqrt{\widetilde{B}}}{\widetilde{A}} \end{cases}$$

It can be checked that the slope of the CAL which is given by $\dfrac{\widetilde{\mu}_T}{\sigma_T}$, is worth $\sqrt{\widetilde{B}}$

The tangency portfolio optimal weights - What is λ equal to?

It is also possible to retrieve the weights of risky assets in the tangency portfolio from the other optimization programs. For instance, using the result from the maximization of expected utility, we end up with the same expression of weights:

$$\begin{cases} \mathbf{w} & = \dfrac{\boldsymbol{\Sigma}^{-1}.\widetilde{\boldsymbol{\mu}}}{\lambda} \\ \\ \boldsymbol{\iota}'.\mathbf{w} & = 1 \end{cases} \implies \mathrm{w}_T = \frac{\boldsymbol{\Sigma}^{-1}\widetilde{\boldsymbol{\mu}}}{\boldsymbol{\iota}'\boldsymbol{\Sigma}^{-1}\widetilde{\boldsymbol{\mu}}} with \lambda_T = \boldsymbol{\iota}'\boldsymbol{\Sigma}^{-1}\widetilde{\boldsymbol{\mu}}$$

λ_T stands for the risk aversion coefficient of the investor holding the tangency portfolio only.

The same result can also be obtained with the Sharpe Ratio maximization.

Financing decision: portfolio selection on the CAL

Then the investor applies the classical mean variance criteria, but not on individual assets, rather on the riskfree asset and the tangency portfolio, to find their optimal weights in his portfolio:

$$\max_{w_T} \quad \mathbb{E}[R_P] - \frac{\lambda}{2}.\sigma_P^2 \quad \text{s.t.} \quad w_T + w_f \quad = 1.$$

If w_T is the portion of his wealth invested in the tangency portfolio, the portfolio return writes: $R_P = w_T.(r_T - r_f) + r_f$

The expected return and variance of return of this portfolio write:

$$\begin{cases} \mathbb{E}[R_P] & = w_T.\mathbb{E}[r_T] + (1 - w_T).r_f \\ \sigma_P^2 & = w_T^2.\sigma_T^2 \end{cases}$$

Portfolio selection on the CAL

Maximizing the objective function of the investor is equivalent to maximizing his expected utility (see infra), and it writes:

$$\mathbb{E}[U(P)] = w_T.(\mathbb{E}[r_T] - r_f) + r_f - \frac{\lambda}{2}.w_T^2.\sigma_T^2$$

We obtain: $$\begin{cases} \dfrac{\delta \mathbb{E}[U(P)]}{\delta w_T} = \mathbb{E}[r_T] - r_f - \lambda.w_T.\sigma_T^2 & = 0 \\ w_T + w_f & = 1 \end{cases}$$

At last, $w_T^* = \dfrac{\mathbb{E}[r_T] - r_f}{\lambda.\sigma_T^2}$

The higher the investor's risk aversion coefficient, the lower the share of his wealth invested in the tangency portfolio.
Individual weights on risky assets can be retrieved by multiplying by w_T^* the vector of optimal weights $\mathbf{w}_T$.

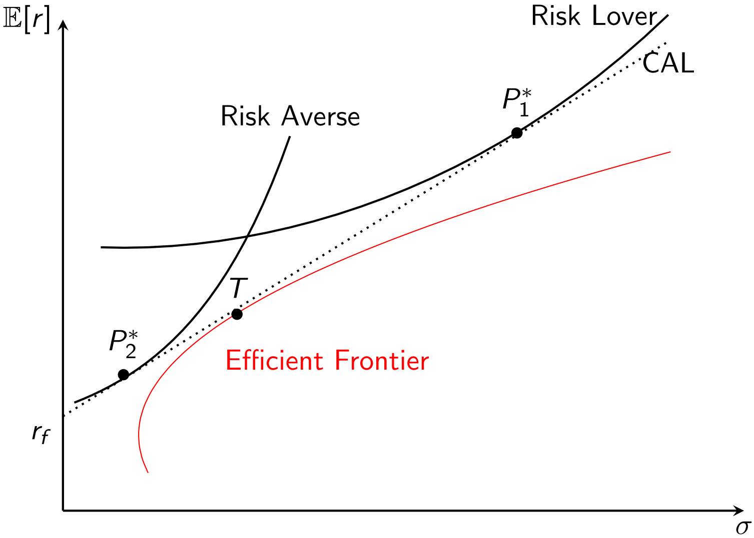
Revisiting positioning on CAL using Market risk aversion coefficient

One can also show that $\frac{\mathbb{E}[r_T] - r_f}{\sigma_T^2} = \lambda_T$, by finding the weights of risky assets in the tangency portfolio using the Sharpe Ratio maximization method.

w_T^* then rewrites $w_T^* = \frac{\lambda_T}{\lambda}$

Accordingly, if the investor is risk averse, the fraction of his wealth invested in the market portfolio is inferior to 1, and larger than 1 if he is risk lover.

Capital Allocation Line and Tangency Portfolio



From variance to other measures of risk

Variance can be a non relevant measure of the risk on a portfolio, and the quadratic utility function can be put into question. Investors might be averse to downside deviations rather than to all deviations of returns around the mean. In this case other measures of risk can replace variance:

- Semi standard deviation, only returns below the mean return will be taken into account
- Shortfall probability, the shortfall being defined with a return threshold
- Expected shortfall

This paves the way for other optimization programs beyond the Markowitz framework.

Adding a short selling constraint

Why impose positive weights in portfolio construction

- Negative weights are usually not relevant in a Strategic Asset Allocation. No benchmark has negative weights
  Short selling can of course be used tactically
- Institutional or regulatory constaints: short selling implies an increase in leverage which can be banned by internal risk management constraints or regulation
- Some assets might not be "shortable" (when there is no repo or derivative market)

Sign-constrained optimization program (no riskfree asset)

- In the absence of a riskfree asset, the program now writes:

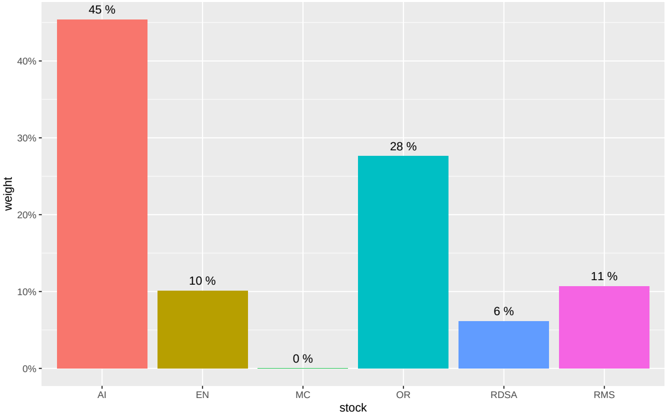
$$\min_{\mathbf{w}} \ \mathbf{w}^{'}.\boldsymbol{\Sigma}\mathbf{w} \quad \text{s.t.} \ \begin{cases} \mathbf{w}^{'}\boldsymbol{\mu} = \mu \\ \mathbf{w}^{'}\boldsymbol{\iota} = 1 \\ \mathbf{w} \geq \mathbf{w_{min}} \end{cases}$$

with $\mathbf{w_{min}}$ a vector of 0's

- There is no closed form solution of the optimization program anymore, but there are numerical solutions, which are found by trial and error

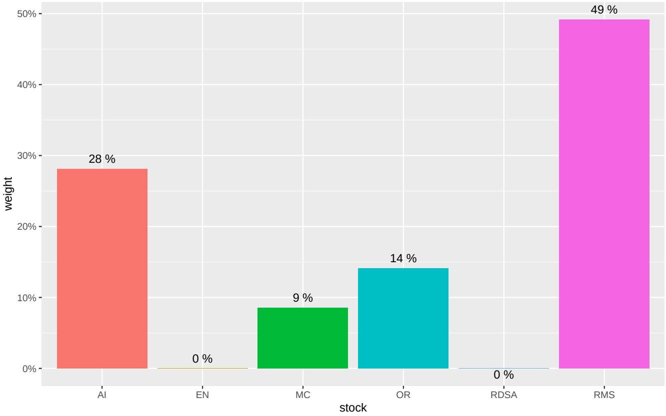
Sign-constrained optimal portfolio weights with a 10% target return

Figure 9: Sign-constrained optimal portfolio weights with a 10% target return



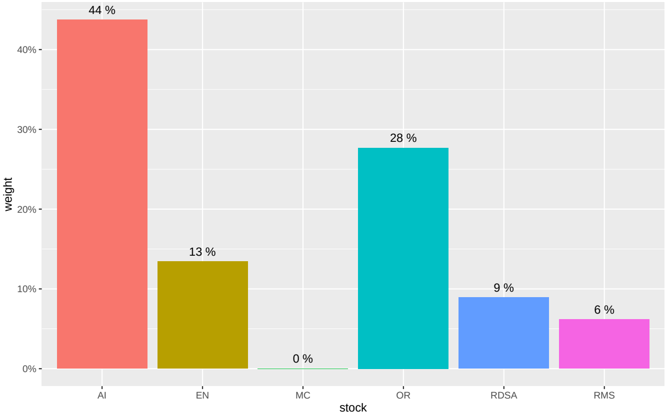
Sign-constrained optimal portfolio weights with a 20% target return

Figure 10: Sign-constrained optimal portfolio weights with a 20% target return



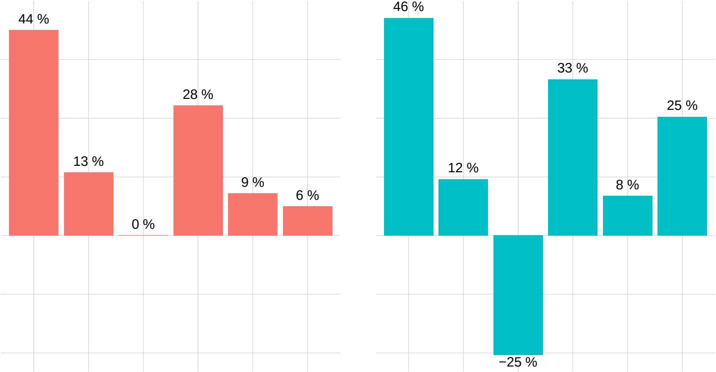
GMVP without short selling

Figure 11: Weights of the sign-constrained GMVP



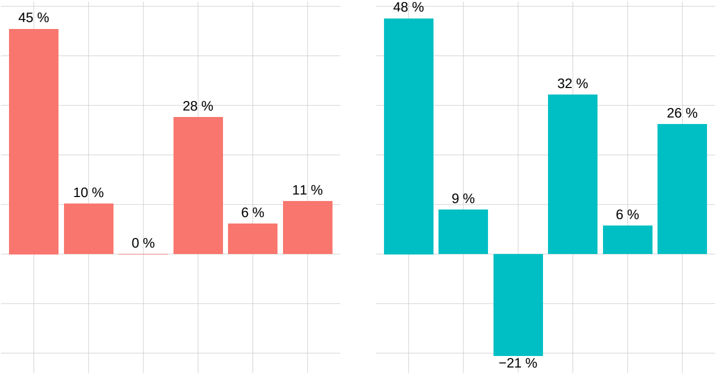
GMVPs compared

Figure 12: Weights of unconstrained and sign-constrained GMVPs



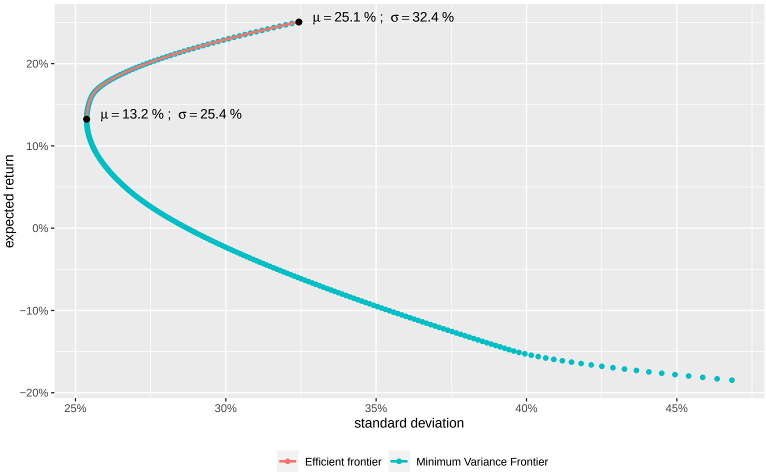
Portfolios with 20% expected return compared

Figure 13: Weights of unconstrained and sign-constrained portfolios with 20% target return



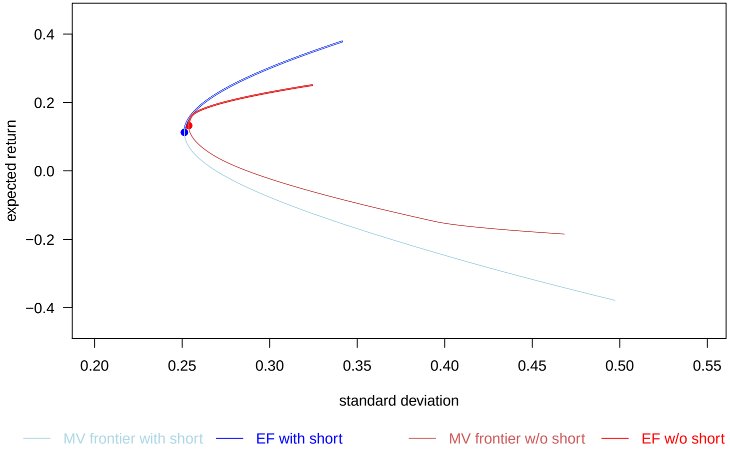
Sign-constrained Minimum Variance and Efficient Frontiers

Figure 14: Sign constrained Minimum Variance and Efficient Frontiers



Minimum Variance and Efficient Frontiers compared

Figure 15: Unconstrained and sign-constrained frontiers



Sign-constrained optimization: new properties of the efficient frontier

- The sign-constrained efficient frontier lies within the unconstrained efficient frontier (below for positive returns and above for negative returns)
  - In the unconstrained set up, by selling short an asset with a low expected return, one can increase to more than 100% the position on a high yielding asset, thus building portfolios with expected returns larger than the highest yielding security.
  - Symmetrically, portfolios of negative returns can be attained by selling short assets with high expected returns and buying assets of lower returns. In the constrained set up, portfolios always have positive returns if all assets have positive returns.
- On the constrained frontier, the portfolio with the largest expected return is 100% invested into the asset with the largest expected return and the portfolio with the lowest expected return is 100% invested into the asset with the lowest expected return.

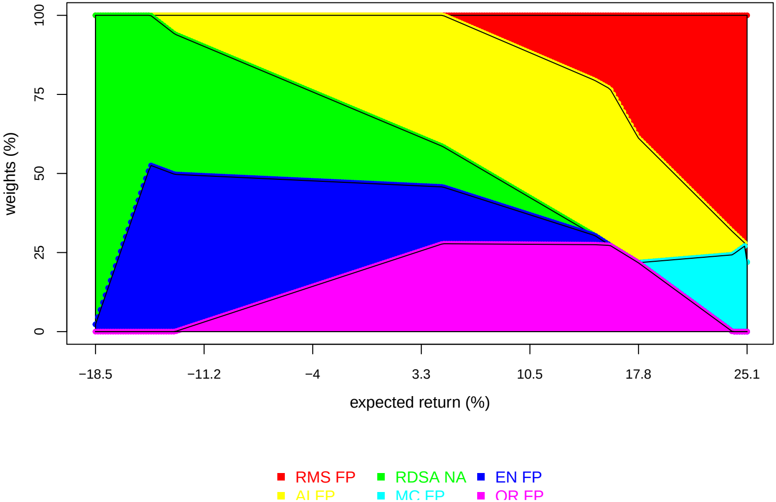
Sign-constrained frontier: Corner Portfolios

Corner portfolios are a new characteristic of the efficient frontier when short positions are forbidden.

- They are points along the efficient frontier at which the weight for one of the constituent goes from positive to zero, or from zero to positive.
- They are relatively few in numbers
- Any minimum variance portfolio can be found uniquely with corner portfolios: the asset weights of any minimum variance portfolio are a positive linear combination of the corresponding weights in the two adjacent corner portfolios that surround it in terms of expected return and standard deviation.
- **The Global Minimum Variance Portfolio and the highest yielding asset are Corner portfolios.**

Weights across the frontier

Figure 16: Weights of sign constrained efficient portfolios



Adding lower and upper bounds on weights

Optimization with bounded weights (no riskfree asset)

- Banning short selling allows to eliminate extreme short positions, but leads to a heavy concentration of positions
- To avoid this, we can add a new constraint of maximum weights on any asset in the portfolio
- The program now writes:

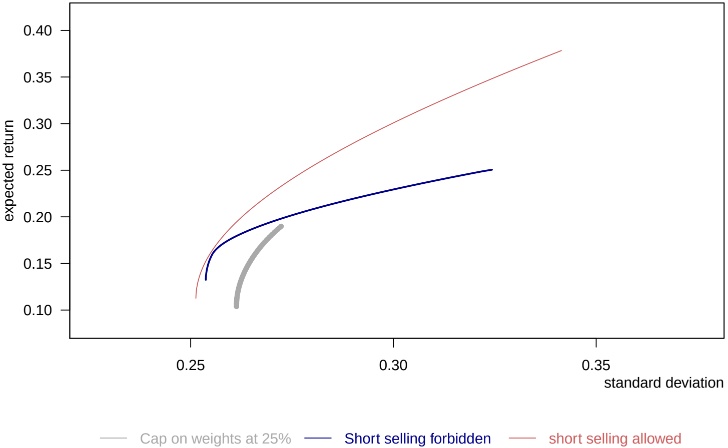
$$\min_{\mathbf{w}} \ \mathbf{w}^{'}.\boldsymbol{\Sigma}\mathbf{w} \quad \text{s.t.} \ \begin{cases} \mathbf{w}^{'}\boldsymbol{\mu} = \mu \\ \mathbf{w}^{'}\boldsymbol{\iota} = 1 \\ \mathbf{w} \geq \mathbf{w}_{min} \\ \mathbf{w} \leq \mathbf{w}_{max} \end{cases}$$

$\mathbf{w}_{max}$ is a vector of maximum weights

- Again, there is no closed form solution to such a program

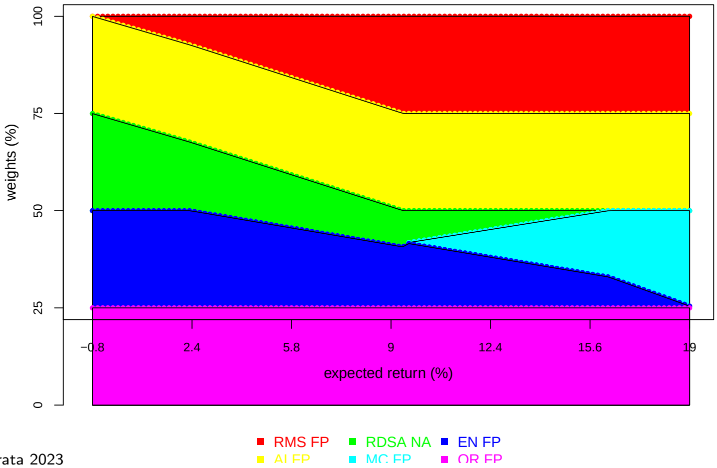
Unconstrained frontier, sign constrained frontier and frontier with cap on weights at 25%

Figure 17: Sign constrained efficient frontier and efficient frontier with caps on weights at 25%



Weights across the sign-constrained efficient frontier with caps on weights at 25%

Figure 18: Weights of sign-constrained efficient portfolios with caps on weights at 25%



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