The Capital Asset Pricing Model

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Outline

- **The Capital Market Line**
- **The Security Market Line**
- **Implied Performance Measures**
- **Extensions of the model**
- **Appendix**

The Capital Market Line

CAPM assumptions

- There are no transaction costs on assets (e.g., commissions or bid-offer spreads) and there are no taxes on income or capital gains
- All assets can be traded publicly, are indefinitely divisible, and can be sold short
- There exists a riskfree asset which can be lent or sold short without any limit
- Investors have mean-variance preferences
- Investors have homogeneous expectations on risk and return of every security
- Markets are perfectly competitive, i.e. investors are price takers. Demand from individual investors cannot influence the equilibrium price

From the Capital Allocation Line to the Capital Market Line

Since all investors have the same expectations on securities, they all share the same Capital Allocation Line

- Which is then named after the **Capital Market Line (CML)**
- Thus all investors allocate part of their wealth in the same tangency portfolio and the remainders in the riskfree asset

What differs between them is only the fraction of their wealth invested in the tangency portfolio.

Introducing the market porfolio and reaching an equilibrium

- **The market portfolio represents all assets supplied in the market**. The proportion of each security in the market portfolio is its market value as a proportion of the market value of all securities.
- To reach an equilibrium, the demand from investors (i.e., the tangency portfolio) should thus equalize with the supply (i.e., the market portfolio): the structure of the tangency portfolio adjusts to match the structure of the market portfolio. Thus, weights of assets in the tangency portfolio eventually match the proportions of assets in the market portfolio, which are exogeneous.
  - The equalization of demand with supply is reached through an adjustment in securities prices, and thus in their expected returns.
  - Thus the efficient frontier is determined by the supply of assets, via variations in expected returns, such that eventually $T = M$.

Equation of the Capital Market Line (CML)

- Remember from previous chapter that the weights of risky assets in the tangency portfolio write:
$\mathbf{w}_T = \frac{\mathbf{\Sigma}^{-1}\widetilde{\boldsymbol{\mu}}}{\lambda_T}$
- At equilibrium, the weights of assets in the tangency portfolio equalize with their weights in the Market portfolio. Thus we have:
$\frac{\mathbf{\Sigma}^{-1}\widetilde{\boldsymbol{\mu}}}{\lambda_T} = \mathbf{w}_M$, and at last $\widetilde{\boldsymbol{\mu}} = \lambda_M.\mathbf{\Sigma}\mathbf{w}_M = \widetilde{\mathbf{\Pi}}$, with $\lambda_T = \lambda_M$

  This is the equation of the Capital Market Line (CML).

  By multiplying both sides of the equation by $\mathbf{w}'_M$, we show that $\widetilde{\Pi}_M = \lambda_M.\sigma_M^2$ and at last $\lambda_M = \frac{\widetilde{\Pi}_M}{\sigma_M^2}$.

  This is the risk aversion coefficient of the market as a whole

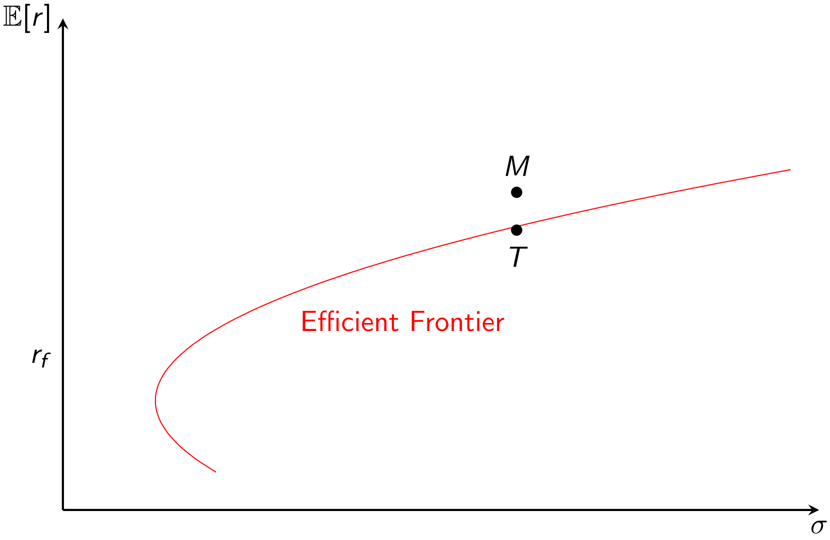
Reaching equilibrium via expected returns

- Thus, equilibrium has been obtained by forcing the weights in portfolio T to be equal to the weights in portfolio M, and $\widetilde{\boldsymbol{\mu}}$ have become $\widetilde{\boldsymbol{\Pi}}$: expected returns have become equilibrium required returns.
- Thus, the efficient frontier has adjusted to supply!

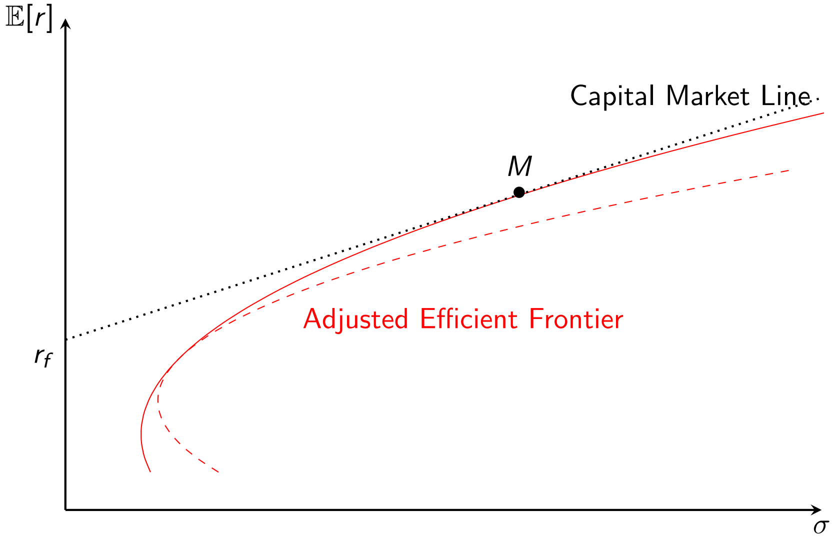
How could equilibrium be achieved?

- If we have for instance for asset i $w_T^i \leq w_M^i$, demand for the asset is too low. It equalizes supply with a rise in its price, which at last increases its weight in the tangency portfolio.
- A rise in the price can only be obtained by a discount rate going down, i.e., $\widetilde{\mu}_i$ has to go down to equalize $\widetilde{\Pi}_i$.

Capital Market Line: before equilibrium



Capital Market Line: at equilibrium



The Security Market Line

From the Capital Market Line to the Security Market Line

Reaching an equilibrium also implies that the line tangent to the efficient frontier only made of risky assets at point $T = M$ should be the CML, as both lines should have the same slope at point M:

- If the slope of the line tangent was lower, some portfolios on the efficient frontier only made of risky assets would stand above the CML for portfolios less risky than M
- If the slope of the line tangent was higher, some portfolios on the efficient frontier only made of risky assets would stand above the CML for portfolios riskier than M

But no portfolio can mean-variance dominate portfolios on the CML, thus the slope can neither be higher nor lower! Thus both slopes should be equal at point $T = M$.

Equation of the Security Market Line (SML)

Introducing an asset i, one shows in appendix that the slope of the tangent to the efficient frontier only made of risky assets at point M is equal to $\sigma_M.\frac{\mathbb{E}(r_M)-\mathbb{E}(r_i)}{\sigma_M^2-\sigma_{i,M}}$
By equalizing it with the slope of the CML, one gets:

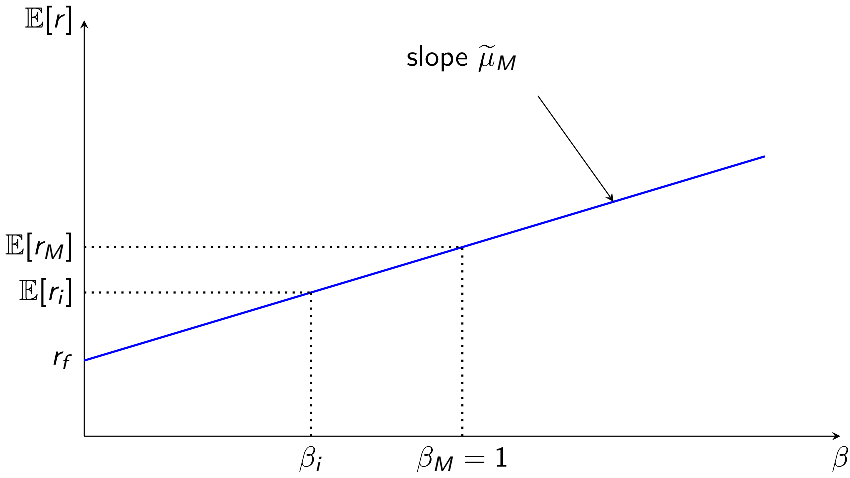
$$\sigma_M.\frac{\mathbb{E}(r_M)-\mathbb{E}(r_i)}{\sigma_M^2-\sigma_{i,M}}=\frac{\mathbb{E}(r_M)-r_f}{\sigma_M}$$

By noting $\beta_{i,M}=\frac{\sigma_{i,M}}{\sigma_M^2}$ the beta of asset i with respect to M, and rearranging the terms, one eventually finds:

$$\mathbb{E}(r_i)-r_f=\beta_{i,M}.(\mathbb{E}(r_M)-r_f) \tag{1}$$

This the equation of the Security Market Line (SML).

The Security Market Line: illustration



Reconciling the two expressions of the CAPM

Can we reconcile the equation of the CML with the equation of the SML? Yes!

We develop $\mathbf{\Sigma w}_M = \begin{pmatrix} \sum_{j=1}^{n} w_j^M.\sigma_{1,j} \\ \sum_{j=1}^{n} w_j^M.\sigma_{2,j} \\ \vdots \\ \sum_{j=1}^{n} w_j^M.\sigma_{n,j} \end{pmatrix} = \left(\sum_{j=1}^{n} w_j^M.\sigma_{i,j}\right)_{(1\leq i\leq n)}$

Yet, $\sum_{j=1}^{n} w_j^M.\sigma_{i,j} = \sum_{j=1}^{n} Cov(r_i, w_j^M.r_j) = Cov(r_i, \sum_{j=1}^{n} w_j^M.r_j) = \sigma_{i,M}$

Reconciling the two expressions of the CAPM

At last, $$\frac{\boldsymbol{\Sigma}\mathbf{w}_M}{\sigma_M^2} = \begin{pmatrix} \frac{\sigma_{1,M}}{\sigma_M^2} \\ \frac{\sigma_{2,M}}{\sigma_M^2} \\ \vdots \\ \frac{\sigma_{n,M}}{\sigma_M^2} \end{pmatrix} = \begin{pmatrix} \beta_{1,M} \\ \beta_{2,M} \\ \vdots \\ \beta_{n,M} \end{pmatrix} = \mathcal{B} \quad \text{and } \widetilde{\boldsymbol{\Pi}} = \mathcal{B}.\widetilde{\pi}_M$$

So the two formulas are equivalent indeed!

Two expressions of the CAPM

Thus the CAPM can be formulated in two equivalent ways:

$$\widetilde{\mathbf{\Pi}} = \mathcal{B}.\widetilde{\pi}_M \quad \text{or} \quad \widetilde{\mathbf{\Pi}} = \lambda_M.\mathbf{\Sigma}\mathbf{w}_M, \quad \text{with} \quad \lambda_M = \frac{\widetilde{\pi}_M}{\sigma_M^2}$$

- The first formula highlights that the required return on a security in excess of the riskfree rate is the product of its quantity of risk (**beta**) times the market price of risk (**risk premium on the market portfolio**)
- The second formula highlights the relationship between the required return in excess of the risk free rate to the supply of the different assets

  In particular, a supply shock on any given asset affects the required returns on all assets
- The CAPM holds for any security as well as for any portfolio

Interpretation of beta

The beta of an asset is a measure of its systematic risk (as opposed to its idiosyncratic risk), i.e. its sensitivity with respect to variation in market prices:

- Securities with a beta superior to 1 have returns of larger amplitude than the market, securities with a beta between 0 and 1 have variations of lower amplitude than the market returns.
- Some securities have negatives betas, and should receive a negative risk premium if the market risk premium is positive

$\beta_M = 1$ by definition
$\beta_i \geq 1$ for a cyclical security
$\beta_i \leq 1$ for a defensive security

Why idisoyncratic risk is not remunerated

- The quantity of risk of a given security is a function of its covariance with the market portfolio.
- Thus investors are only rewarded for bearing the systematic risk on an asset:
  - This is because the systematic risk cannot be diversified away
  - The non-systematic, or idiosyncratic risk on an asset can be eliminated at no cost. Each investor can self-insure with portfolio diversification
  - The idiosyncratic risk embedded in each security is a source of return as well, but it is not factored in the security's required return
  - The CAPM does not ignore the presence of idiosyncratic risk and its impact on return, but assumes it is not rewarded ex ante

Implied performance measures

Usages of SML

The Security Market Line (SML) becomes a reference:

- for security valuation: calculation of required return
- for detecting investment opportunities: comparison of market implied return with required return
- for performance appraisal: comparison of realized return with required return.

Market implied return versus CAPM required return : ex ante alpha

If a security does not plot on the SML, then it is mispriced in the CAPM framework

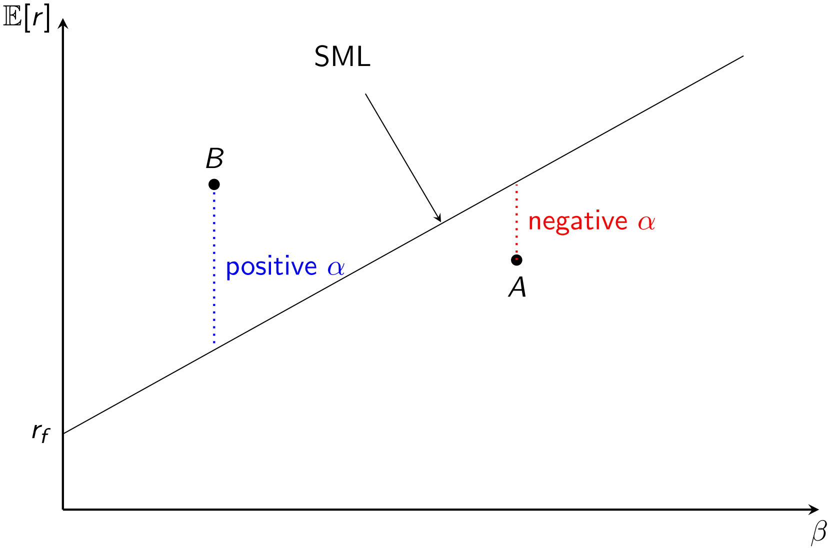
- One can compare the market implied expected return (for instance, from an inverted discounted cash flows model) to the CAPM required return to get the ex ante alpha:

$$\mathbb{E}[\alpha_s] = \mu_s - \pi_s = \mu_s - (r_f + \beta_s.(\pi_M - r_f))$$

μ_s = market implied expected return on security s
β_s = beta of security s with respect to market portfolio
π_M = expected return on the market portfolio at equilibrium
r_f = riskfree rate

- A security with a positive ex-ante alpha is undervalued with respect to the CAPM and should be bought until the market implied expected return equal the CAPM required return.

Ex ante alpha = graphical representation



Ex-post return in excess of CAPM required return: Jensen's Alpha

- Alpha can also be computed ex-post, as the difference between the realized return and the CAPM required return. It is referred to as **Jensen's Alpha** or **abnormal return**

$$\alpha_s = \overline{r_s} - \pi_s = \overline{r_s} - (r_f + \beta_s.(\overline{r_M} - r_f))$$

$\overline{r_s}$ = realized return on security s
β_s = beta of security s with respect to market portfolio
$\overline{r_M}$ = realized return on the market portfolio
r_f = riskfree rate

- It can be computed at the security's level (to assess a mispricing) or at the portfolio level (to assess a portfolio manager's skill)

Misconceptions of Alpha

There are some misconceptions of Alpha (see Baz, 2024).
Alpha is not a Risk Premium:

- **It is not a compensation for taking exposure to systematic risk**, as Alpha is uncorrelated with market movements
- **It is not compensation for taking exposure to idiosyncratic risk**, as Alpha has no variance (it is not "risky"). Idiosyncratic risk generates variability in a stock's return

Alpha is a risk-adjusted Excess Return, but the raw Excess Return on a portfolio over its benchmark is not necessarily Alpha!

The required return from other models can also serve to compute Alpha

Alpha as a systematic risk adjusted excess return

Alpha rewrites:

$$\alpha_i = \mu_i - \pi_i = \mu_i - r_f - \beta_i * (\pi_M - r_f) = \tilde{\mu_i} - \beta_i * \tilde{\pi}_M \qquad (2)$$

Thus it can be interpreted as an excess return of a security compared to its benchmark. This benchmark would be the risk-adjusted return on the market portfolio, the adjustment being for systematic risk only.

From Jensen's Alpha to Treynor Ratio

The realized performance on a security (or a portfolio) can be compared to its realized risk

- The Treynor Ratio is the ratio of the realized excess return of a portfolio to its systematic risk only:

$$TR_P = \frac{\overline{r_P} - r_f}{\beta_P}$$

$\overline{r_P}$ is the average realized return on the portfolio
β_P is the portfolio's beta
r_f is the riskfree rate

- The Treynor ratio can be linked to Alpha the following way:

$$TR_P = \frac{\tilde{\mu}_P}{\beta_P} = \frac{\tilde{\mu}_P - \beta_P * \tilde{\pi}_M}{\beta_P} + \tilde{\pi}_M = \frac{\alpha_P}{\beta_P} + \tilde{\pi}_M$$

Modigliani & Modigliani measure

Portfolio P can also be assessed by comparing its mean realized return to portfolio M after adjusting for total risk.

- This is measured by creating a fictive portfolio P^* which is made of r_f and P and with the same standard deviation than M.
- M^2 is defined as the average realized return on portfolio P^* in excess of portfolio M:

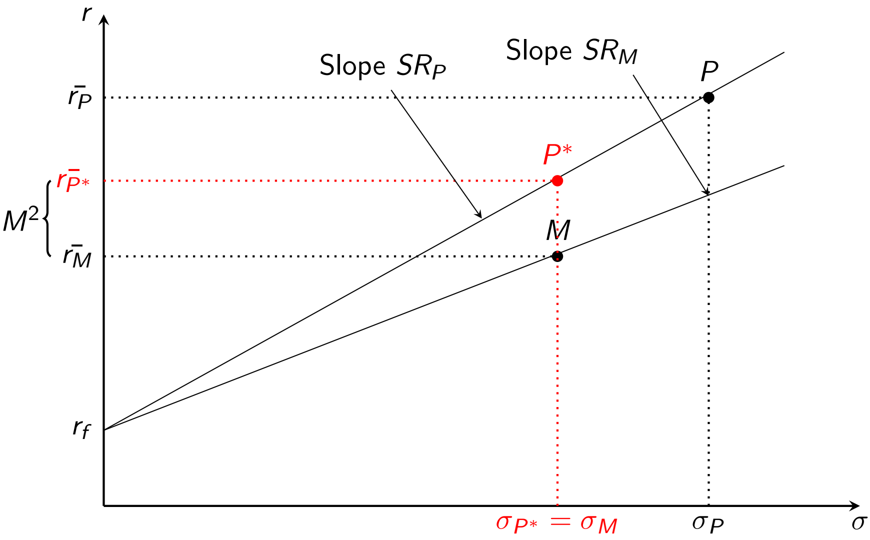
$$M^2 = \bar{r_{P^*}} - \bar{r_M} = (w_f.r_f + w_P.\bar{r_P}) - \bar{r_M} =$$
$$(w_P.(\bar{r_P} - r_f) + r_f) - \bar{r_M} = w_P.(\bar{r_P} - r_f) - (\bar{r_M} - r_f).$$

To have $\sigma_{P^*} = \sigma_M$, we need $w_P = \frac{\sigma_M}{\sigma_P}$ (and $w_f = 1 - \frac{\sigma_M}{\sigma_P}$).
Thus $M^2 = \frac{\sigma_M}{\sigma_P}.(\bar{r_P} - r_f) - (\bar{r_M} - r_f)$.

In this measure this is the portfolio's return which is adjusted for its risk to compare it with the benchmark's return.

It rewrites: $M^2 = \sigma_M * SR_P - \tilde{\mu}_M = \sigma_M * (SR_P - SR_M)$

M^2 measure = graphical representation



Recap on all measures

We summarize the following measures in the table below:

Table 1: Performance measures recap

	Performance Measure	**Adjustment for**
Jensen's Alpha	Excess Return	systematic risk
Treynor Ratio	Risk Premium	systematic risk
Sharpe Ratio	Risk Premium	total risk
M^2 Measure	Excess Return	total risk

Measures can be split between:

- Risk-adjusted returns (Sharpe Ratio, Treynor Ratio) versus risk-adjusted excess returns (Jensen's Alpha, Modigliani & Modigliani measure)
- Measures which adjust for systematic risk (Jensen's Alpha, Treynor Ratio) versus for total risk (M^2, Sharpe Ratio)

Model extensions

CAPM extensions: relaxing certain assumptions

Different lending and borrowing rates

- It is reasonable to think that investors can lend at the riskfree rate, for instance by purchasing Treasuries. However, it is less obvious how they can borrow at the risk free rate as individual investors' risk profile differs from the one of issuers such as governments.
- When the borrowing and the lending rates differ, the Capital Market Line will be made of two line segments of different slopes.

CAPM extensions: relaxing certain assumptions

Zero Beta CAPM

- An alternative model to the CAPM built by Black in 1972. It does not necessitate the existence of a risk free asset. There exist certain portfolios totally uncorrelated with the market portfolio. Their variance in returns is positive though, as they embed some idiosyncratic risk.
- The covariance between the Zero Beta portfolio and the Market portfolio is the same as for the riskfree asset. If the expected return on the Z-Beta portfolio is superior to the risk free rate of return, then the slope of the CML obtained by combining the Z Beta portfolio and the Market portfolio will be lower than the one of the CML obtained with the riskfree asset.

CAPM extensions: relaxing certain assumptions

Transaction costs

- The CAPM assumes the absence of transaction costs. In a market without transaction costs, investors will exploit any observed mispricing, i.e. any security whose implied return is departing from the CAPM required return will be traded accordingly. With transaction costs, investors only take advantage of mispricings of greater amplitude than the transaction costs.
- The SML becomes a corridor of two lines, and not a line anymore.

Appendix

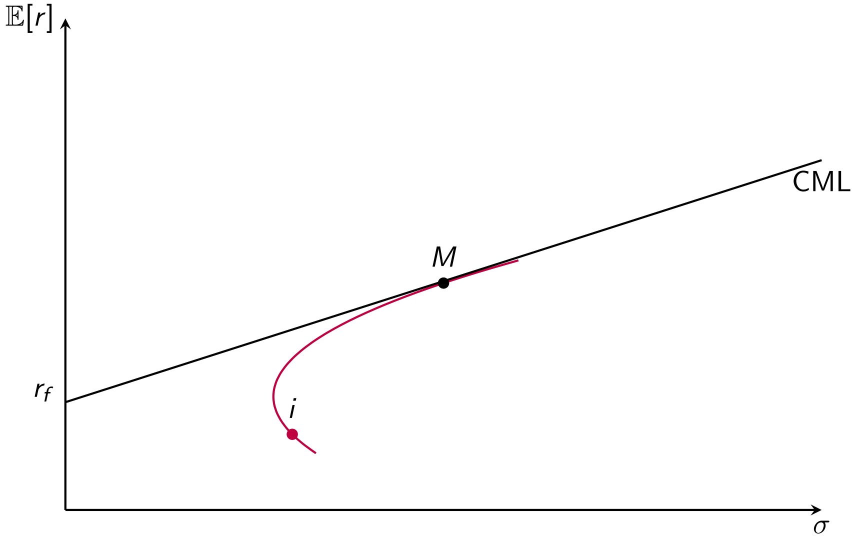
CAPM derivation

We derive the CAPM by introducing an "excess demand" on any asset i in the Market Portfolio.
The expected return and standard deviation on asset i are noted μ_i and σ_i, the covariance of its returns with those of M is $Cov(r_i, r_M) = \sigma_{i,M}$.

Let us assume that an investor forms a portfolio made of M and i (M already contains asset i) in proportions w_M and $w_i = 1 - w_M$.

Varying the proportions in i and M allows us to draw the curve of all attainable portfolios in the mean-standard deviation space.

Capital Market Line and curve passing through i and M



CAPM derivation

The CML should be tangent to this curve at point M. Otherwise the curve made by these portfolios i and M would have tangent lines with slopes lower than the slope of the CML, which is by construction not possible as the CML mean-variance dominates any portfolio.

The expected return and the standard deviation of the investor's portfolio then write:

$$\begin{cases} \mathbb{E}(r_P) & = w_i.(\mathbb{E}(r_i) - \mathbb{E}(r_M)) + \mathbb{E}(r_M) \\ \sigma_P & = \left(w_i^2.\sigma_i^2 + (1-w_i)^2.\sigma_M^2 + 2.w_i.(1-w_i).\sigma_{i,M}\right)^{\frac{1}{2}} \end{cases} \tag{3}$$

(4)

The expression of the slope of that curve at any point writes:

$$\frac{\delta \mathbb{E}(r_P)}{\delta \sigma_P} = \frac{\frac{\delta \mathbb{E}(r_P)}{\delta w_i}}{\frac{\delta \sigma_P}{\delta w_i}}$$

CAPM derivation

We develop both elements:

$$\begin{cases} \frac{\delta \mathbb{E}(r_P)}{\delta w_i} & = \mathbb{E}(r_i) - \mathbb{E}(r_M) \\ \frac{\delta \sigma_P}{\delta w_i} & = \frac{1}{2} . \frac{2.w_i.\sigma_i^2 + 2.(w_i - 1).\sigma_M^2 + 2.(1 - 2.w_i).\sigma_{i,M}}{\sigma_P} \end{cases} \tag{5}$$

At market equilibrium, as all investors hold the tangency portfolio only (in varying proportions of their wealths though), the excess demand for asset i is equal to 0, thus $w_i = 0$. The slope of that curve then writes:

$$\frac{\delta \mathbb{E}(r_P)}{\delta \sigma_P}|_{w_i=0} = \frac{\frac{\delta \mathbb{E}(r_P)}{\delta w_i}}{\frac{\delta \sigma_P}{\delta w_i}}|_{w_i=0} = \frac{\mathbb{E}(r_i) - \mathbb{E}(r_M)}{\frac{\sigma_{i,M} - \sigma_M^2}{\sigma_M}} \tag{6}$$

CAPM derivation

To prevent any arbitrage, this slope must also be equal to the slope of the CML, thus:

$$\frac{\mathbb{E}(r_i) - \mathbb{E}(r_M)}{\frac{\sigma_{i,M} - \sigma_M^2}{\sigma_M}} = \frac{\mathbb{E}(r_M) - r_f}{\sigma_M} \tag{7}$$

We can then draw the equation of the SML.

CAPM alternative derivation

To derive the equation of the CAPM, it is also possible to start with the equation of the Efficient Frontier. We have seen that it can be obtained by combining two portfolios located on it, for instance by combining the GMVP and M.

In chapter 1, we have expressed the expected return on a portfolio of two assets as a function of its standard deviation. We reuse that expresson here. If i stands for the GMVP, then we have

$$\mathbb{E}(r_p) = \sqrt{\frac{\sigma_P^2 . (\mathbb{E}(r_M) - \mathbb{E}(r_i))^2}{\sigma_M^2 + \sigma_i^2 - 2.\sigma_{M,i}} - D} - \frac{B}{2.A}$$

CAPM alternative derivation

Let us compute the slope of the tangent of this curve at any point and then at point M:

$$\frac{\delta\mathbb{E}(r_P)}{\delta\sigma_P} = \frac{1}{2}.\left(\sigma_P^2.\frac{(\mathbb{E}(r_M)-\mathbb{E}(r_i))^2}{\sigma_M^2+\sigma_i^2-2.\sigma_{M,i}} - D\right)^{-\frac{1}{2}}.2.\sigma_P.\frac{(\mathbb{E}(r_M)-\mathbb{E}(r_i))^2}{\sigma_M^2+\sigma_i^2-2.\sigma_{M,i}}$$

$$= \sigma_P.\frac{(\mathbb{E}(r_M)-\mathbb{E}(r_i))^2}{\sigma_M^2+\sigma_i^2-2.\sigma_{M,i}}.\frac{1}{\mathbb{E}(r_p)+\frac{B}{2.A}}$$

$$\frac{\delta\mathbb{E}(r_P)}{\delta\sigma_M}\Big|_{\mathbb{E}(r_P)=\mathbb{E}(r_M)} = \sigma_P.\frac{(\mathbb{E}(r_M)-\mathbb{E}(r_i))^2}{\sigma_M^2+\sigma_i^2-2.\sigma_{M,i}}.\frac{1}{\mathbb{E}(r_M)+\frac{B}{2.A}}$$

CAPM alternative derivation

It can also be shown that:

$$\mathbb{E}(r_M) + \frac{B}{2.A} = \frac{(\mathbb{E}(r_M) - \mathbb{E}(r_i)) \,.\, (\sigma_M^2 - \sigma_{i,M})}{\sigma_M^2 + \sigma_i^2 - 2.\sigma_{M,i}}$$

At last, $\frac{\delta \mathbb{E}(r_P)}{\delta \sigma_P}\Big|_{\mathbb{E}(r_P)=\mathbb{E}(r_M)} = \sigma_M . \frac{\mathbb{E}(r_M) - \mathbb{E}(r_i)}{\sigma_M^2 - \sigma_{i,M}}$

Remember that in chapter 1:

$$\begin{cases} A &= \dfrac{\sigma_A^2 + \sigma_B^2 - 2.\sigma_{A,B}}{(\mathbb{E}(r_A) - \mathbb{E}(r_B))^2} \\ \\ B &= \dfrac{2.[(\mathbb{E}(r_A) + \mathbb{E}(r_B)).\sigma_{A,B} - \mathbb{E}(r_A).\sigma_B^2 - \mathbb{E}(r_B).\sigma_A^2]}{(\mathbb{E}(r_A) - \mathbb{E}(r_B))^2} \end{cases}$$

CAPM alternative derivation

If the first asset is the Market Portfolio and the second asset is the GMVP then:

$$\frac{B}{2.A} = \frac{[(\mathbb{E}(r_M) + \mathbb{E}(r_{GMVP})).\sigma_{M,GMVP} - \mathbb{E}(r_M).\sigma^2_{GMVP} - \mathbb{E}(r_{GMVP}).\sigma^2_M]}{\sigma^2_M + \sigma^2_{GMVP} - 2.\sigma_{M,GMVP}}$$

At last,

$$\mathbb{E}(r_M) + \frac{B}{2.A}$$

$$= \frac{\begin{array}{c}\mathbb{E}(r_M).\sigma^2_M + \mathbb{E}(r_M).\sigma^2_{GMVP} - 2.\mathbb{E}(r_M).\sigma_{M,GMVP} \\ - \mathbb{E}(r_M).\sigma^2_{GMVP} - \mathbb{E}(r_{GMVP}).\sigma^2_M + (\mathbb{E}(r_M) + \mathbb{E}(r_{GMVP})).\sigma_{M,GMVP}\end{array}}{\sigma^2_M + \sigma^2_{GMVP} - 2.\sigma_{M,GMVP}}$$

$$= \frac{(\mathbb{E}(r_M) - \mathbb{E}(r_{GMVP})).\sigma^2_M + (\mathbb{E}(r_{GMVP}) - \mathbb{E}(r_M)).\sigma_{M,GMVP}}{\sigma^2_M + \sigma^2_{GMVP} - 2.\sigma_{M,GMVP}}$$

References

Sharpe W. (1964), "Capital asset prices: A theory of market equilibrium under conditions of risk," *Journal of Finance*, Vol. 19(3), pp. 425-442.

Baz J. (2024), "The Alpha Equation: Myths and Realities, Discerning risk-adjusted returns and investment skill " *Pimco*.