A conservative diffusion scheme

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This document describes the implementation of a conservative Smagorinsky scheme for a tracer m (which could represent the mixing ratio of a moisture species, for instance). The UM documentation (UMDP 028) describes the diffusion equation as:

$$\frac{\partial m}{\partial t} + \frac{1}{\rho}\boldsymbol{\nabla}\cdot(\rho\nu\boldsymbol{\nabla}m) = 0. \tag{1}$$

In reality the density is ignored in the implementation, so that

$$\frac{\partial m}{\partial t} + \boldsymbol{\nabla}\cdot(\nu\boldsymbol{\nabla}m) = 0. \tag{2}$$

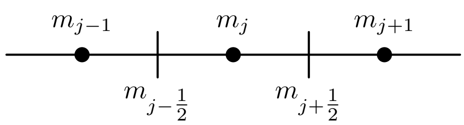
However, in LFRic this is not currently implemented through a conservative operator, and is simply

$$\frac{\partial m}{\partial t} + \nu\nabla^2 m = 0, \tag{3}$$

which also neglects the spatial variance of ν.

Notation

The degrees of freedom (DoFs) are annotated in 1D as follows:



Current Implementation

The current LFRic diffusion operator is implemented as:

$$\Delta m_j = \Delta t\ \nu_j\left(m_{j-1} - 2m_j + m_{j+1}\right)\left(\frac{2}{\Delta x_{j-\frac{1}{2}} + \Delta x_{j+\frac{1}{2}}}\right)^2. \tag{4}$$

New Implementation

The conservative diffusion operator is implemented through the follow steps, taking m and ν at $\mathbb{W}_3$ points (assumed to be on a shifted mesh).

1. Compute flux of m at $\mathbb{W}_2^H$ points. This uses linear reconstructions to (a) average ν to faces, and (b) compute the difference of m at faces. The field S is the face area, while V is the cell volume.

$$F_{j-\frac{1}{2}} = \tfrac{1}{2}S_{j-\frac{1}{2}}\left(\nu_j + \nu_{j-1}\right)\left(m_j - m_{j-1}\right)/\Delta x_{j-\frac{1}{2}}, \qquad F_{j+\frac{1}{2}} = \tfrac{1}{2}S_{j+\frac{1}{2}}\left(\nu_{j+1} + \nu_j\right)\left(m_{j+1} - m_j\right)/\Delta x_{j+\frac{1}{2}}. \tag{5}$$

2. Take the divergence of the flux:

$$\Delta m_j = \Delta t\left(F_{j+\frac{1}{2}} - F_{j-\frac{1}{2}}\right)/V_j, \tag{6}$$

   which yields the final result of

$$\Delta m_j = \Delta t\left[\tfrac{1}{2}S_{j+\frac{1}{2}}\left(\nu_{j+1} + \nu_j\right)\left(m_{j+1} - m_j\right)/\Delta x_{j+\frac{1}{2}} - \tfrac{1}{2}S_{j-\frac{1}{2}}\left(\nu_j + \nu_{j-1}\right)\left(m_j - m_{j-1}\right)/\Delta x_{j-\frac{1}{2}}\right]/V_j, \tag{7}$$

In the case of a regular mesh, $S_{j+\frac{1}{2}} = S_{j-\frac{1}{2}} = S$, $\Delta x_{j+1/2} = \Delta x_{j-1/2} = \Delta x$ and $S/V_j = 1/\Delta x$. If constant ν is assumed, (7) reduces to (4), demonstrating that the new approach has the same order of accuracy as the old approach, but now includes the spatial-variation in ν while also ensuring conservation.