Parallel SDC for Navier-Stokes equations

Fully implicit SDC using the monolithic scheme:

The two-dimensional incompressible Navier-Stokes equations are given as follows

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} &= -\mathbf{u}\cdot\nabla\mathbf{u} + \nu\Delta\mathbf{u} - \nabla p + \mathbf{g}, \\ 0 &= \nabla\cdot\mathbf{u}. \end{aligned} \tag{1}$$

For the space discretization, we utilize the mixed finite element approach. The matrix form of a weak formulation for equation (1) can be written as

$$\begin{aligned} \int_\Omega \frac{\partial \mathbf{u}}{\partial t}\mathbf{v}d\Omega &= -\int_\Omega \mathbf{u}\cdot\nabla\mathbf{u}\mathbf{v}d\Omega - \nu\int_\Omega \nabla\mathbf{u}\nabla\mathbf{v}d\Omega + \int_\Omega p\nabla\cdot\mathbf{v}d\Omega + \int_\Omega \mathbf{g}\mathbf{v}d\Omega, \\ 0 &= \int_\Omega \nabla\cdot\mathbf{u}q d\Omega. \end{aligned} \tag{2}$$

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The weak form (2) can be rewritten in matrix-vector form as follows

$$\begin{aligned} [\mathtt{M}]\frac{d\mathbf{u}}{dt} &= -[\mathtt{C}_{\mathbf{u}}]\mathbf{u} - [\mathtt{k}]\mathbf{u} + [\mathtt{B}]p + [\mathtt{M}]\mathbf{g}, \\ \mathbf{0} &= [\mathtt{B}^T]\mathbf{u} \end{aligned} \tag{3}$$

Thus we have the following semi-discrete Navier-Stokes equation rewritten in a block matrix form

$$\begin{pmatrix} [\mathtt{M}] & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \frac{d\mathbf{u}}{dt} \\ \frac{dp}{dt} \end{pmatrix} = \begin{pmatrix} -[\mathtt{C}_{\mathbf{u}}] - [\mathtt{k}] & -[\mathtt{B}] \\ -[\mathtt{B}^T] & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} + \begin{pmatrix} [\mathtt{M}]\mathbf{g} \\ 0 \end{pmatrix}. \tag{4}$$

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The presence of a singular mass matrix in the left hand side — mass matrix $[\mathtt{M}]$ acting only on the velocity component — reflects the incompressibility constraint inherent in the system. The system can be expressed more compactly in the form

$$[\alpha]\frac{d\mathbf{w}}{dt} = [\beta]\mathbf{w} + \gamma, \tag{5}$$

where the state vector $\mathbf{w}$, system matrices $[\alpha]$ and $[\beta]$, and forcing term γ are defined as

$$\mathbf{w} = \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix}, \quad [\alpha] = \begin{pmatrix} [\mathtt{M}] & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad [\beta] = \begin{pmatrix} -[\mathtt{C}_{\mathbf{u}}] - [\mathtt{k}] & -[\mathtt{B}] \\ -[\mathtt{B}^T] & \mathbf{0} \end{pmatrix}, \quad \text{and} \quad \gamma = \begin{pmatrix} [\mathtt{M}]\mathbf{g} \\ 0 \end{pmatrix}. \tag{6}$$

Here, the singular structure of $[\alpha]$, which enforces the divergence-free condition on the velocity field, results in a differential-algebraic system. Nevertheless, we will treat equation (5) as an initial-value problem for the purpose of numerical integration. Thus, the equation (5) can be expressed as

$$[\alpha]\frac{d\mathbf{w}}{dt} = f\left(t, \mathbf{w}\right). \tag{7}$$

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Therefore the SDC preconditioned iteration for the problem (7) is given by

$$[\alpha]\mathbf{w}_m^{k+1} = [\alpha]\mathbf{w}_0 + \Delta t \sum_{j=1}^{M} q_{m,j} f(\tau_j \mathbf{w}, j^k) + \Delta t \sum_{j=1}^{m} q_{m,j}^{\Delta} \left(f(\tau_j, \mathbf{w}_j^{k+1}) - f(\tau_j, \mathbf{w}_j^k) \right) \tag{8}$$

Substituting $f(\tau_j, \mathbf{w}_j^k) = [\beta]\mathbf{w}_j^k + \gamma_j$, the iteration can be rewritten as

$$[\alpha]\mathbf{w}_m^{k+1} = [\alpha]\mathbf{w}_0 + \Delta t \sum_{j=1}^{M} q_{m,j} \left([\beta]\mathbf{w}_j^k + \gamma_j \right) + \Delta t \sum_{j=1}^{m} q_{m,j}^{\Delta} [\beta] \left(\mathbf{w}_j^{k+1} - \mathbf{w}_j^k \right) \tag{9}$$