Fiber Intersections

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1 Fiber Intersections

We are interested in finding the intersection of two fibers. A fiber in $SO(3)$ is the set of rotation matrices taking a fixed vector $\mathbf{c}$ to a fixed vector $\mathbf{s}$, *i.e.* the set of matrices R such that $R\mathbf{c} = \mathbf{s}$.

Two fibers can be skew (meaning they don't intersect), or they can intersect. We are interested in determining whether they intersect, and if they do, how to find the unique rotation R that lies on both fibers.

We consider a system of equations:

$$R\,\mathbf{c}_1 = \mathbf{s}_1, \tag{1}$$
$$R\,\mathbf{c}_2 = \mathbf{s}_2. \tag{2}$$

where $\mathbf{c}_1$, $\mathbf{c}_2$, $\mathbf{s}_1$, $\mathbf{s}_2$ are 3-vectors, and R is a 3x3 matrix.

1.1 Conditions for a Solution

Non-Parallel Requirement The vectors $\mathbf{c}_1$ and $\mathbf{c}_2$ must not be parallel, i.e.,

$$\|\mathbf{c}_1 \times \mathbf{c}_2\| > 0.$$

If they are parallel, either the system has no solution (they are parallel fibers), or it is underdetermined and has infinitely many solutions (they are the same fiber).

Consistency Condition Because R is a rotation matrix, it preserves inner products:

$$\langle R\,\mathbf{c}_1,\, R\,\mathbf{c}_2 \rangle = \langle \mathbf{c}_1, \mathbf{c}_2 \rangle.$$

Substituting equations (1) and (2), a necessary condition for a solution to exist is

$$\langle \mathbf{s}_1, \mathbf{s}_2 \rangle = \langle \mathbf{c}_1, \mathbf{c}_2 \rangle.$$

If this condition is not satisfied, no solution exists. If it is satisfied and the non-parallel requirement is also satisfied, the there is a unique solution, and a closed formula is available.

1.2 Computing the Solution

Define the matrix C as the matrix whose columns are $\mathbf{c}_1$, $\mathbf{c}_2$, and their cross product:

$$C = \left[\, \mathbf{c}_1 \mid \mathbf{c}_2 \mid \mathbf{c}_1 \times \mathbf{c}_2 \,\right].$$

Because $\mathbf{c}_1$ and $\mathbf{c}_2$ are not parallel, $\mathbf{c}_1 \times \mathbf{c}_2 \neq \mathbf{0}$, so C is invertible.

Similarly, define the matrix S from the target vectors:

$$S = \left[\, \mathbf{s}_1 \mid \mathbf{s}_2 \mid \mathbf{s}_1 \times \mathbf{s}_2 \,\right].$$

By construction, $R\,C = S$, since

$$\begin{aligned} R\,\mathbf{c}_1 &= \mathbf{s}_1, \\ R\,\mathbf{c}_2 &= \mathbf{s}_2, \\ R\,(\mathbf{c}_1 \times \mathbf{c}_2) &= \mathbf{s}_1 \times \mathbf{s}_2, \end{aligned}$$

Right-multiplying both sides by C^{-1} gives

$$\boxed{R = S\,C^{-1}.}$$

1.3 Application

Our intended application is to finding orientations that deliver spots on two separate diffraction rings. In this case, there will typically be many right hand sides, $(\mathbf{s}^i)$, for a single pair of $\mathbf{c}$-vectors. In this case, $\mathbf{c}_1$ and $\mathbf{c}_2$ correspond to fixed crystallographic directions (HKLs) for each ring. The $\mathbf{s}_i$ are the components of the (HKLs) in a fixed spatial reference frame. They correspond to and are determined from spots on the detector.

The procedure for finding solutions for a large number of right hand sides is as follows:

1. Check that $\|\mathbf{c}_1 \times \mathbf{c}_2\| > \varepsilon_{\|}$. If not, return indicating the system does not have a unique solution.
2. Compute $C = [\, \mathbf{c}_1 \mid \mathbf{c}_2 \mid \mathbf{c}_1 \times \mathbf{c}_2 \,]$ and its inverse C^{-1}.
3. For each index i:
   - (a) Check the consistency condition $|\langle \mathbf{s}_1^{(i)}, \mathbf{s}_2^{(i)} \rangle - \langle \mathbf{c}_1, \mathbf{c}_2 \rangle| \leq \varepsilon_c$. If it fails, skip to the next i.
   - (b) Compute $S^{(i)} = [\, \mathbf{s}_1^{(i)} \mid \mathbf{s}_2^{(i)} \mid \mathbf{s}_1^{(i)} \times \mathbf{s}_2^{(i)} \,]$.
   - (c) Compute $R^{(i)} = S^{(i)}\,C^{-1}$ and record i and $R^{(i)}$.
4. Return the list of valid indices and their corresponding rotation matrices.