Analysis of Optical Modes

A Finite Difference Approach

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- Study how the structure of a waveguide affects the distribution of the EM field within it

Introduction to Waveguides

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In a very simple sense, a waveguide is exactly that, a structure that guides waves.

Rather than allowing the wave energy to diffuse in all directions a waveguide confines the wave, and thus its energy, allowing the transfer of energy over distances not possible without a waveguide.

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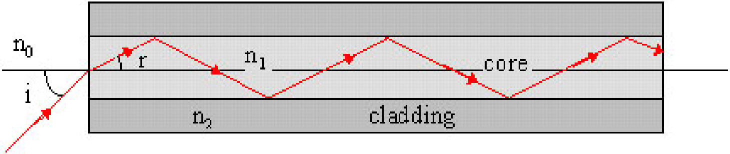

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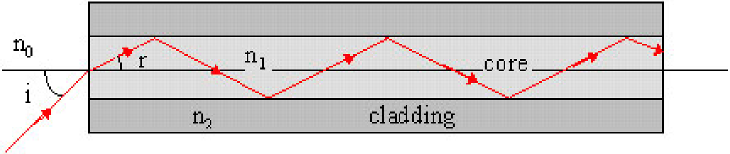

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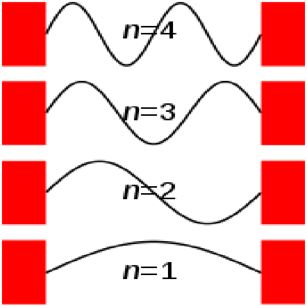
This necessity to satisfy boundary conditions means that the distribution of the field inside the structure is restricted to certain patterns.

These allowed patterns are called **modes** and they can be thought of as analogous to the discretisation of a particle wavepacket in potential.



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- This approach is crucial in understanding the behaviour of an EM field in a waveguide mainly because there is no analytical modal solution for higher than 1D.
- Numerical techniques allow us to study this field in a very scalable way.

Finite Difference Methods

There are several different methods that can be used in the numerical analysis of this problem.
The most popular two methods are Finite Element Methods (FEM) and Finite Difference Methods (FDM).
The choice generally depends on the problem being studied

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This project focuses completely on FD methods.
This is due to there being no major benefit in using FE methods over FDMs for this particular system and FD methods tend to require less of a mathematical foundation to implement efficiently.

Wave Equation

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From electrodynamics we know that interacting electric and magnetic fields oscillate and can be described by a wave equation.
From Maxwell's equation's it can be shown that in a medium with a relative permittivity ϵ_r and a relative permeability $\mu_r = 1$, the electric and magentic fields are governed by the following equations.

$$\nabla^2 \boldsymbol{E} + \nabla \left(\frac{\nabla \epsilon_r}{\epsilon_r} \cdot \boldsymbol{E} \right) + k_0^2 \epsilon_r \boldsymbol{E} = 0 \tag{1}$$

$$\nabla^2 \boldsymbol{H} + \frac{\nabla \epsilon_r}{\epsilon_r} \times (\nabla \times \boldsymbol{H}) + k_0^2 \epsilon_r \boldsymbol{H} = 0 \tag{2}$$

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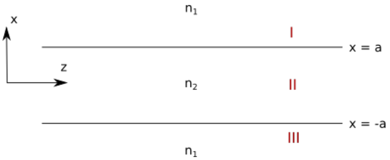

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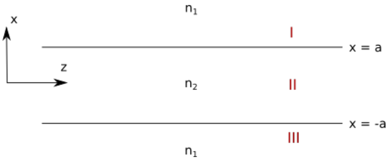

This waveguide is chosen as it has a continuous permittivity except at the boundaries between media. This allows for the great simplification of letting $\nabla \epsilon_r = 0$.
This waveguide also captures the fundamental properties of a waveguide with minimal complexity.

TE and TM Modes

When we talk about allowed modes there are two main types we are interested in; what are known as Transverse Electric (TE) and Transverse Magnetic (TM) modes. TE modes have no electric field in the direction of propagation $E_z = 0$ and TM modes have no magnetic field in the direction of propagation $H_z = 0$.

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Using this knowledge along with two of Maxwell's equations for EM waves

$$\nabla \times \boldsymbol{E} = -i\omega\mu_0\boldsymbol{H} \tag{3}$$

$$\nabla \times \boldsymbol{H} = i\omega\epsilon_0\epsilon_r\boldsymbol{E} \tag{4}$$

Allows the vector wave equations (1) and (2) to be simplified to a scalar form.

Scalar Wave Equations

$$\frac{d^2 E_y}{dx^2} + k_0^2 \left(\epsilon_r - n_{eff}^2\right) E_y = 0 \tag{5}$$

$$\frac{d^2 H_y}{dx^2} + k_0^2 \left(\epsilon_r - n_{eff}^2\right) H_y = 0 \tag{6}$$

The above equations can be solved quite easily analytically and using the boundary condition that the tangential component of both fields are continuous across the boundary between lossless dielectrics yields solutions that vary sinusoidally within guide and decay exponentially outside.

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The parameter n_{eff} is called the 'effective index' and is very important in the confinement of waves. It will be discussed in the next slide.

Effective Index

The effective index n_{eff} is defined as the ratio between the wave vector in a vacuum and the component of the wave vector in the direction of propagation in the given medium, in this case $n_{eff} = \frac{k}{k_z} n$ where n is the refractive index of the medium.

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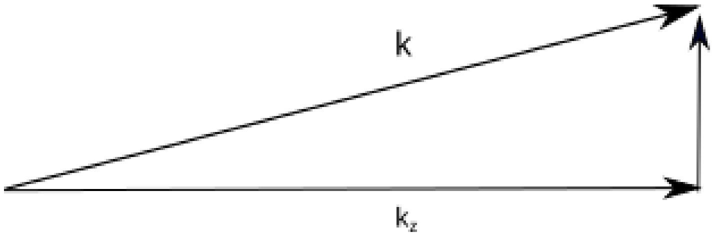

Analytical Solutions to 1D Case

Below are the analytical solutions to the first two TE modes using light of wavelength $\lambda = 1.55\mu m$ confined in guide of width $2a = 4\mu m$.

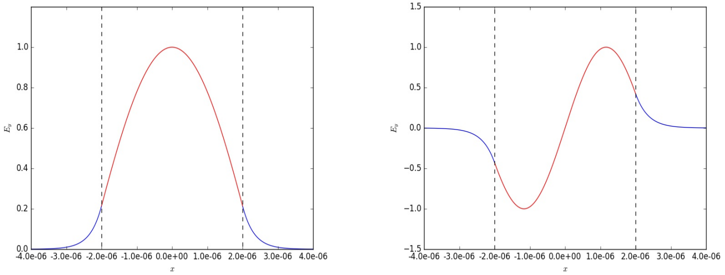

Figure: First two TE modes

Finite Difference Approximation

Now that we have an analytical solution we want to design an algorithm to numerical compute the solution and compare results.
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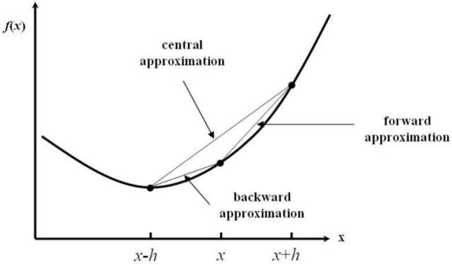
There are several ways to implement this method but all are based on using local Taylor Expansions to approximate derivatives at discrete points on the interval of interest.



Finite Difference Approximation

As we just need to approximate second order derivatives we are going to use the central difference method.
This is called 'central' as it works by taking Taylor expansions a step h either side of each discrete point on the interval.

$$E_y(x+h) = E_y(x) + E_y'(x)h + \frac{E_y''(x)}{2}h^2 + O(h^3)$$
$$E_y(x-h) = E_y(x) - E_y'(x)h + \frac{E_y''(x)}{2}h^2 - O(h^3)$$

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$$E_p'' = \frac{1}{h^2}\left(E_{p+1} - 2E_p + E_{p-1}\right)$$

Finite Difference Approximation

If we make the substitution into our scalar wave equation and rearrange, we are left with a linear equation in E.

$$\frac{E_{p+1}}{(k_0 h)^2} + \frac{E_{p-1}}{(k_0 h)^2} + \left(n_p^2 - \frac{2}{(k_0 h)^2} \right) E_p = n_{eff}^2 E_p \tag{7}$$

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$$\tilde{\boldsymbol{A}}\boldsymbol{E} = n_{eff}^2 \boldsymbol{E}$$

Reduction to Eigenvalue Equation

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As the interval we are dealing with was quite a lot larger than the guide itself, boundary conditions were chosen such that the field went to zero at the interval boundary.
Therefore the finite difference matrix $\tilde{\boldsymbol{A}}$ has the form

$$\tilde{\boldsymbol{A}} = \begin{bmatrix} n_0^2 - \frac{2}{(k_0 h)^2} & \frac{1}{(k_0 h)^2} & 0 & \dots & 0 \\ \frac{1}{(k_0 h)^2} & n_1^2 - \frac{2}{(k_0 h)^2} & \frac{1}{(k_0 h)^2} & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \frac{1}{(k_0 h)^2} & n_{n-1}^2 - \frac{2}{(k_0 h)^2} & \frac{1}{(k_0 h)^2} \\ 0 & \dots & 0 & \frac{1}{(k_0 h)^2} & n_n^2 - \frac{2}{(k_0 h)^2} \end{bmatrix}$$

Solving Eigenvalue Problem

As this is a symmetric tridiagonal matrix it isn't very computationally complex and is easily solvable on a standard laptop.
If we compare the first two TE modes with the analytical results we can see that they agree very well.

Analytical vs Numerical Comparison

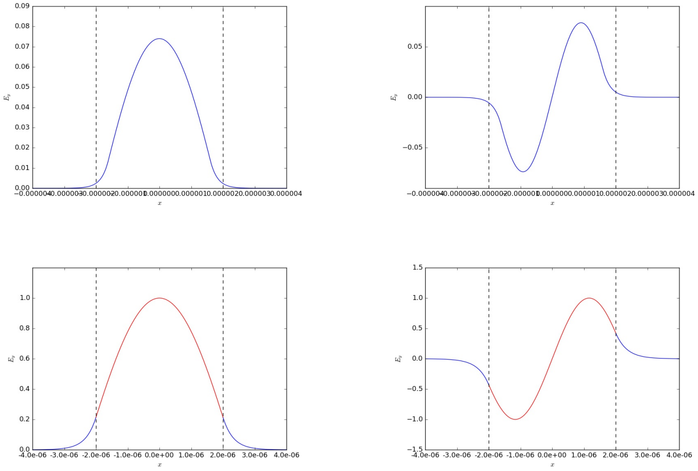

Figure: Comparison of first two TE modes for numerical (top) and analytical (bottom) solutions

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- The width of the waveguide
- The difference in refractive index across the boundary between media

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If we consider the case of a particle in a square well, these properties are analogous to:

- The energy of the particle
- The width of the well
- The height of the well walls (potential step)

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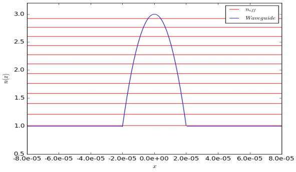

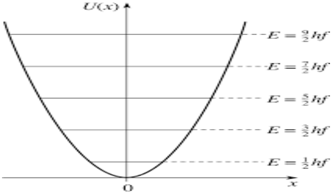

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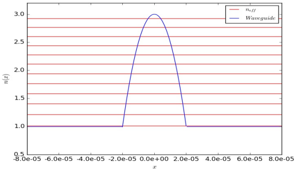

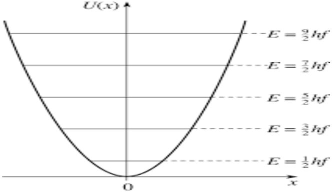

Note: here a wide waveguide was used to make the comparison clearer

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In the 2D case we don't allow $\nabla \epsilon_r = 0$, instead we assume that the components of the fields are decoupled.

This leads to what are know as the semivectorial wave equations

Semivectorial Wave Equations

In order to clarify this explanation, we will just discuss the semivectorial equation for the E_x field in was is now called the Quasi-TE mode (due to the decoupling assumption)

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial}{\partial x}\left(\frac{1}{\epsilon_r}\frac{\partial \epsilon_r}{\partial x}E_x\right) + \frac{\partial^2 E_y}{\partial y^2} + (k_0^2\epsilon_r - \beta^2)E_x = 0$$

Here the term β used by convention to represent the component of the wave vector in the direction of propagation k_z.

2D Finite Difference Method

The finite difference method in two dimensions works in the exact same way as in the 1D case but now more care is needed as we need to consider 4 points around each discrete field point (2 in either direction)

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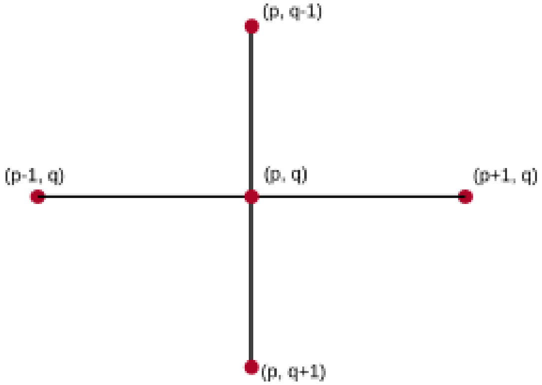

2D Finite Difference Method

Applying the same central difference approximations at each point adjacent to (p, q) we are left with an equation of the form.

$$\alpha_l E_{p-1,q} + \alpha_r E_{p+1,q} + \alpha_t E_{p,q-1} + \alpha_b E_{p,q+1} \\ + \alpha_c E_{p,q} + \left(k_0^2 \epsilon_r(p,q) - \beta^2\right) E_{p,q} = 0$$

2D Finite Difference Method

$$
\begin{aligned}
\alpha_l &= \frac{1}{(\delta x)^2}\frac{2\epsilon_r(p-1,q)}{\epsilon_r(p,q)+\epsilon_r(p-1,q)} \\
\alpha_r &= \frac{1}{(\delta x)^2}\frac{2\epsilon_r(p+1,q)}{\epsilon_r(p,q)+\epsilon_r(p+1,q)} \\
\alpha_t &= \frac{1}{(\delta y)^2} \\
\alpha_b &= \frac{1}{(\delta y)^2} \\
\alpha_c &= -\frac{4}{(\delta x)^2}+\alpha_l+\alpha_r-\alpha_t-\alpha_b
\end{aligned}
$$

Implementation of 2D FD Method

$$\tilde{\boldsymbol{A}} = \begin{bmatrix} n_0^2 - \frac{2}{(k_0 h)^2} & \frac{1}{(k_0 h)^2} & 0 & \dots & 0 \\ \frac{1}{(k_0 h)^2} & n_1^2 - \frac{2}{(k_0 h)^2} & \frac{1}{(k_0 h)^2} & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \frac{1}{(k_0 h)^2} & n_{n-1}^2 - \frac{2}{(k_0 h)^2} & \frac{1}{(k_0 h)^2} \\ 0 & \dots & 0 & \frac{1}{(k_0 h)^2} & n_n^2 - \frac{2}{(k_0 h)^2} \end{bmatrix}$$