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Final Report

Analysis of Optical Modes Using the Finite Difference Method

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Declaration

This report was written entirely by the author, except where stated otherwise. The source of any material not created by the author has been clearly referenced. The work described in this report was conducted by the author, except where stated otherwise.

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Date

Abstract

This report sets out to model and analyse the different distributions (or modes) of an electromagnetic field in several waveguide structures. To begin with, the one dimensional (1D) case in a slab waveguide is discussed. In this case it is possible to find analytical solutions of the the field distribution. These solutions are found and discussed in terms of particle confinement in a quantum mechanical well. In order to extend our analysis to include more complex waveguide structures a finite difference (FD) algorithm was developed to numerically solve to the modal structures.

The next part of the report deals with extending the work done to the two dimensional (2D) case for a core waveguide and later, a strip loaded waveguide. In the 2D case there are no exact analytical solutions and so certain approximate methods are discussed before moving on to developing a FD algorithm to solve for the modes in two dimensions. The final part of this report deals with the 2D case and how the modal structure changes based on the size and shape of the confining waveguide.

Contents

1 Introduction

In recent years the use of light to transfer information has become increasingly important. Today we see this most obviously in the form of optical fibre technology that has been speeding up our Internet connection. Up until quite recently however, mainstream use of light as a signal carrier has been done with the large scales in mind, think of optical fibre as an internet connection. However, as technology continues to improve at a staggering pace, advancements in electronics have begun to become limited by the properties of the materials used. This has led to a great interest in the field of photonics which seeks to improve upon the capabilities of current electronics by making use of the nature of light.

Across the field of photonics one of the most important devices in use is the waveguide. Waveguides are analogous to wires in electronics, they provide a method to reliably transfer information, however in the case of waveguides the transferal occurs via light rather than electrons. One of the main benefits of using light rather than electric current is that it travels at the speed of light, meaning the transfer of information can happen on timescales much smaller than those possible with conducting wire. There is also the benefit of not losing energy through heating which is an issue with copper electronics. The benefits of photonics have made it an interesting topic of study for quite some time but it is only with the improvement of technology, especially the in production of small scale devices, that we now have the capability to use the nature of light in an efficient manner. The use of photonics has exploded in recent decades due to technological advancement and there is now a huge amount of potential for waveguides, and photonics in general, to become the new benchmark of integrated circuitry [2].

The basic theory of waveguides is not a very new theory; in fact the idea of guiding waves (both sound and light) goes back to not too long after Maxwell's work revolutionised the way we think about light in the 19th century [1]. The basic premise behind waveguides is the maintaining of a field with minimal loss by confining the propagation to a single direction rather than allowing the wave to spread out in three dimensions as it would naturally. In optics, this is done via total internal reflection at the walls as the wave propagates. An figure illustrating this is shown below in *Figure 1.*

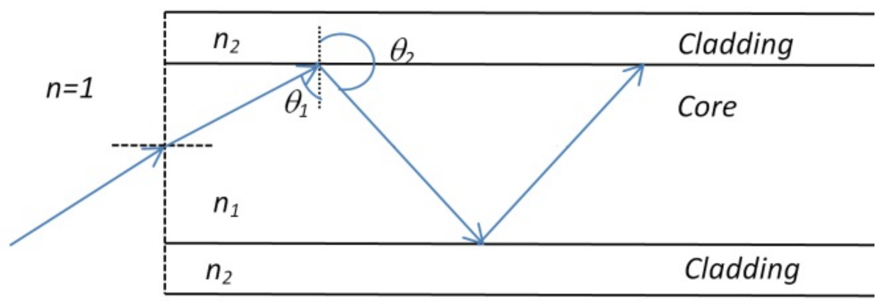

Figure 1: Total Internal Reflection in an Optical Fibre

Viewing light as a beam like in *Figure 1* it is very easy to visualise the optic confinement as a the bouncing back and forth of the light between walls. However, in order to analyse it as a field, Maxwell's governing equations need to be used.

2 Theory

2.1 Electromagnetic Waves

$$\nabla \times \boldsymbol{E} = -\tfrac{\partial \boldsymbol{B}}{\partial t}$$
$$\nabla \times \boldsymbol{H} = \tfrac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{J}$$
$$\nabla \cdot \boldsymbol{B} = 0$$
$$\nabla \cdot \boldsymbol{D} = \rho$$

Figure 2: Maxwell's Equations

In order to analyse how optical waveguides function it is first necessary to understand the properties of electromagnetic fields. The main property we are interested in its wave property, which is a direct result of Maxwell's Equations, see *Figure* **??** above. If we assume the electric and magnetic fields oscillate with a single angular frequency ω we can write the respective fields in the form shown below.

$$\boldsymbol{E}(\boldsymbol{r}, t) = Re\left\{\tilde{\boldsymbol{E}}(\boldsymbol{r}, t) exp(i\omega t)\right\} \tag{1}$$
$$\boldsymbol{H}(\boldsymbol{r}, t) = Re\left\{\tilde{\boldsymbol{H}}(\boldsymbol{r}, t) exp(i\omega t)\right\} \tag{2}$$

Assuming that $\rho = 0$ (no charge density) and $\mu_r = 1$, this wave form can be used to write those Maxwell's equations involving $\boldsymbol{E}$ and $\boldsymbol{D}$ in the form below.

$$\nabla \times \boldsymbol{E} = -i\omega \boldsymbol{B} = -i\omega\mu_0 \boldsymbol{H}$$
$$\nabla \cdot (\epsilon \boldsymbol{E}) = 0 \iff \nabla \cdot (\epsilon_r \boldsymbol{E}) = 0$$

Applying the curl both sides of the top equation and using the vector formula $\nabla \times (\nabla \times \boldsymbol{A}) = \nabla (\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}$ the equation can be rewritten as.

$$\begin{aligned} \nabla (\nabla \cdot \boldsymbol{E}) - \nabla^2 \boldsymbol{E} &= -i\omega\mu_0 \nabla \times \boldsymbol{H} \\ &= -i\omega\mu_0 \frac{\partial}{\partial t}(\epsilon \boldsymbol{E}) \\ &= -i\omega\mu_0\epsilon_0\epsilon_r \frac{\partial}{\partial t}(\boldsymbol{E}) \qquad \because \epsilon \neq \epsilon(t) \end{aligned}$$

Taking into account the wave form (1), the right hand side can be written as

$$\omega^2 \mu_0 \epsilon_0 \epsilon_r \boldsymbol{E}$$

Which, when the simplification $\frac{\omega}{k_0} = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$ is used, can be written succinctly as,

$$k_0^2 \epsilon_r \boldsymbol{E} \tag{3}$$

As for the left hand side of the equation, the knowledge that $\nabla \cdot (\epsilon_r \boldsymbol{E}) = 0$ is used along with the vector identity

$$\nabla \cdot (\epsilon_r \boldsymbol{E}) = \nabla \epsilon_r \cdot \boldsymbol{E} + \epsilon_r \nabla \cdot \boldsymbol{E} = 0$$

to obtain the below expression for the divergence of $\boldsymbol{E}$.

$$\nabla \cdot \boldsymbol{E} = -\frac{\nabla \epsilon_r}{\epsilon_r} \cdot \boldsymbol{E} \tag{4}$$

Substituting expressions (3) and and (4) back into the original vector equation yields.

$$-\nabla \left(\frac{\nabla \epsilon_r}{\epsilon_r} \cdot \boldsymbol{E} \right) - \nabla^2 \boldsymbol{E} = k_0^2 \epsilon_r \boldsymbol{E}$$

This means that for a medium with a relative permittivity ϵ_r the vector wave equation in $\mathbf{E}$ is given by.

$$\nabla^2 \boldsymbol{E} + \nabla \left(\frac{\nabla \epsilon_r}{\epsilon_r} \cdot \boldsymbol{E} \right) + k_0^2 \epsilon_r \boldsymbol{E} = 0 \tag{5}$$

This is quite a complex formula, however an important simplification of this formula exists when the relative permittivity ϵ_r is constant. In this case the above formula is reduced to the Helmholtz equation, with $k = k_0 \sqrt{\epsilon_r}$ the wave vector for this medium specifically.

$$\nabla^2 \boldsymbol{E} + k^2 \boldsymbol{E} = 0 \tag{6}$$

The derivation of the equivalent vector wave equation in $\mathbf{H}$ is very similar just using slightly different vector identities. Without going through the derivation, the final vector equation governing the H-field in a medium with relative permittivity ϵ_r is

$$\nabla^2 \boldsymbol{H} + \frac{\nabla \epsilon_r}{\epsilon_r} \times (\nabla \times \boldsymbol{H}) + k_0^2 \epsilon_r \boldsymbol{H} = 0 \tag{7}$$

In the case where ϵ_r is constant, this equation simplifies exactly to the Helmholtz equation in $\mathbf{H}$.

$$\nabla^2 \boldsymbol{H} + k^2 \boldsymbol{H} = 0 \qquad k = k_0 \sqrt{\epsilon_r} \tag{8}$$

The report focuses on waveguides which have a uniform structure in the direction of propagation, as confinement is only desired in the other two dimensions. Here and throughout the rest of the report the direction of propagation is taken to be in the z-direction. Therefore it's assumed that both the E and H-fields act as a planar wave in the z-direction, meaning $\boldsymbol{E}(x, y, z) = \boldsymbol{E}(x, y)e^{-i\beta z}$. This means that the derivative of either field in the z-direction can be written as

$$\frac{\partial}{\partial z} = -i\beta$$

Here β is the z-directed component of the wave vector. Using this insight the Laplace operator $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ can be rewritten in the form.

$$\begin{aligned}\nabla^2 &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \beta^2 \\ &= \nabla_\perp^2 - \beta^2\end{aligned}$$

With the perpendicular Laplacian defined as $\nabla_\perp^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. As β is a component of the wave vector in the medium, we can defined what's called the effective refractive index n_{eff} such that

$$\begin{aligned}n_{eff} &= \frac{\beta}{k_0} \\ &= \frac{\lambda_0 \beta}{2\pi}\end{aligned}$$

Because the effective index is dependent on the z-component of the wave vector in the medium, it is not just dependent on the medium itself but also the geometry of the waveguide and the distribution of the field (mode) inside the guide. This makes it an important parameter in the confinement of the wave. Using this definition and the expanded Laplace operator the governing vector equations for the E and H-fields in a waveguide with a relative permittivity ϵ_r can be written as.

$$\nabla_\perp^2 \boldsymbol{E} + \nabla\left(\frac{\nabla\epsilon_r}{\epsilon_r} \cdot \boldsymbol{E}\right) + k_0^2\left(\epsilon_r - n_{eff}^2\right)\boldsymbol{E} = 0 \tag{9}$$

$$\nabla_\perp^2 \boldsymbol{H} + \frac{\nabla\epsilon_r}{\epsilon_r} \times (\nabla \times \boldsymbol{H}) + k_0^2\left(\epsilon_r - n_{eff}^2\right)\boldsymbol{H} = 0 \tag{10}$$

And similarly the respective Helmholtz equations become.

$$\nabla_\perp^2 \boldsymbol{E} + k_0^2\left(\epsilon_r - n_{eff}^2\right)\boldsymbol{E} = 0 \tag{11}$$

$$\nabla_\perp^2 \boldsymbol{H} + k_0^2\left(\epsilon_r - n_{eff}^2\right)\boldsymbol{H} = 0 \tag{12}$$

2.2 Electromagnetic Field Inside a Waveguide

To introduce waveguides in one dimension, this section will begin with the simplest case, a slab waveguide. A symmetric kind of this wave guide can be seen in *Figure 3*.

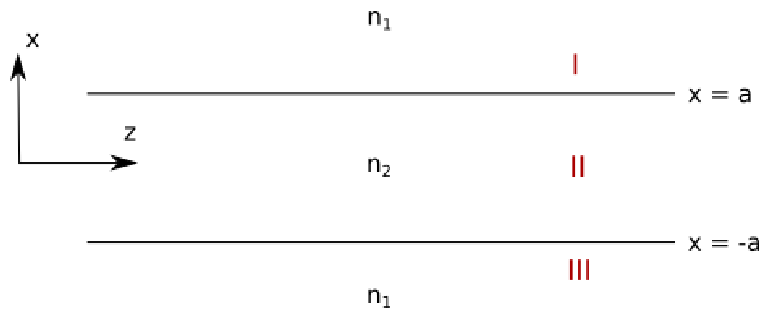

Figure 3: Slab Waveguide

For the 1D case, the waveguide is considered to be uniform in the y-direction. This allows for the great simplification that derivatives in this direction go to zero $\frac{\partial}{\partial y} = 0$. Therefore, the perpendicular Laplacian can be reduced to just the second order derivative in x. Therefore, equations (11) and (12) become.

$$\frac{d^2\boldsymbol{E}}{dx^2} + k_0^2\left(\epsilon_r - n_{eff}^2\right)\boldsymbol{E} = 0 \tag{13}$$
$$\frac{d^2\boldsymbol{H}}{dx^2} + k_0^2\left(\epsilon_r - n_{eff}^2\right)\boldsymbol{H} = 0 \tag{14}$$

Once reduced to the above form the separate components must be considered. The allowed components are given by Maxwell's equations; taking the first two equations in *Figure 2* and using the the wave form of the fields in time yields.

$$\nabla \times \boldsymbol{E} = -i\omega\mu_0\boldsymbol{H}$$
$$\nabla \times \boldsymbol{H} = i\omega\epsilon_0\epsilon_r\boldsymbol{E}$$

In order to see the relationships between the separate components clearly these equations can be expanded into their scalar form. This is done on the next page.

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -i\omega\mu_0 H_x \tag{15}$$
$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -i\omega\mu_0 H_y \tag{16}$$
$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -i\omega\mu_0 H_z \tag{17}$$

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = i\omega\epsilon_0\epsilon_r E_x \tag{18}$$
$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = i\omega\epsilon_0\epsilon_r E_y \tag{19}$$
$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = i\omega\epsilon_0\epsilon_r E_z \tag{20}$$

Here the relative permeability is assumed to be($\mu_r = 1$). The wave equations (13) and (14) allow for two different polarisations called transverse modes. These modes are split into what is known as the Transverse Electric (TE) mode and the Transverse Magnetic (TM) mode. As the names suggest TE modes allow only the magnetic field in the direction of propagation and the TM modes allow just the electric field in the direction of propagation. Therefore the distribution of the fields in the waveguide depends on which modes are being studied.

2.2.1 TE Mode

In the TE mode there is no electric field in the longitudinal (z) direction, therefore $E_z = 0$. Substituting this into equation (20) and remembering $\partial/\partial y = 0$ yields, $\partial H_y/\partial x = 0$. This implies H_y is constant and so it can be chosen to be zero. Making the substitution $E_z = H_y = 0$ into equation (16) yields $\partial E_x/\partial z = 0$, again as this shows E_x to be constant it is chosen to be zero. These restrictions yield.

$$E_z = E_x = H_y = 0$$

Thus the wave equation in **E** simplifies to.

$$\frac{d^2 E_y}{dx^2} + k_0^2\left(\epsilon_r - n_{eff}^2\right) E_y = 0 \tag{21}$$

This is a simple scalar wave equation and can easily be solved. The viable solutions depend on the the sign of $\epsilon_r - n_{eff}^2$ this dependence is what causes the confinement of the wave in a waveguide. Because $\mu_r = 1$ the refractive index within a particular region is just given by $n = \epsilon_r^2$. The solutions are given on the next page.

$$E_y(x) = A\ exp(-\gamma x) \qquad \text{with } \gamma = k_0\sqrt{n_{eff}^2 - n_1^2} \qquad \text{Region I} \tag{22}$$

$$E_y(x) = \begin{cases} C\ cos(hx) & \text{Symmetric} \\ D\ sin(hx) & \text{Antisymmetric} \end{cases} \qquad \text{with } h = k_0\sqrt{n_2^2 - n_{eff}^2} \qquad \text{Region II} \tag{23}$$

$$E_y(x) = B\ exp(\gamma x) \qquad \text{with } \gamma = k_0\sqrt{n_{eff}^2 - n_1^2} \qquad \text{Region III} \tag{24}$$

The reason there exists symmetric and antisymmetric fields in Region II is just due to the symmetry of the waveguide in the x-direction. Next we can use the boundary conditions at $x = +/-a$ to solve for the constants A,B,C and D. In order to find these boundary conditions it is the separate components of the waveguide are assumed to be lossless dielectrics, where no conduction occurs. The transverse E-field is continuous across the boundary between mediums and because there is no surface current so is the H-field. The two field can be related by the equation (17) which reduces to.

$$H_z(x) = \frac{i}{\omega\mu_0}\frac{\partial E_y}{\partial x}$$

Therefore continuity in the H_z-field is just continuity in the x-derivative of the E_y-field. Equating they E_y-fields and their x-derivatives at each boundary yields two cases, one symmetric and one antisymmetric. The two cases are as follows.

Symmetric	**Antisymmetric**
$A = B$	$A = -B$
$h\ tan(ha) = \gamma$	$h\ cot(ha) = -\gamma$

In order to find both $h$andγ one can look bcak to equations (22) and (23) to see that.

$$h^2 + \gamma^2 = k_0^2\left(n_2^2 - n_1^2\right)$$
$$\implies (ha)^2 + (\gamma a)^2 = k_0^2 a^2\left(n_2^2 - n_1^2\right)$$

Multiplying the expression for γ in terms of h by a (half the width of the waveguide) and substituting into the above expression gives a separate equation for both the symmetric and the antisymmetric case. With some algebra these equations can be simplified to.

$$(ha) = \left(k_0 a\sqrt{n_2^2 - n_1^2}\right) cos(ha) \qquad \text{Symmetric}$$
$$(ha) = \left(k_0 a\sqrt{n_2^2 - n_1^2}\right) sin(ha) \qquad \text{Antisymmetric}$$

Making the substitutions, $x = ha$ and $r = k_0 a\sqrt{n_2^2 - n_1^2}$ these equations can be written in the clear and simple form.

$$x = rcos(x) \qquad \text{Symmetric} \tag{25}$$

$$x = rsin(x) \qquad \text{Antisymmetric} \tag{26}$$

These are nonlinear equations of the form $x = f(x)$. When in this form a simple iterative solver $x_{n+1} = f(x_n)$ can be used. This requires an initial guess to be made, however this isn't too difficult when the right hand side and left hand side of the above equations are plotted together.

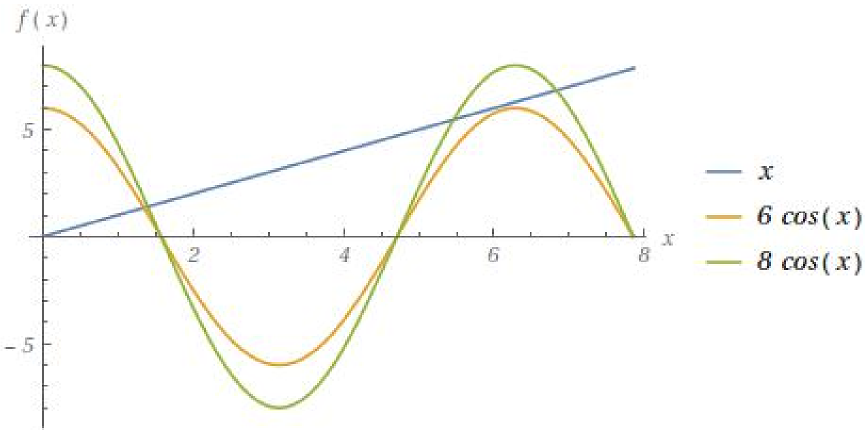

Figure 4: Plot describing solutions to symmetric case; $x = rcos(x)$

A plot like this not only allows for a reasonable initial condition to be chose but it also shows how the solution's uniqueness depends on the value of r. In the above plot there are two possible curves plotted, one for $r = 6$ and one for $r = 7$. From this we can see that for some r such that $6 < r < 7$, two of the solutions combine and disappear leaving just one solution left. Generalising this it can be seen that increasing r increases the number of solutions. These solutions correspond to what are known as 'modes'. Each mode represents a field distribution in the waveguide so increasing the number of solutions physically means more modes can 'fit' into the waveguide. Notice that increasing the size of the waveguide, decreasing the wavelength of the light in the waveguide or increasing the difference between the respective refractive indices all correspond to an increase in r. Therefore it makes sense that increasing r introduces more solutions as it makes it easier for more modes to fit into the waveguide. On the next page TM modes will be discussed.

2.2.2 TM Mode

The derivation of the analytical solutions to the TM modes are very similar to that show previously for the TE mode. With this in mind this section will go through the derivation much more briefly, pointing out the main key differences. The governing equation for the TM mode is given by equation (14), taking $H_z = 0$ ($E_z = 0$ for TE mode) and $\partial/\partial y = 0$ as done previously, the equations (15)-(20) can be used to derive the relation $H_x = H_z = E_y = 0$. This allows for the reduction of the vector equation (14) to the following scalar form.

$$\frac{d^2 H_y}{dx^2} + k_0^2 \left(\epsilon_r - n_{eff}^2\right) H_y = 0 \tag{27}$$

This equation has the exact same form as the TE equivalent. Therefore, the boundary conditions will be the source of the differences between the two types of mode. The general solutions are therefore the same as previously.

$$H_y(x) = A\ exp(-\gamma x) \qquad \text{with } \gamma = k_0\sqrt{n_{eff}^2 - n_1^2} \qquad \text{Region I} \tag{28}$$

$$H_y(x) = \begin{cases} C\ cos(hx) & \text{Symmetric} \\ D\ sin(hx) & \text{Antisymmetric} \end{cases} \qquad \text{with } h = k_0\sqrt{n_2^2 - n_{eff}^2} \qquad \text{Region II} \tag{29}$$

$$H_y(x) = B\ exp(\gamma x) \qquad \text{with } \gamma = k_0\sqrt{n_{eff}^2 - n_1^2} \qquad \text{Region III} \tag{30}$$

Again we have that both fields are continuous at the boundaries between the waveguide media. For the TM mode however, continuity in the electric field means continuity in the following expression, derived from equation (20).

$$E_z = \frac{i}{\omega\epsilon_0 n^2(x)}\frac{\partial H_y}{E_z} \qquad \text{where } n^2(x) = \epsilon_r(x)$$

Therefore, due to the $n^2(x)$ term, the boundary conditions imply that there must be continuity in the derivative divided by the relative permittivity for that medium rather than just in the derivative; as was the case for the TE modes. Applying the same procedure as previously but with the new boundary conditions, it isn't difficult to show that the following conditions hold.

Symmetric	**Antisymmetric**
$A = B$	$A = -B$
$\frac{n_1^2}{n_2^2} h\ tan(ha) = \gamma$	$\frac{n_1^2}{n_2^2} h\ cot(ha) = -\gamma$

Again using the same method as previously, involving the substitutions $x = ha$ and $r = k_0 a\sqrt{n_2^2 - n_1^2}$ the problem can be simplified to two nonlinear equations in x.

$$x = \frac{r}{\sqrt{1 + \left(\frac{n_1}{n_2}\right)^4 tan^2(x)}} \qquad \text{Symmetric} \qquad (31)$$

$$x = \frac{r}{\sqrt{1 + \left(\frac{n_1}{n_2}\right)^4 cot^2(x)}} \qquad \text{Antisymmetric} \qquad (32)$$

These equations are slightly more complex than those for the TE mode but are still relatively easy to solve using a simple iterative solver algorithm such as the Newton-Raphson method.

Below are plots of the first two modes in both the TE and TM case. Note that the amplitude of the wave, given by the constants C and D in the equations (23) and (29) can be chosen arbitrarily, it is just the relationships between constants that are important. In the plots below the the constants C and D are chosen to be unitary for simplicity and the physical parameters are $\lambda = 1.55\mu m$, $2a = 4\mu m$, $n_2 = 3.1$, $n_1 = 3.0$.

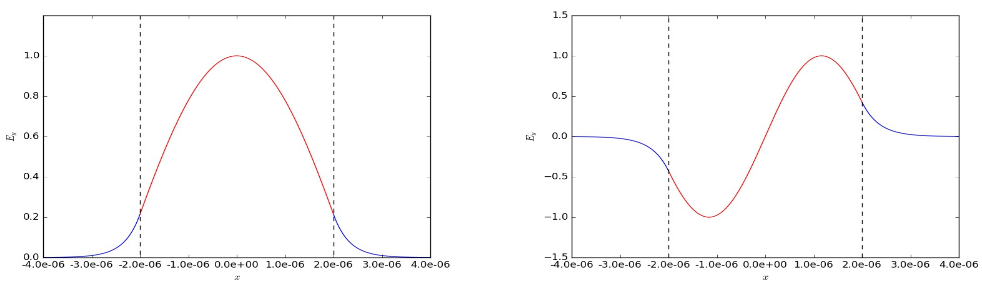

Figure 5: First two analytical solutions for TE modes

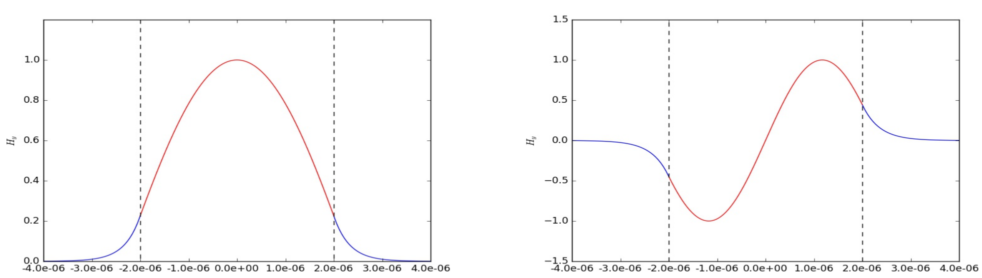

Figure 6: First two analytical solutions for TM modes

3 Numerical Simulation of 1D Case

As seen previously the one dimensional system is quite easily solved analytically. However, this is the simplest case and so in order to study more complex waveguide geometries and waveguides in higher dimensions an understanding of numerical methods is needed.

There are several ways in which to solve a system like this; two popular methods are the finite element method (FEM) and the finite difference method (FDM). Of these two methods there is no immediate preference in electrodynamics and often the efficiency and accuracy of the method depends strongly on the problem being solved. However, FD methods are very powerful and generally require a less rigorous background in mathematics to understand and implement. Therefore in cases where a lot FE methods offer no real benefit FD methods are preferred. This is the main reason FD methods were used in this project.

3.1 Finite Difference Method in 1D

The finite difference method is based on using a local Taylor expansion to approximate derivative over short steps. The basic idea is that as the step size approaches zero the approximation approaches the analytical solution. This method allows for the transformation of a differential equation to a system of linear equations which can then be solved by a computer.

The equation of interest is the scalar wave equation discussed previously. As this equation is the same for both the TE and TM modes, this report will just derive it for the TE wave equation (21). Therefore the second derivative of the E_y-field needs to be approximated. This is done by taking Taylor expansions by a step h either side of a the point x.

$$E_y(x+h) = E_y(x) + E_y'(x)h + \frac{E_y''(x)}{2}h^2 + O(h^3)$$
$$E_y(x-h) = E_y(x) - E_y'(x)h + \frac{E_y''(x)}{2}h^2 - O(h^3)$$

Adding these two expressions removes the first derivatives, and terms of order $O(h^3)$ leaving.

$$E_y''(x) = \frac{1}{h^2}\left(E_y(x+h) - 2E_y(x) + E_y(x-h)\right) + O(h^2)$$

Because the electric field being dealt with is always in the y-direction for the TE mode, the subscript y can be dropped from notation. The notation can be simplified further by treating the E-field as a discrete function. The E-field on the p^th step will now be written as $E(x_0 + ph) = E_p$, where x_0 is some starting point. This not only simplifies the equations but also makes programming the algorithm simpler as this is how the computer treats the field. Using this notation the previous expression can be rewritten as the following.

$$E''_p = \frac{1}{h^2}\left(E_{p+1} - 2E_p + E_{p-1}\right) \tag{33}$$

This approximates the second derivative up to a truncation error or order $O(h^2)$. Substituting this expression into the scalar wave equation (21) and discretising the refractive index, as done with the field itself, yields the following finite difference expression for the wave equation, after some rearranging.

$$\frac{E_{p+1}}{(k_0h)^2} + \frac{E_{p-1}}{(k_0h)^2} + \left(n_p^2 - \frac{2}{(k_0h)^2}\right)E_p = n_{eff}^2 E_p \tag{34}$$

This equation is just a linear equation and for a given step size across an interval of interest, the above defines a single equation in a system of linear equations that solve for every discrete point on that interval. This system can thus be written in the matrix form $\boldsymbol{A}E = n_{eff}^2 E$ where $\mathbf{A}$ is the finite difference matrix defined by (34) and E is an eigenvector representing evenly spaced points on the interval of interest.

It is important to note that this is just one way of defining the FD approximation to the scalar wave equation and it's called the central difference method. The method described above is only for a fixed step size h. A variable step size method can be used to save computer memory but there is a cost in accuracy and in the 1D case memory usage is not a problem. Having derived the FD approximation to the scalar wave equation

3.1.1 Boundary Conditions

References

[1] J. W. S. Rayleigh, *The Theory of Sound*, Macmillan, London (1877)

[2] L. Thylén and L. Wosinski, *Integrated photonics in the 21st century*, Photonics Research, **2** (2014) 75-81