Chapter 7
Finite Impulse Response (FIR) Filter Design

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7.1 Introduction

- This chapter concerns with design of FIR filter
  - Begin with specifications
  - Through coefficient calculation
  - The analyses of finite wordlength effects and implementations

7.1.1 Summary of Key Characteristics Features of FIR Filter

- The basic FIR filter is characterized by the following two equations

$$y(n)=\Sigma_{k=0}^{N-1}h(k)\mathrm{x}(n-k)$$

$$H(\mathrm{z})=\Sigma_{k=0}^{N-1}h(k)z^{-k}$$

- Having an exactly linear phase response
- Easily implemented
- Suffering less from the finite wordlength effects

7.1.2 Linear Phase Response and Its Implications

- Mathematically, the phase and group delays are defined as

$$T_p = -\theta(\omega)/\omega$$

$$T_g = -\, d\theta(\omega)/d\omega$$

- A filter is said to have a linear phase response if its phase response satisfies one of the following relationships

$$\theta(\omega) = -\alpha\omega$$

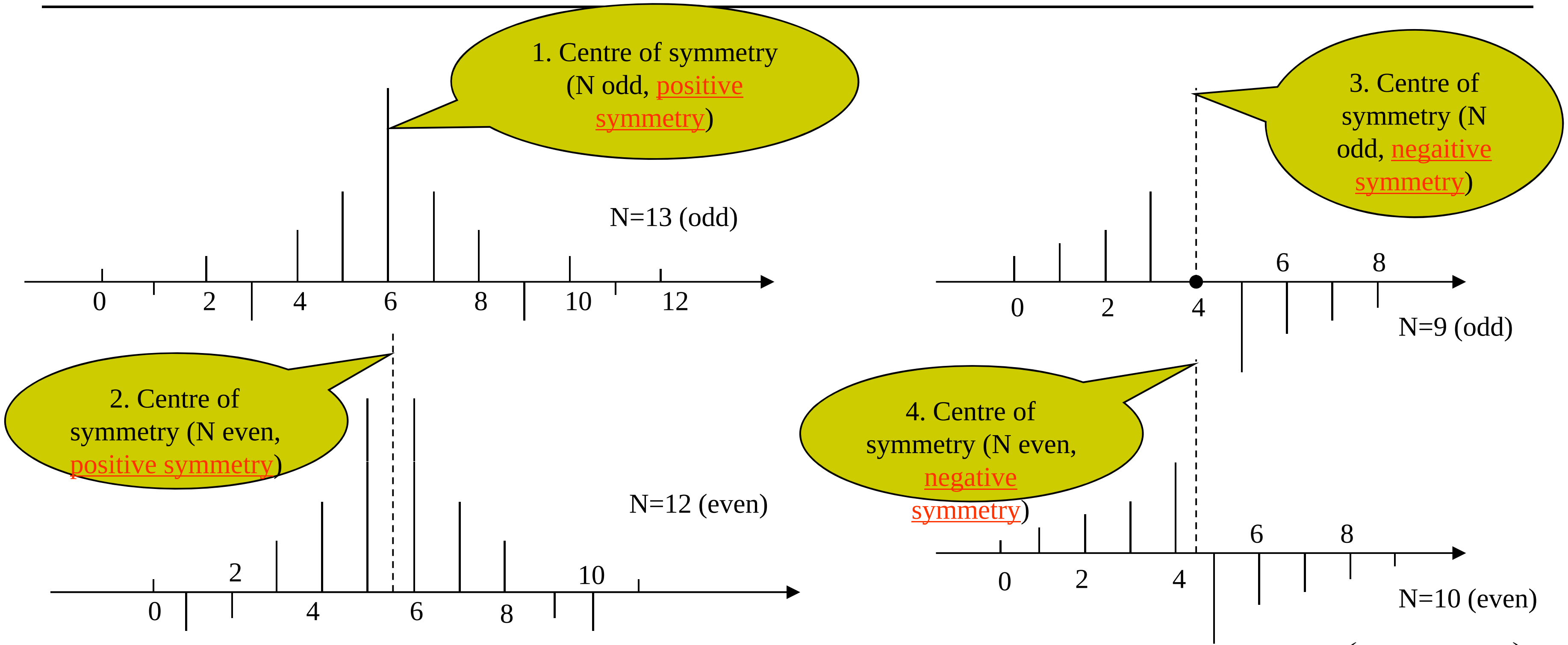
$$\theta(\omega) = \beta - \alpha\omega$$

where α and β are constant

Linear Phase Response Implications

- The phase and group delays are a useful measure of how the filter modifies the phase characteristics of a signal
- Nonlinear phase will cause a phase distortion which is undesired in many application, e.g., music, data transmission, video, and biomedical.

7.1.3 Types of Linear Phase FIR Filter



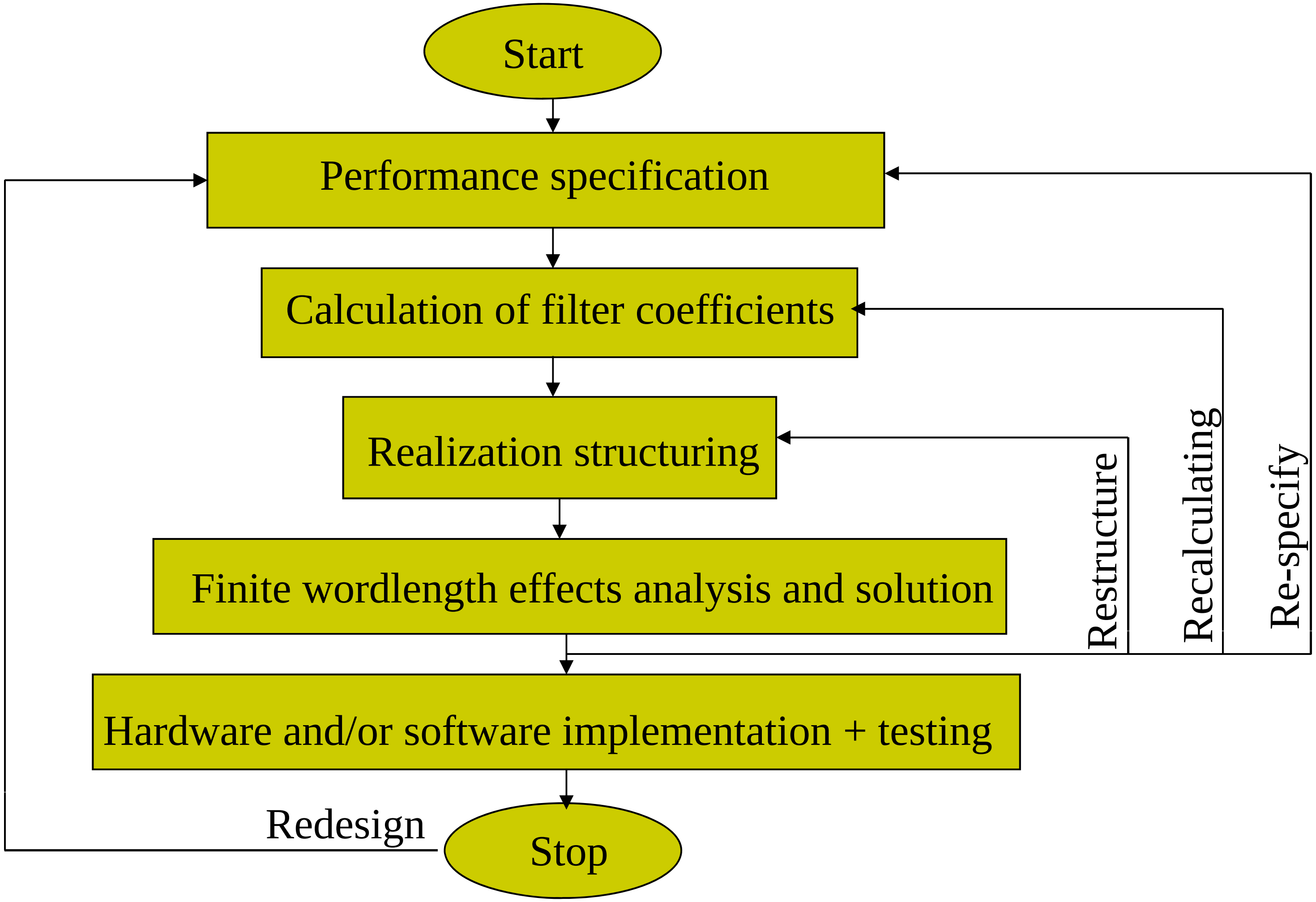
Type 1 and 2 phase delay $T_p = \left(\frac{N-1}{2}\right)T$

Where T is the sampling period

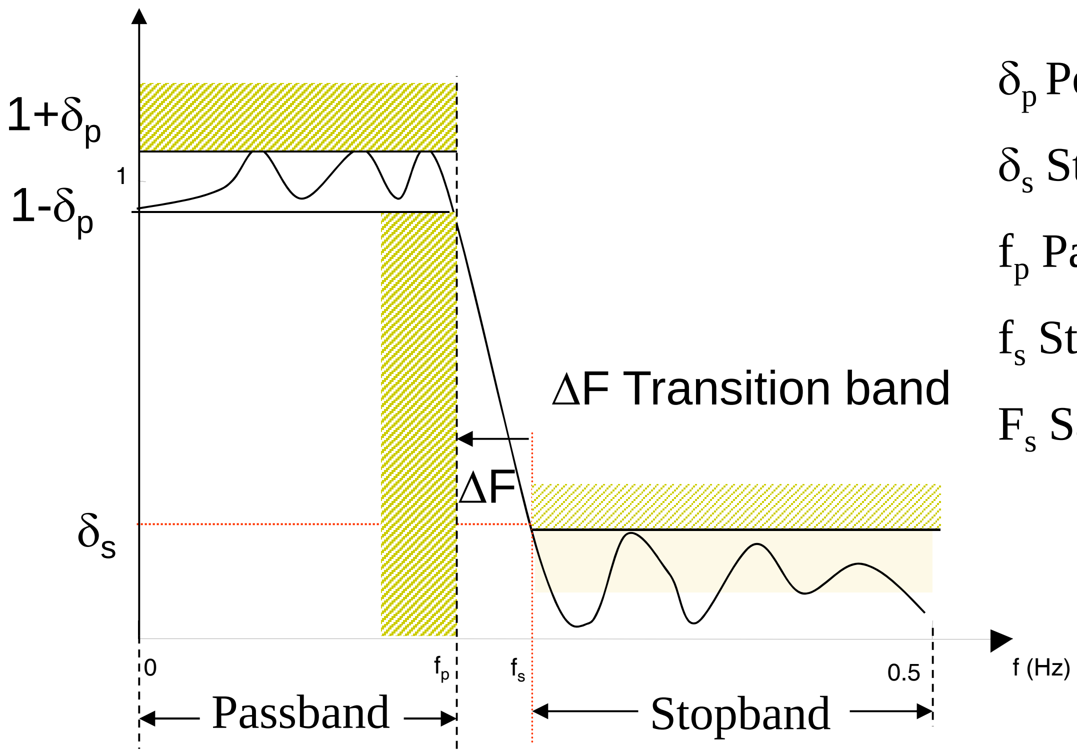
Type 3 and 4 group delay $T_p = \left(\frac{N-1-\pi}{2}\right)T$

Where T is the sampling period

7.2 FIR Filter Design



7.3 Filter Specifications



δ_p Peak passband deviation (or ripples)

δ_s Stopband deviation

f_p Passband edge frequency

f_s Stopband edge frequency

F_s Sampling frequency

7.4 FIR Coefficient Calculation Methods

- FIR equations

$$y(n)=\Sigma_{k=0}^{N-1}h(k)\mathrm{x}(n-k)$$

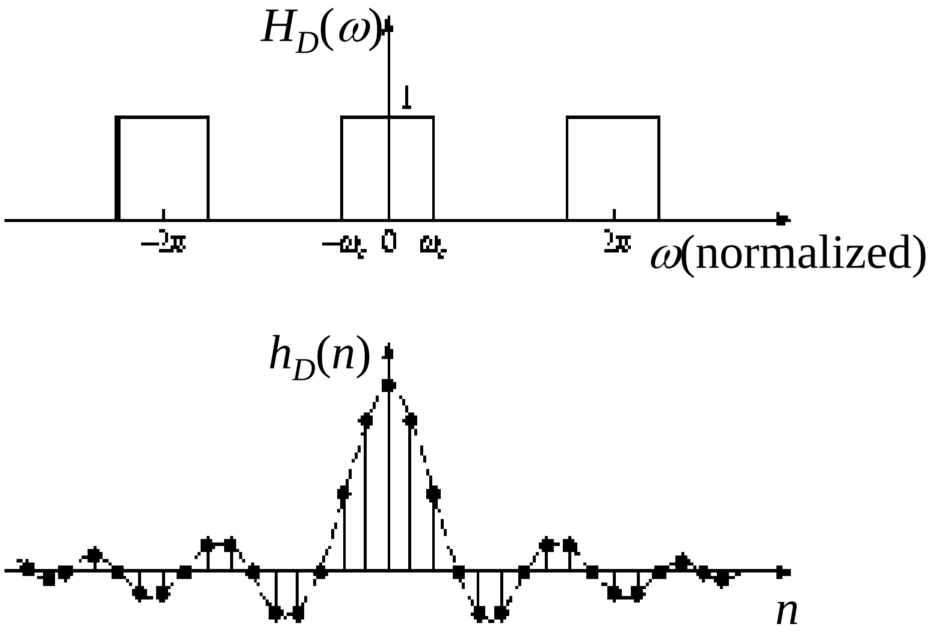
$$H(\mathrm{z})=\Sigma_{k=0}^{N-1}h(k)z^{-k}$$

- The aim of most FIR coefficient calculation (or approximation) methods is to obtain values of h(n) such that the resulting filter meets the design requirements

7.5 Windowing Method

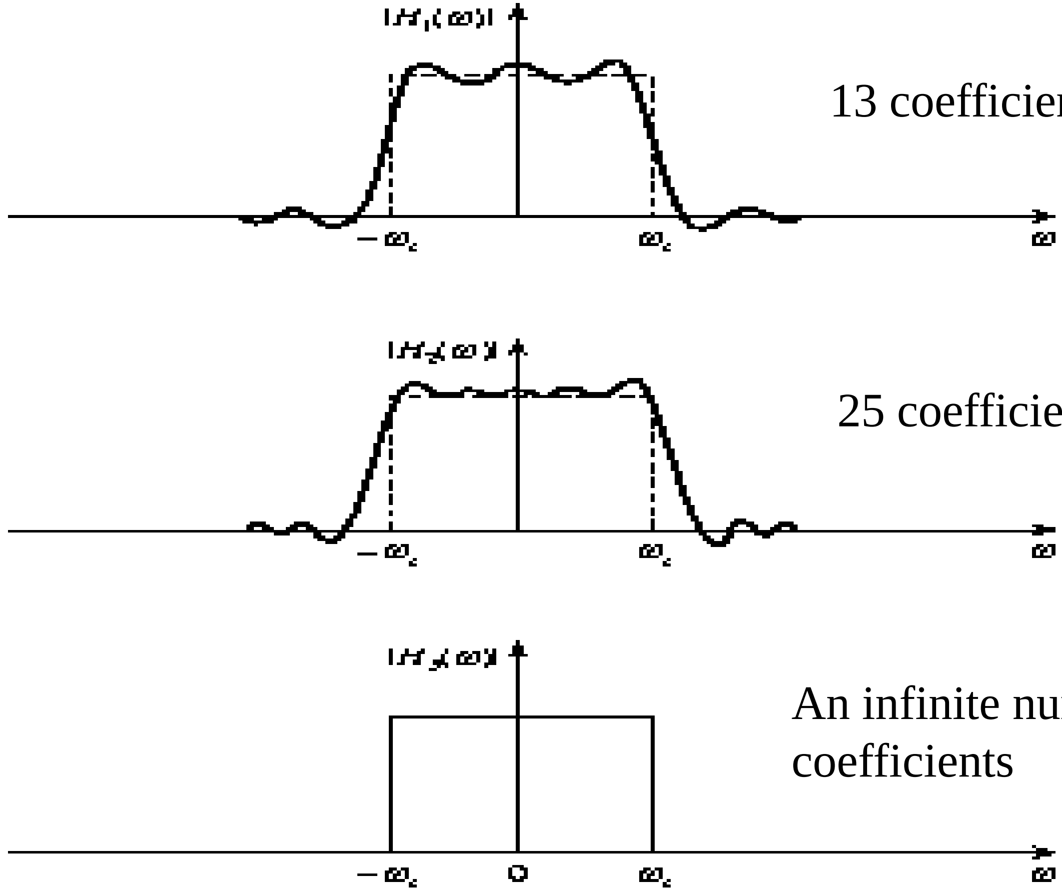
- FIR filter are almost entirely restricted to discrete-time implementations.
- The design techniques for FIR filters are based on directly approximating the desired frequency response of the discrete-time system.
- Most techniques for approximating the magnitude response of an FIR system assume a linear phase constraint, thereby avoiding the problem of spectrum factorization that complicates the direct design of IIR filters.
- The windowing technique is the simplest method of FIR filter design.
- This method generally begins with an ideal desired frequency response, $H_d(e^{jw})$, and evaluates its corresponding impulse response, $h_d[n]$. Then, the desired impulse response, $h[n]$, will be obtained by truncating $h_d[n]$. with selected window function, $w[n]$.

Direct Approximation of the Ideal Frequency Response of a Lowpass Filter



- Problems
  - Not an FIR
  - Introducing undesirable ripples and overshoots-the Gibb's phenomenon
- Solutions
  - Direct truncation of $h_D(n)$
  - Leading to overshoots and ripples

Effects on Frequency Response of Truncating the Ideal Impulse



13 coefficients

25 coefficients

An infinite number of coefficients

The Truncation Approach

$H_d(e^{j\omega}) = \Sigma_{n=-\infty}^{\infty} h_d[n]e^{-j\omega n}$

$h_d[n] \quad = (1/2p) \int_{-p}^{p} H_d(e^{j\omega})e^{j\omega n}d\omega$

$h[n] \quad = h_d[n]w[n] = \begin{cases} h_d[n], & 0 \le n \le M, \\ 0, & \text{otherwise.} \end{cases}$

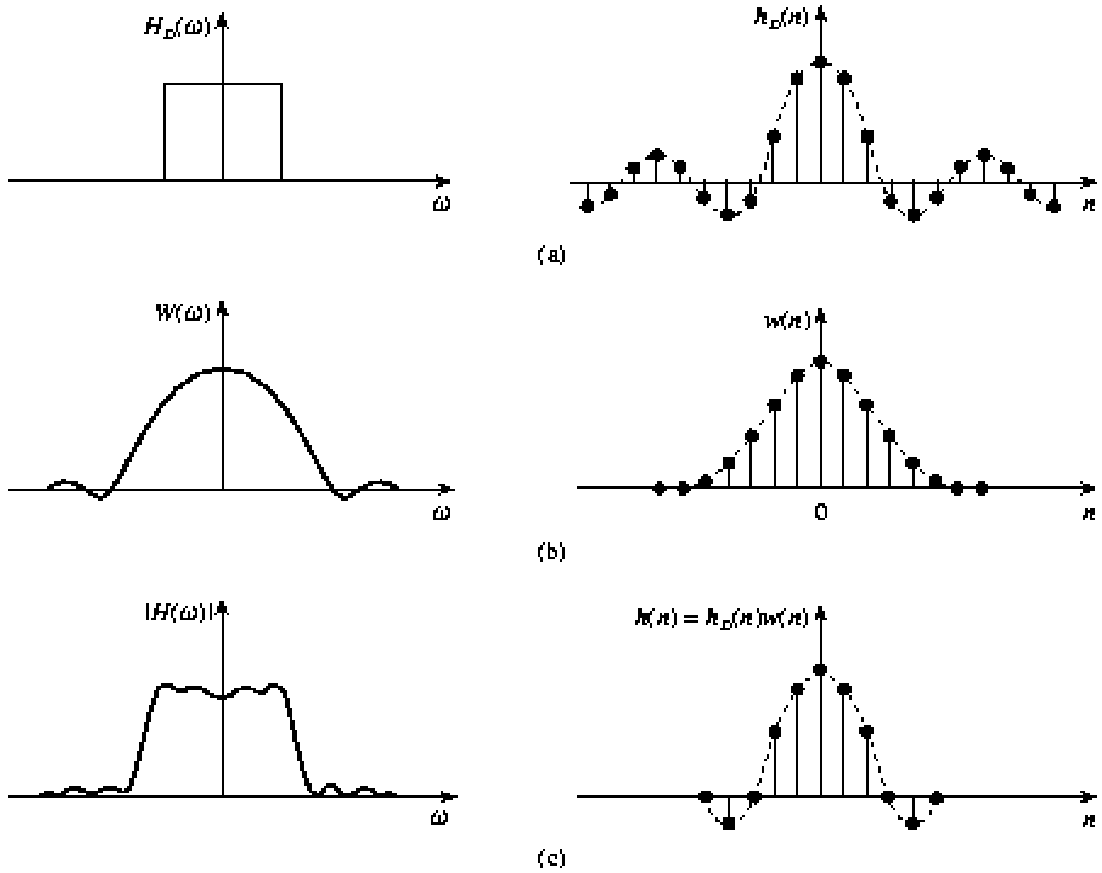
$H(e^{j\omega}) \; = (1/2p) \int_{-p}^{p} H_d(e^{j\omega})W(e^{j(\omega-q)})dq$.

- That is $H(e^{j\omega})$ is the periodic convolution of the desired ideal frequency response with the Fourier transform of the window function.
- Thus, the frequency response $H(e^{j\omega})$ will be a "smeared" version of the desired response $H_d(e^{j\omega})$.
- In the case of the rectangular window:

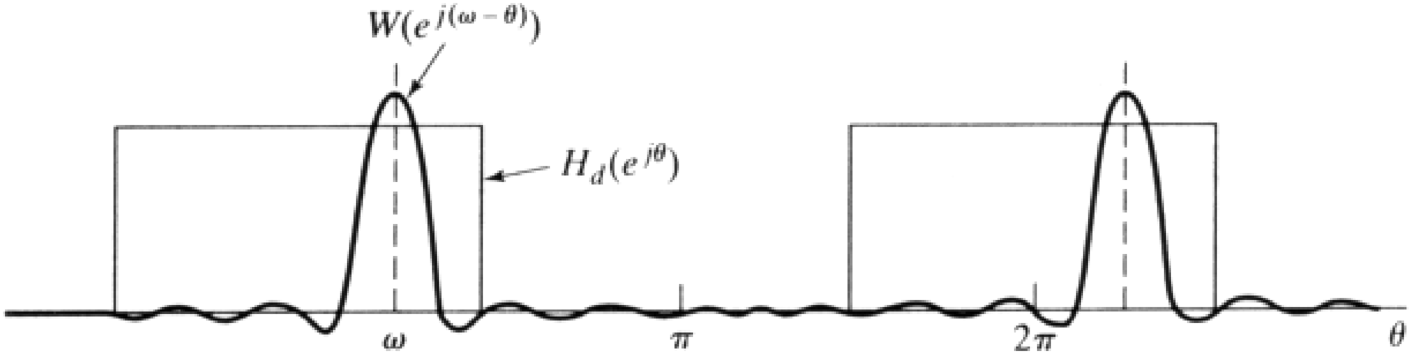
  $\boxed{W(e^{j\omega}) \; = \Sigma_{n=0}^{M} e^{-j\,\omega n} \; = e^{-j\omega M/2}\{\sin(\omega[M+1]/2\}/\{\sin(\omega/2)\}.}$

  As M increases, the width of the main lobe decreases.
- The main lobe is the region between the first zero-crossings on either side of the origin.
- Gibbs Phenomenon

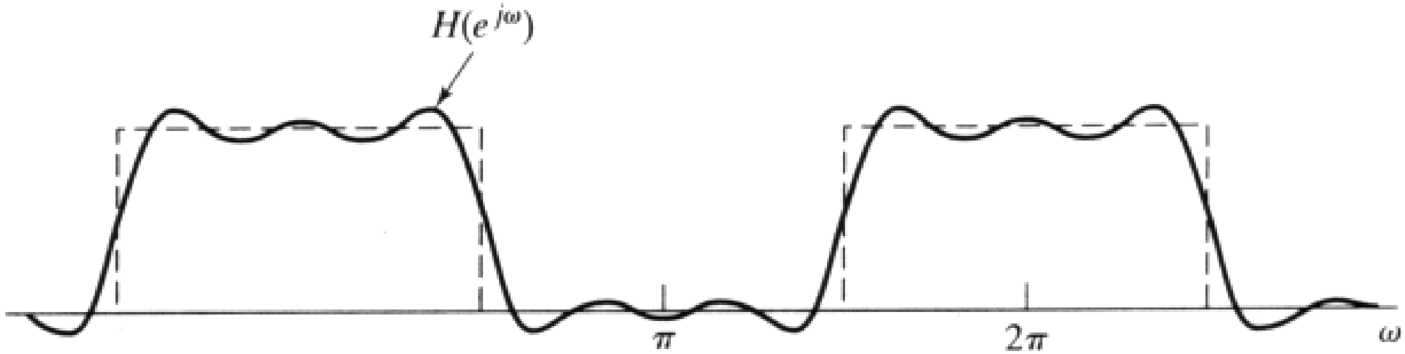
An Illustration for Determining the Filter Coefficients



Convolution Process for Truncating the Ideal Impulse Response

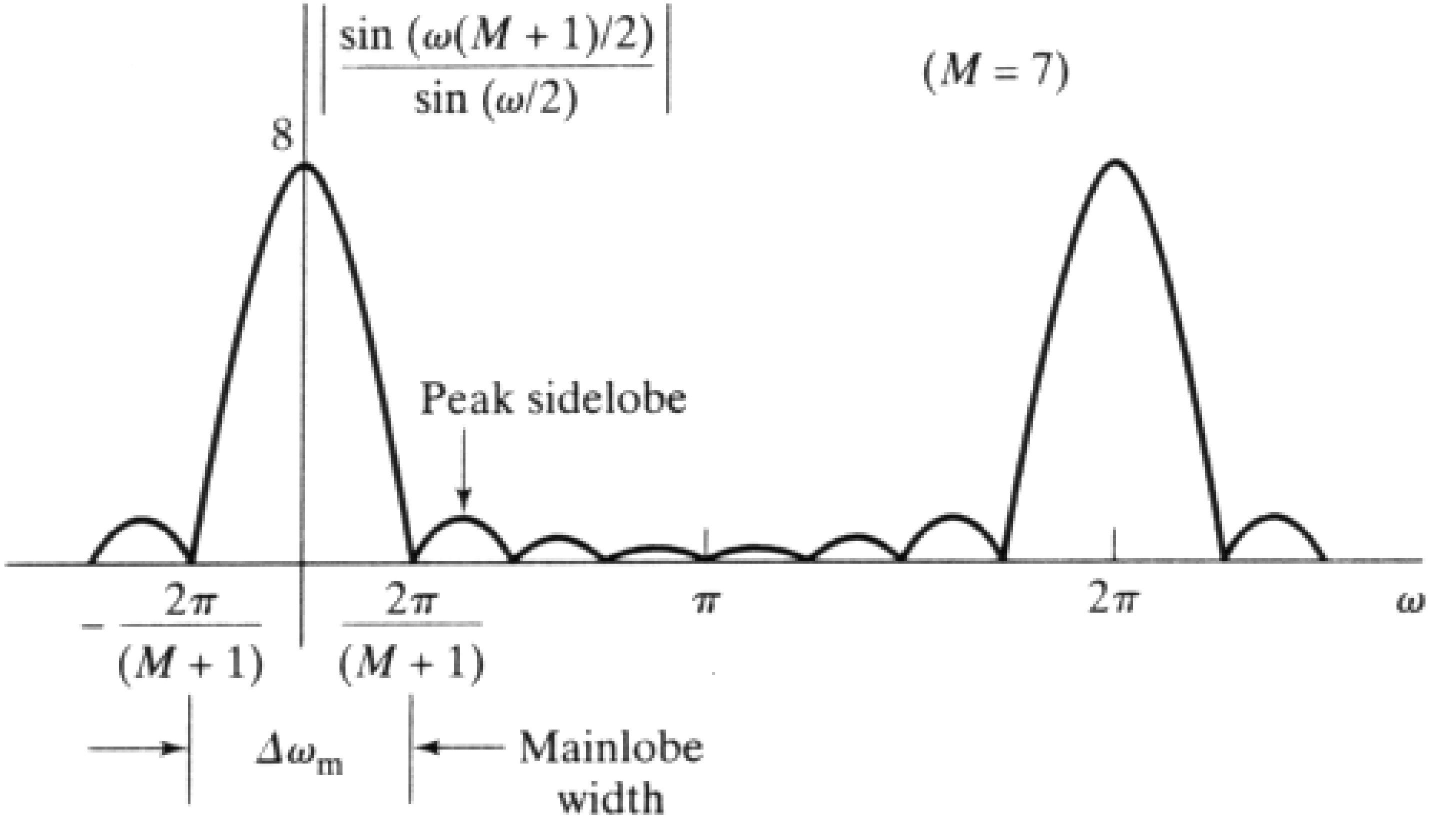


Convolution process implied by truncation of the ideal impulse response



Typical approximation resulting from windowing the ideal impulse response

7.5.1 Some Common Window Functions



Magnitude of the Fourier transform of a rectangular window, M = 7.

Window Functions for FIR Filter Design

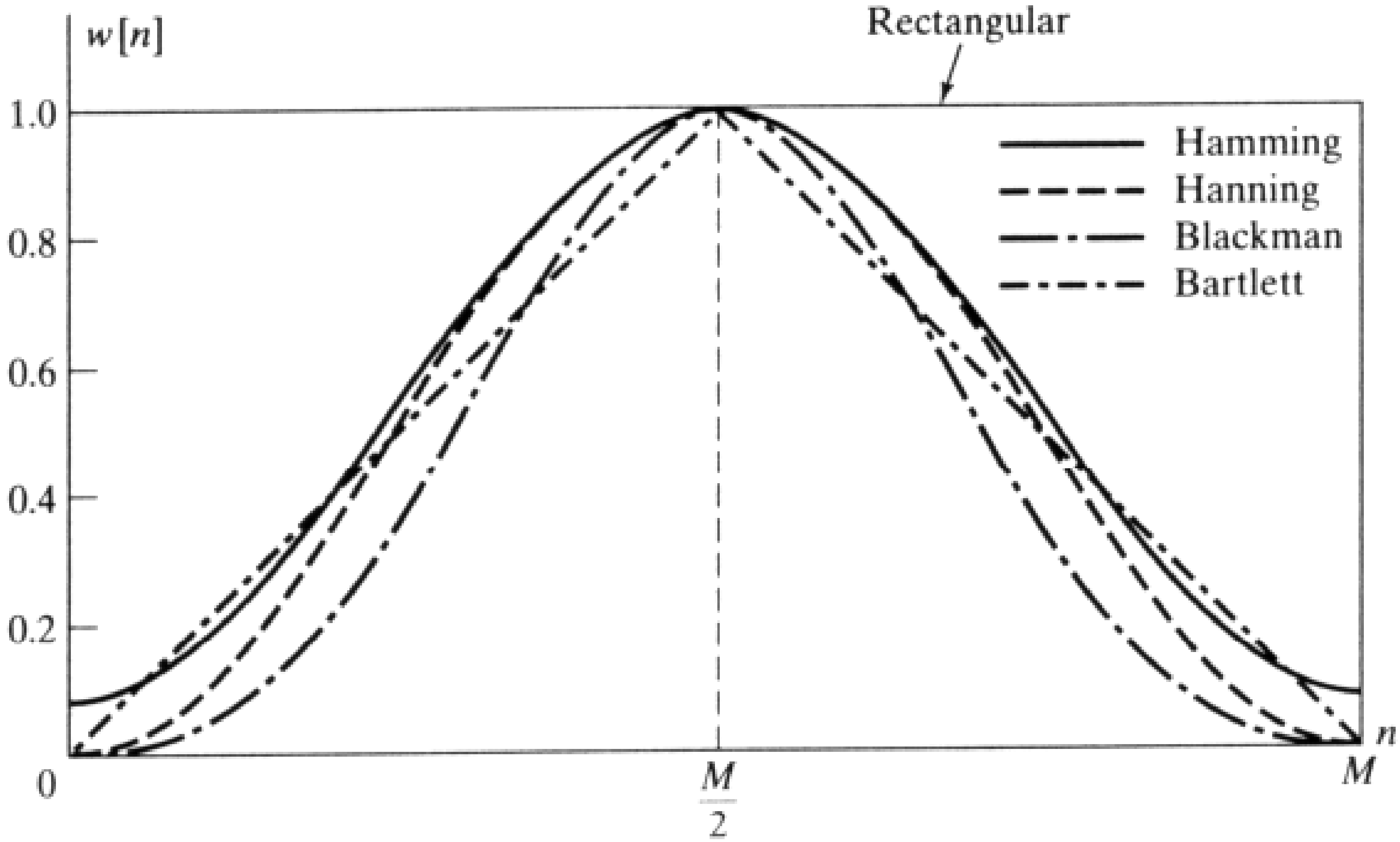
Window Type	Time-Domain Sequence
Rectangular	$w[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$
Bartlett (Triangular)	$w[n] = \begin{cases} 2n/M, & 0 \le n \le M/2 \\ 2-2n/M, & M/2 < n \le M \\ 0, & \text{otherwise} \end{cases}$
Hanning	$w[n] = \begin{cases} 0.5 - 0.5\cos(2pn/M), & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$
Hamming	$w[n] = \begin{cases} 0.54 - 0.46\cos(2pn/M), & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$
Blackman	$w[n] = \begin{cases} 0.42 - 0.5\cos(2pn/M) + 0.08\cos(4pn/M), & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$
Kaiser	$w[n] = \begin{cases} I_0[\beta(1 - \{(n-a)/a\}^2)^{1/2}]/I_0(\beta), & 0 \le n \le M,\ a = M/2 \\ 0, & \text{otherwise} \end{cases}$

$$I_0(x) = 1 + \sum_{k=1}^{L} \left[\frac{(x/2)^k}{k!} \right]^2$$

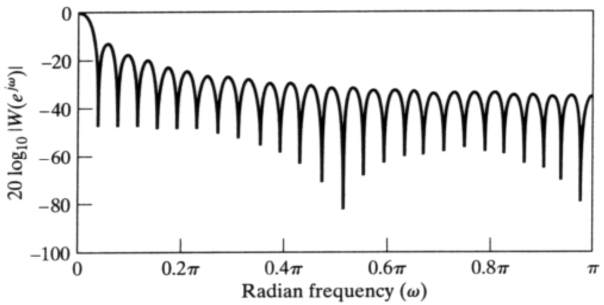
Typical L<25

$I_0(.)$ is zero order modified Bessel function of the first kind, β is window shape parameter.

Shape of commonly used window functions

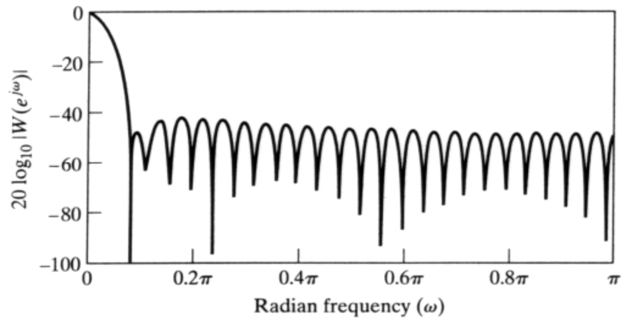


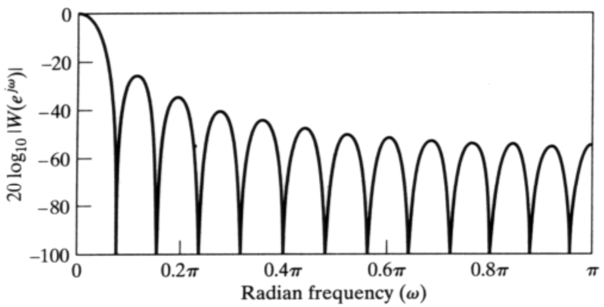
Log magnitude of Fourier transform of window functions



Rectangular
M = 50

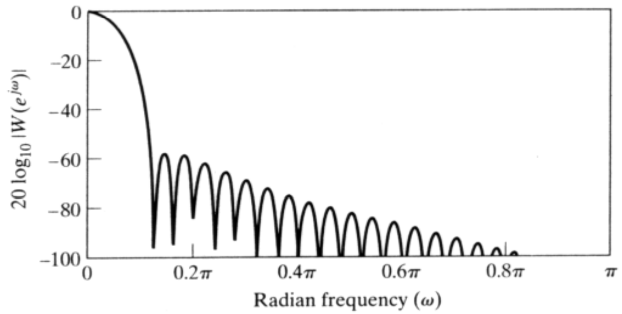
Hamming
M = 50

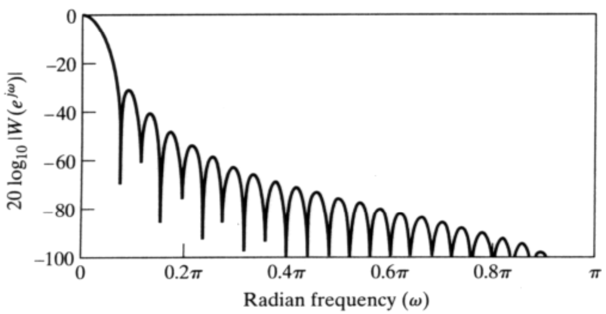




Bartlett
M = 50

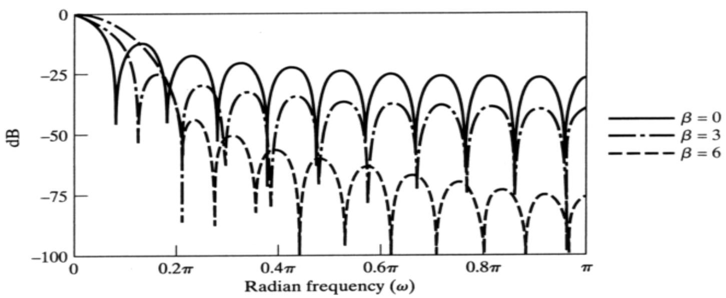
Blackman
M = 50





Hanning
M = 50

Kaiser
M = 20



Notes

- The width of the main lobe and the relative side-lobe amplitude depend primarily on the window length M and the shape (amount of tapering) of the window.
- Through the choice of the shape and duration of the window, we can control the properties of the resulting FIR filter:
  - The windows with the smaller side lobes yield better approximations of the ideal response at a discontinuity.
  - The smaller width of the main lobe which can be achieved by increasing M yield the narrower transition regions.
- Kaiser [1974] has developed a simple formalization of the window method using Kaiser window.
- Kaiser window overcomes the disadvantage occurred in using other window because we must try different windows and adjust their length by trial and error method.
- Filters designed by the window method inherently have $\delta p = \delta s$, so must use the smaller value of ripple in the design procedure.

Frequency-Domain Characteristics of Window Functions

Type of window	Peak side lobe amplitude, dB	Approximation width of main lobe	Peak approximation error, $20\log_{10}\delta$, dB	Equivalent Kaiser window, β	Transition width of equivalent Kaiser window, $\Delta\omega$
Rectangular	-13	$4\pi/(N+1)$	-21	0	$1.81\pi/N$
Bartlett	-25	$8\pi/N$	-25	1.33	$2.37\pi/N$
Hanning	-31	$8\pi/N$	-44	3.86	$5.01\pi/N$
Hamming	-41	$8\pi/N$	-53	4.86	$6.27\pi/N$
Blackman	-57	$12\pi/N$	-74	7.04	$9.19\pi/N$
Kaiser			-50 -70 -90	4.54 6.76 8.96	$4.86\pi/N$ $8.64\pi/N$ $11.42\pi/N$

$\Delta\omega = \omega_s - \omega_p$

Transition width = width of the main lobe

Kaiser Window Function

- Window shape parameter, β

$$\beta = \begin{cases} 0.1102(A - 8.7), & A > 50 \\ 0.5842(A - 21)^{0.4} + 0.07886(A - 21), & 21 \le A \le 50 \\ 0.0, & A < 21 \end{cases}$$

$$A = -20\log_{10}\delta,\ \delta = \min(\delta_p, \delta_s)$$

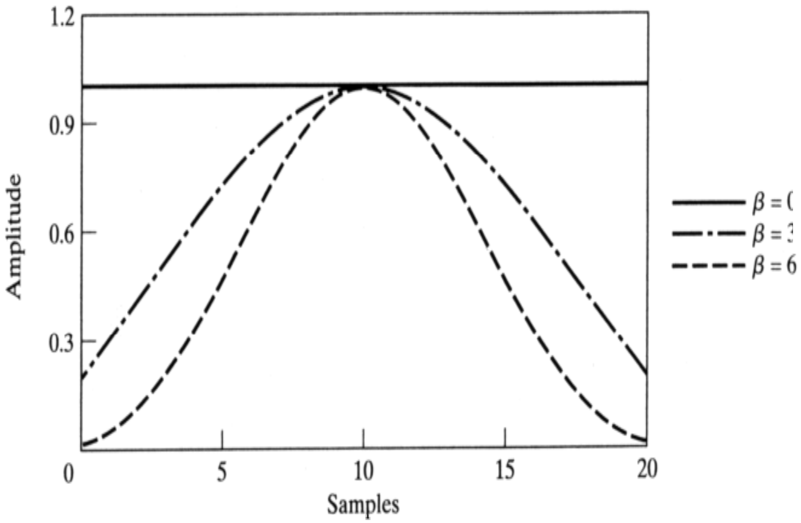
The number of filter coefficients, N, is given by

$$N \ge (A - 7.95)/14.36\Delta f$$

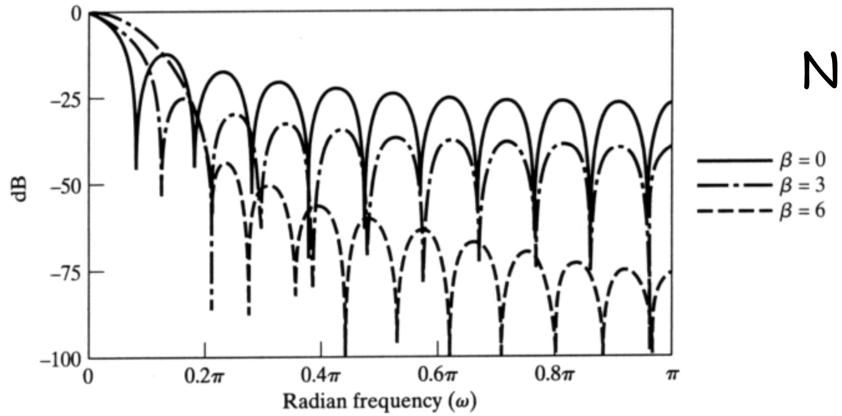
$\Delta f = f_s - f_p$ is the normalized transition width.

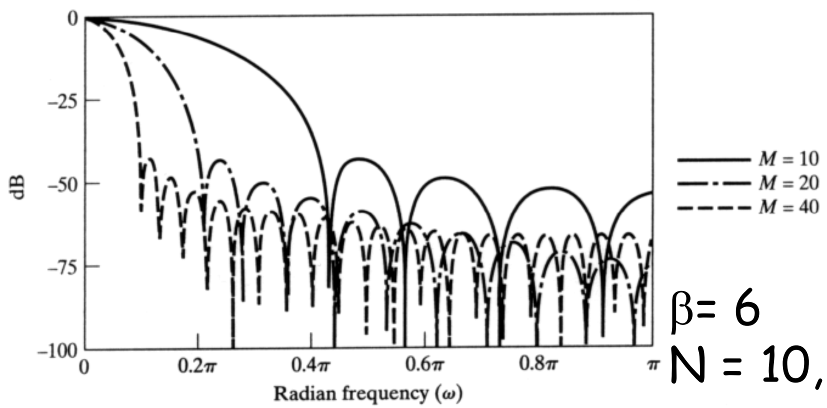
Kaiser window shapes and their frequency characteristics

N = 20



N = 20





β= 6
N = 10, 20, 40.

7.5.2 Windowing Algorithm

- **Step 1**: define specifications of desired filter.
- **Step 2**: evaluate the system function $H_d(e^{j\omega})$ from step 1.
- **Step 3**: evaluate the impulse response sequence $h_d[n]$ as

$$h_d[n] = [1/2p] \int_{-p}^{p} H_d(e^{j\omega})e^{j\omega n}dw$$

where $H_d(e^{j\omega}) = \Sigma_{n=-\infty}^{\infty} h_d[n]e^{-j\omega n}$

- **Step 4**: obtain finite duration sequence $h[n]$ from $h_d[n]$ as

$$h[n] = \begin{cases} h_d[n]w[n], & 0 \le n \le N \\ 0, & \text{otherwise} \end{cases}$$

where $w[n]$ is a selective window function to meet the attenuation requirement, so

$$H(e^{j\omega}) = [1/2p] \int_{-p}^{p} H_d(e^{jq})W(e^{j(\omega-q)})dq$$

- **Step 5**: verify the result. If it does not meet requirement, return to step 4 by reselection of window width (N) and/or type (w[n]).

Summary of Ideal Impulse responses for Standard Frequency Selective Filters

Filter Type	Ideal Impulse Response, $h_d[n]$ $h_d[n],\ n \neq 0$	$h_d[0]$
Lowpass	$2f_c[\sin((n-N/2)\omega_c)/(n-N/2)\omega_c]$	$2f_c$
Highpass	$-2f_c[\sin((n-N/2)\omega_c)/(n-N/2)\ \omega_c]$	$1-2f_c$
Bandpass	$2f_2[\sin((n-N/2)\omega_2)/(n-N/2)\omega_2]$ $- 2f_1[\sin((n-N/2)\omega_1)/(n-N/2)\omega_1]$	$2(f_2 - f_1)$
Bandstop	$2f_1[\sin((n-N/2)\omega_1)/(n-N/2)\omega_1]$ $- 2f_2[\sin((n-N/2)\omega_2)/(n-N/2)\omega_2]$	$1-2(f_2 - f_1)$

Where f_c, f_1 and f_2 are passband or stopband edge frequencies, and N is the filter length.

Example: Kaiser window design of a lowpass filter

- Consider the lowpass digital filter specifications:

  $0.99 \le |H(e^{j\omega})| \le 1.01, \qquad |\omega| \le 0.4\pi,$

  $|H(e^{j\omega})| \le 0.001, \qquad 0.6\,\pi \le |\omega|\ .$

  Using the design formulas for the Kaiser window to design an FIR lowpass filter to meet prescribed specifications.
- First, we set $\delta = 0.001$.
- Next, the cutoff frequency of the ideal lowpass filter is

  $$\omega_c = (\omega_p + \omega_s)/2 = 0.5\,\pi\ .$$
- To determine the parameters of the Kaiser window, we first compute

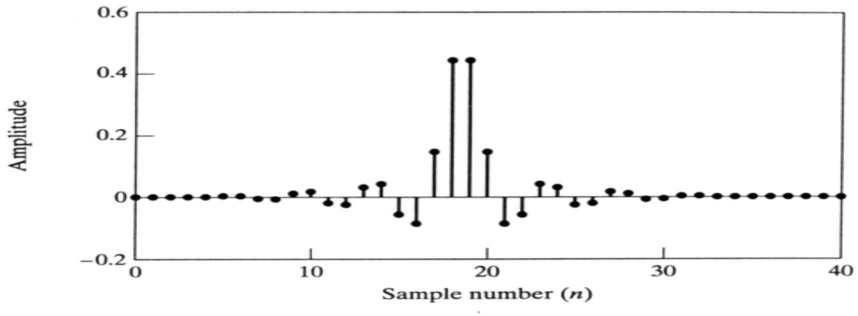
  $$\Delta\omega = \omega_s - \omega_p = 0.2\,\pi\ , \qquad A = -20\log_{10}\delta = 60.$$

  Then, $\beta = 5.653$, $N = 37$.
- The impulse response of the filter is:

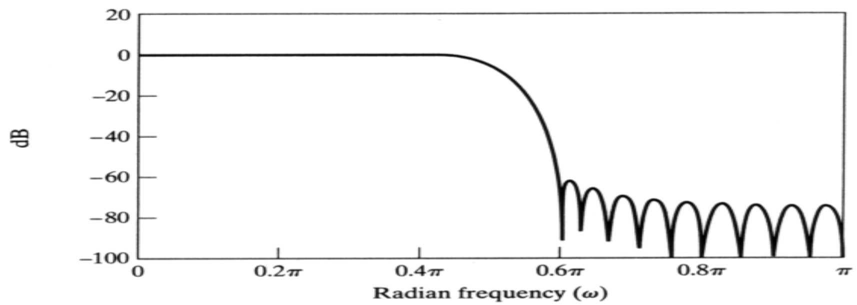
  $$h[n] = \begin{cases} \{\sin(\omega_c[n-a])/\ \pi(n-a)\}\{I_0[\beta(1-[(n-a)/a]^2)^{1/2}/I_0(\beta)\}, & 0 \le n \le N, \\ 0, & \text{otherwise.} \end{cases}$$

  where $a = N/2 = 18.5$

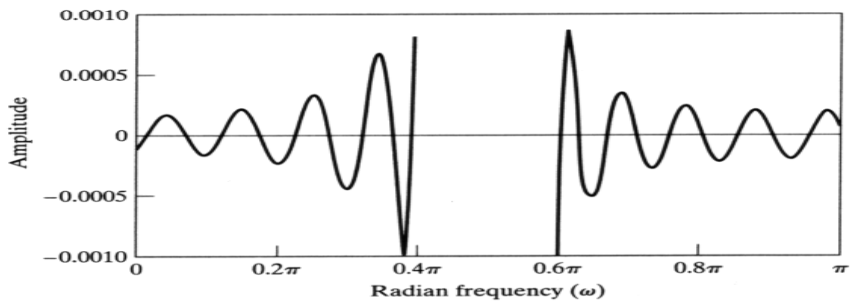
The Response Functions of Lowpass Filter for Kaiser Windows of β = 5.653 and N = 37.



Impulse response



Log magnitude



Approximation error

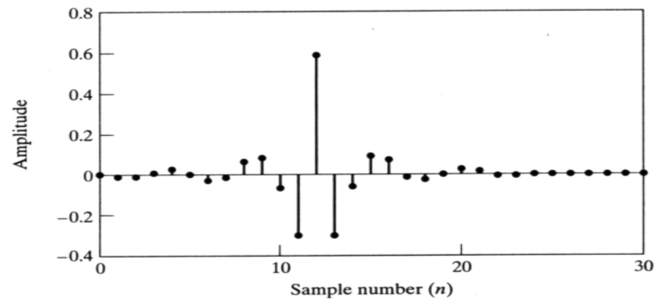
Notes

- Increasing the order of N may lead to more unsatisfactory result.
- Type II FIR linear-phase systems are generally not appropriate approximations for either highpass or bandstop filters.

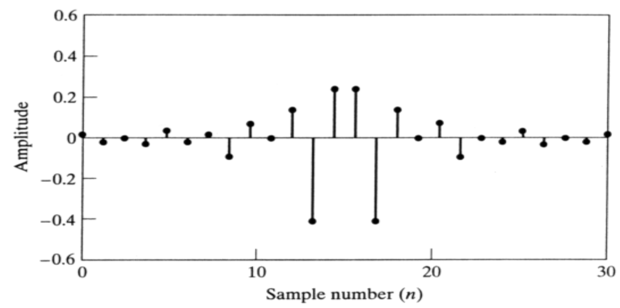
The response function for type I and II FIR highpass filters.

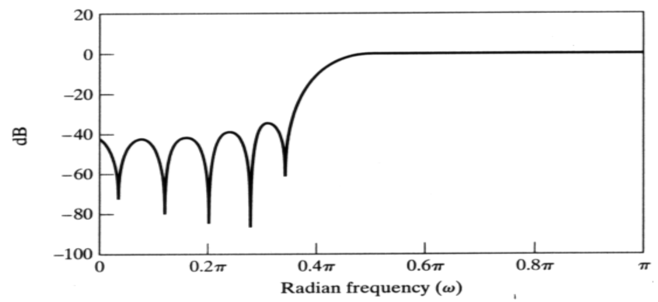
Type I: M = 24

Type II: M = 25

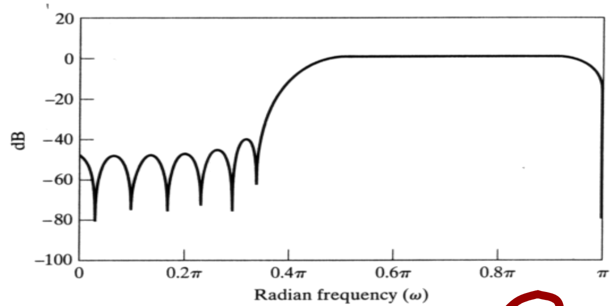


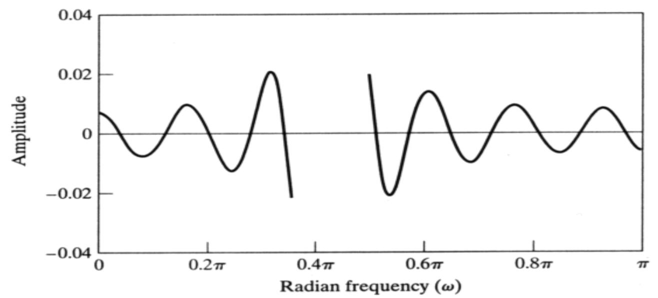
Impulse response



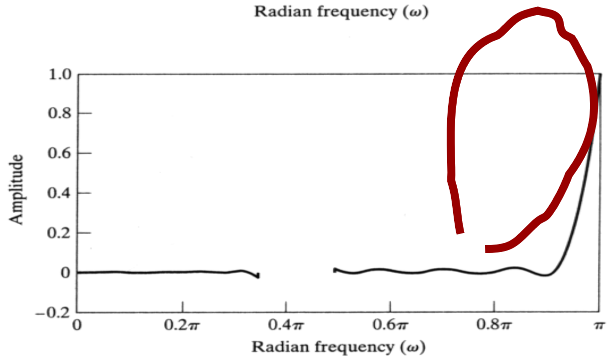


Log magnitude





Approximation error



Window Functions for FIR Filter Design

Window Type	Time-Domain Sequence
Rectangular	$w[n] = \begin{cases} 1, & 0 \le n \le N \\ 0, & \text{otherwise} \end{cases}$
Bartlett (Triangular)	$w[n] = \begin{cases} 2n/N, & 0 \le n \le N/2 \\ 2-2n/N, & N/2 < n \le N \\ 0, & \text{otherwise} \end{cases}$
Hanning	$w[n] = \begin{cases} 0.5 - 0.5\cos(2\pi n/N), & 0 \le n \le N \\ 0, & \text{otherwise} \end{cases}$
Hamming	$w[n] = \begin{cases} 0.54 - 0.46\cos(2\pi n/N), & 0 \le n \le N \\ 0, & \text{otherwise} \end{cases}$
Blackman	$w[n] = \begin{cases} 0.42 - 0.5\cos(2\pi n/N) + 0.08\cos(4\pi n/N), & 0 \le n \le N \\ 0, & \text{otherwise} \end{cases}$
Kaiser	$w[n] = \begin{cases} I_0[\beta(1 - \{(n-\alpha)/\alpha\}^2)^{1/2}]/I_0(\beta), & 0 \le n \le N,\ \alpha = N/2 \\ 0, & \text{otherwise} \end{cases}$

$I_0(.)$ is zero order modified Bessel function of the first kind, β is window shape parameter.

Example:
Windowing Method → Hamming Window

- Obtain the coefficients of an FIR lowpass filter to meet the specifications given below using the window method.

passband edge frequency	1.5 kHz
transition width	0.5 kHz
stopband attenuation	> 50 dB
sampling frequency	8 kHz

- We select $h_d[n]$ for lowpass filter which is given by:

$$h_d[n] = \begin{cases} 2f_c[\sin((n-N/2)\omega_c)]/[(n-N/2)\omega_c] = \sin(n\omega_c)/n\pi, & n \neq N/2, \\ 2f_c, & n = N/2 \end{cases}$$

- From characteristics table, it indicates that only the Hamming, Blackman or Kaiser ($\beta = 4.54$) windows will satisfy the stopband attenuation requirements. If we use the Hamming window for simplicity.

$\Delta f = 0.5k/8k = 0.0625 \rightarrow 8\pi/N = \Delta\omega$ (pp.21 slide) → take $N = 64$

And $h[n] = h_d[n]w[n]$, $w[n] = \begin{cases} 0.54 - 0.46\cos(2\pi n/N), & 0 \le n \le N, \\ 0, & \text{otherwise.} \end{cases}$

$h[0] = h[64]$	$= h_d[0]w[0]$	$= 0x0.08$	$= 0$
$h[1] = h[63]$	$= h_d[1]w[1]$	$= -0.01007x0.08222$	$= -0.00083$
$h[2] = h[62]$	$= h_d[2]w[2]$	$= -0.00406x0.08884$	$= -0.00036$
$h[3] = h[61]$	$= h_d[3]w[3]$	$= 0.00913x0.09981$	$= 0.00091$
$h[4] = h[60]$	$= h_d[4]w[4]$	$= 0.00804x0.11502$	$= 0.00093$
$h[5] = h[59]$	$= h_d[5]w[5]$	$= -0.00655x0.13432$	$= -0.00088$
$h[6] = h[58]$	$= h_d[6]w[6]$	$= -0.01131x0.15752$	$= -0.00178$
…			
$h[30] = h[34]$	$= h_d[30]w[30]$	$= 0.06091x0.99116$	$= 0.06037$
$h[31] = h[33]$	$= h_d[31]w[31]$	$= 0.31219x0.99779$	$= 0.31150$
$h[32] = h[32]$	$= h_d[32]w[32]$	$= 1x1$	$= 1$

Where f_c will be chosen to the center of the transition band $= f_c + \Delta f/2 =$
$[1.5k + 0.5k/2]/8k = 0.21875$

Example: Windowing Method → Kaiser Window

- Design an FIR digital filter to meet the following specifications:

passband	150～250 Hz
transition width	50 Hz
passband ripple	0.1 dB
stopband attenuation	60 dB
sampling frequency	1 kHz

  Obtain the filter coefficients using the window method.
- Compare the ripples: $20\log_{10}(1+\delta_p) = 0.1$ dB → $\delta_p = 0.0115$
  and $-20\log_{10}\delta_s = 60$ dB → $\delta_s = 0.001 < \delta_p$
  Thus $d = \min(\delta_p, \delta_s) = 0.001$ → $A = -20\log_{10}\delta = 60$ dB
- The attenuation requirements (60 dB) can only be met by the Kaiser or the Blackman window. If we select the Kaiser window, the number of coefficients is $N \geq (A-7.95)/(14.36\Delta f) = 72.44$ → $N = 73$, and $\beta = 0.1102(A-8.7) = 5.65$. where $\Delta f = 50/1k = 0.05$.

Notes

- When we select the Blackman window:

  $\Delta\omega = 12\pi/N \rightarrow N = 120$

- It is clearly that the complexity of the designed filter using the Blackman window is nearly 2 times greater than that using the Kaiser window.

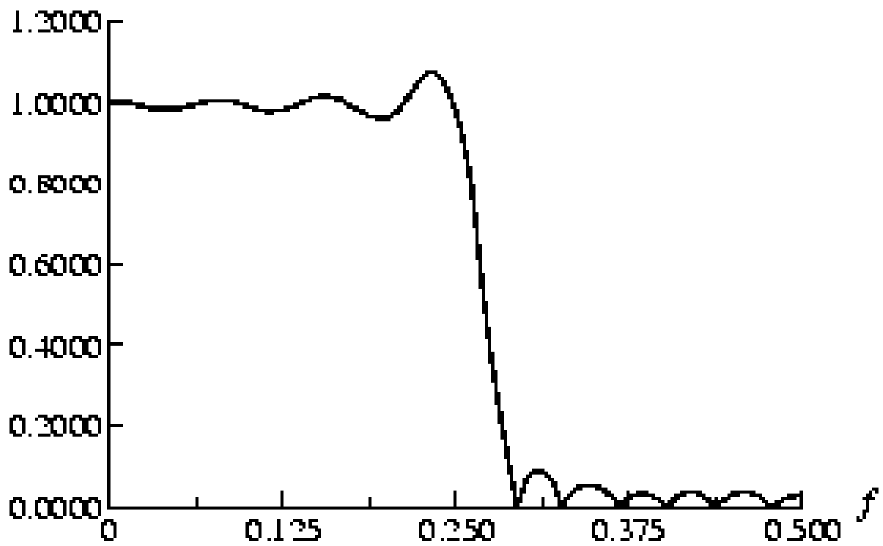
Advantages and Disadvantages of the Window Method

- Simplicity
  - It is simple to apply and simple to understand. It involves a minimum amount of computational effort, even for the more complicated Kaiser window.
- Lack of flexibility
  - Both the peak passband and stopband ripples are approximately equal, so that the designer may end up with either too small a passband ripple or too large a stopband attenuation.
- Imprecision
  - Because of the effect of convolution of the spectrum of the window function and the desired response, the passband and stopband edge frequencies cannot be precisely specified.
- Clumsy (trial and error technique)
  - For a given window (except the Kaiser) the maximum ripple amplitude in the filter response is fixed regardless of how large we make N. Thus the stopband attenuation for a given window is fixed. Thus, for a given attenuation specification, the filter designer must find a suitable window.
- Lack of capability
  - In some applications, the expression for the desired filter response, $H_d(w)$, will be too complicated for $h_d[n]$ to be obtained analytically. In these cases $h_d[n]$ may be obtained via the frequency sampling method before the window function is applied.

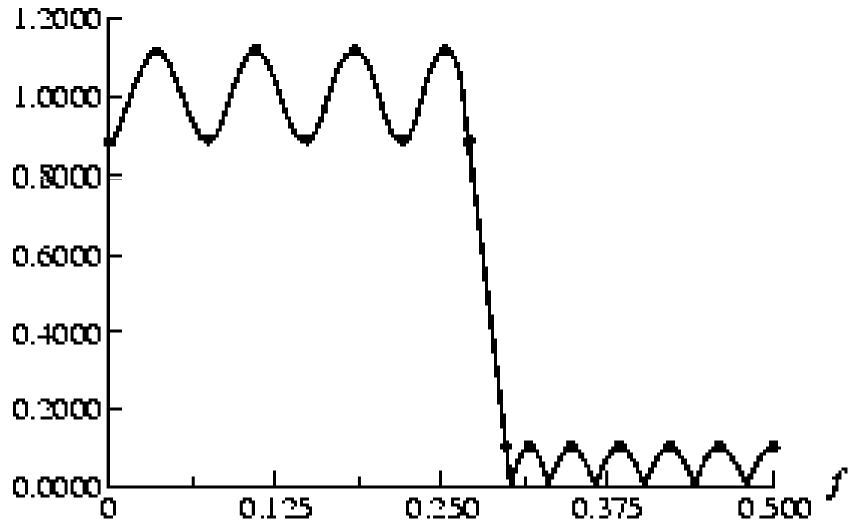
7.6 The Optimal Method

- The optimal method is based on the concept of "equiripple" passband and stopband.
- The window method and the frequency sampling method have a major problem that is the lack of precise control of the critical frequencies such as ω_p and ω_s .
- The filter design method selected to implement the optimal design is formulated as a *Chebyshev approximation problem*.
- It will be viewed as an optimum design criterion in the sense that the *weighted approximation error* between the desired frequency response and the actual frequency response is *spread evenly* across the passband and evenly across the stopband of the filter minimizing the maximum error.
- The resulting filter designs have *ripples in both* the passband and stopband.

Basic Concepts

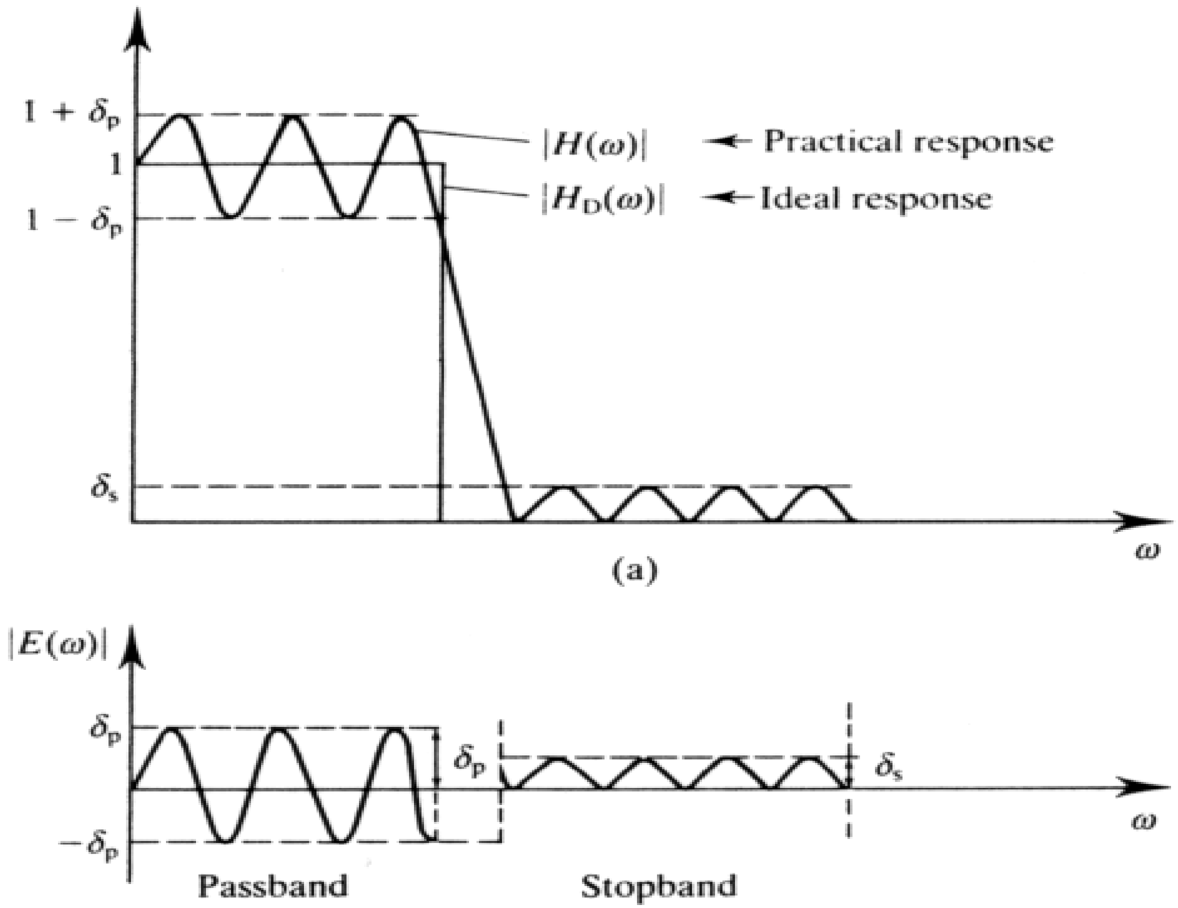


Window Filter



Optimal filter

Frequency Response of an Optimal Lowpass Filter



Optimal Approximation of FIR Filters

- Alternation Theorem and Polynomials
  - It provides a <u>necessary and sufficient condition</u> for a polynomial to satisfy in order that it is the polynomial that minimizes the maximum weighted error for a given order.
- The <u>Parks-McClellan Algorithm</u>
  - Parks and McClellan [1972] applied the alternation theorem to the optimum approximation of FIR filter design problem.

7.7 Frequency Sampling Method

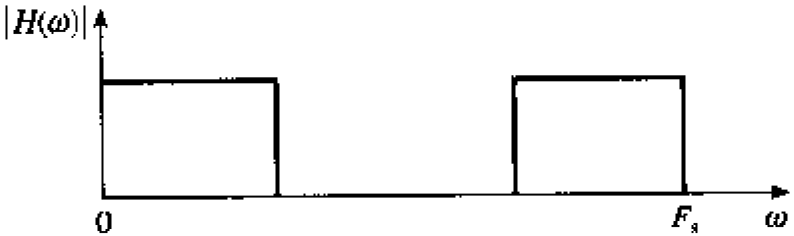
- This method allows us to design nonrecursive FIR filters for both standard frequency selective filters (lowpass, highpass, bandpass) and filters with arbitrary frequency response.
- It also allows recursive implementation of FIR filters, leading to computationally efficient filters.
- From the DFT, $h[n] = (1/N)\Sigma_{k=0}^{N-1} H[k]e^{j2\pi nk/N}$, it can be shown that for linear phase filters, with positive symmetrical impulse response and for N even:

$$h[n] = (1/N)[\Sigma_{k=0}^{(N/2)-1} 2|H[k]|\cos|(2\pi k(n - \alpha)/N)| + H(0)]$$

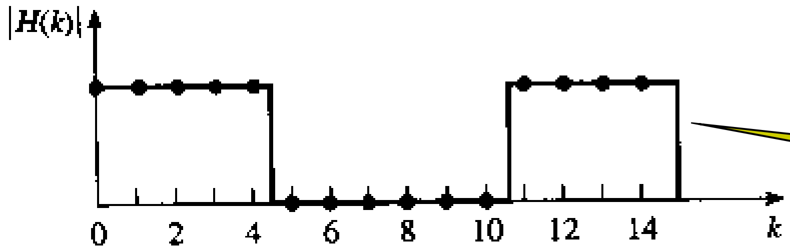
where $\alpha = (N - 1)/2$. For N odd, it becomes:

$$h[n] = (1/N)[\Sigma_{k=0}^{(N-1)/2} 2|H[k]|\cos|(2\pi k(n - \alpha)/N)| + H(0)]$$

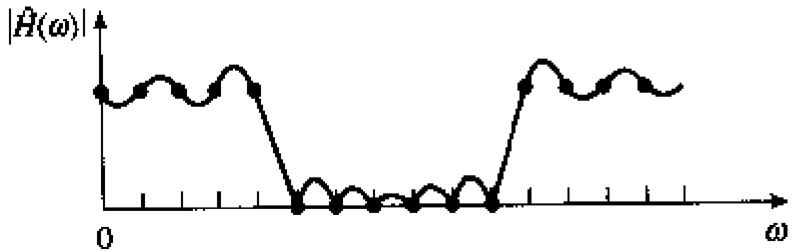
7.7.1 Nonrecursive Frequency Sampling Filters



(a)



(b)

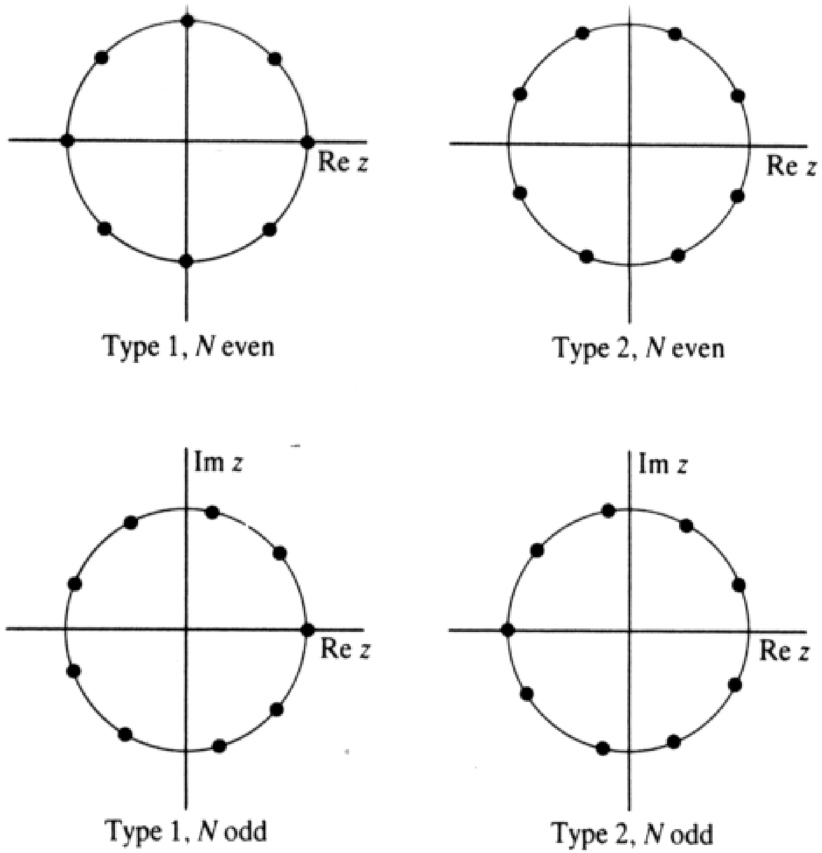


(c)

The frequency response of the designed filter

Obtaining
$h[n] = (1/N)\Sigma_{k=0}^{N-1} H[k]e^{j2\pi nk/N}$

The 4 possible z-plane sampling grids for the two types of frequency sampling filters.



Example 1: Frequency Sampling Method

- Consider a lowpass FIR filter with the following specifications:

  passband 0-5 kHz
  sampling frequency 18 kHz
  filter length 9

  Obtain the filter coefficients using the frequency sampling method.
- The frequency samples are taken at intervals of kF_s/N, that is at intervals of 18/9 = 2 kHz. Thus the frequency samples are given by

  $|H[k]| = 1$ at $k = 0, 1, 2$
  $= 0$ at $k = 3, 4$
- Because N is even, then:

  $h[0] = h[8] = 7.2522627 \times 10^{-2}$
  $h[1] = h[7] = -1.1111111 \times 10^{-1}$
  $h[2] = h[6] = -5.9120987 \times 10^{-2}$
  $h[3] = h[5] = 3.1993169 \times 10^{-1}$
  $h[4] = 5.5555556 \times 10^{-1}$.

Example 2: Frequency sampling method

- Determine the coefficients of a linear-phase FIR filter of length M = 15 which has a symmetric unit sample response and a frequency response that satisfies the conditions

$H[2\pi k/15] =$	1 ,	k = 0, 1, 2, 3
	0.4 ,	k = 4
	0 ,	k = 5, 6, 7

- Since h[n] is symmetric and the frequencies are selected to correspond to the case of Type I and because N is even

h[0]	=	h[14]	=	-0.014112893
h[1]	=	h[13]	=	-0.001945309
h[2]	=	h[12]	=	0.04000004
h[3]	=	h[11]	=	0.01223454
h[4]	=	h[10]	=	-0.09138802
h[5]	=	h[9]	=	-0.01808986
h[6]	=	h[8]	=	0.3133176
h[7]			=	0.52

Optimizing the Amplitude Response

- For a lowpass filter, the stopband attenuation increases, approximately, by 20 dB for each transition band frequency sample [Rabiner et al., 1970], with a corresponding increase in the transition width:

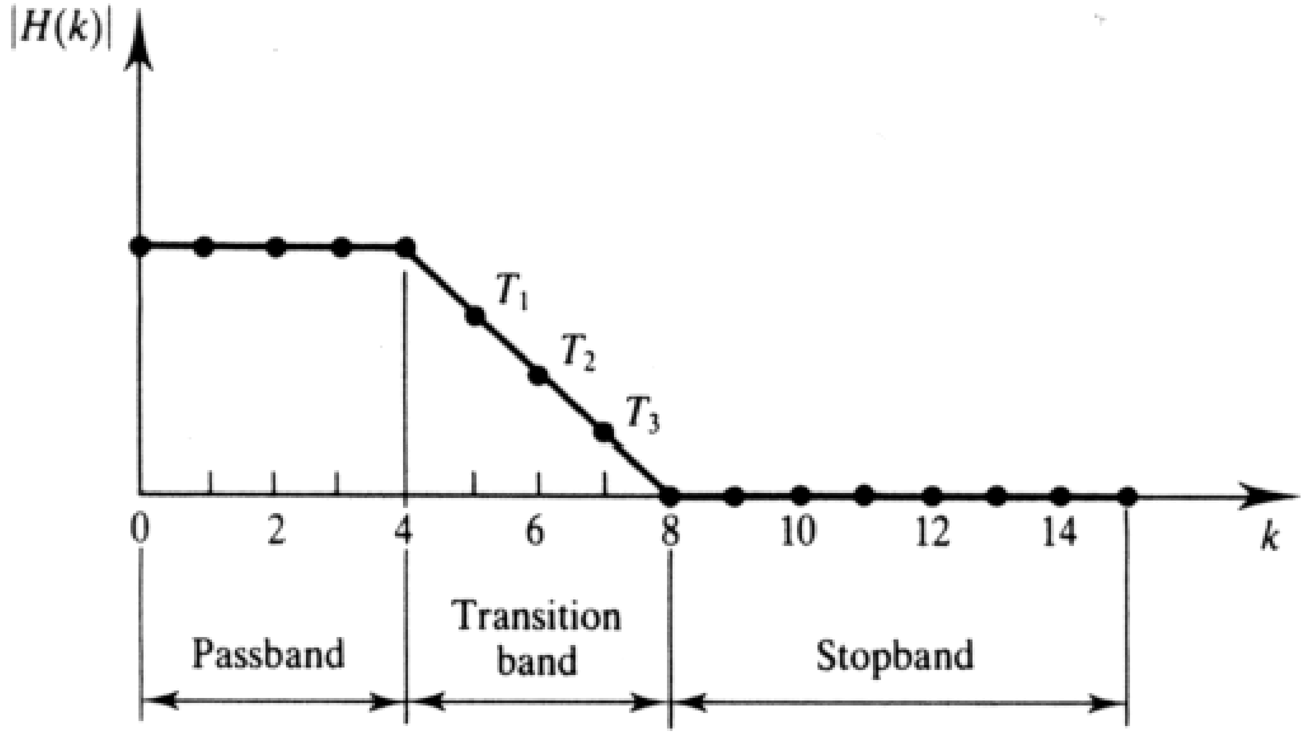
  Approximate stopband attenuation (25 + 20M) dB

  Approximate transition width $(M + 1)F_s/N$

  where M is the number of transition band frequency samples and N is the filter length.

  - For one transition frequency sample: $0.250 < T_1 < 0.450$
  - For two transition frequency samples: $0.040 < T_1 < 0.150$
    $0.450 < T_2 < 0.650$
  - For three transition frequency samples: $0.003 < T_1 < 0.035$
    $0.100 < T_2 < 0.300$
    $0.550 < T_3 < 0.750$

Lowpass filter frequency samples including three transition band samples

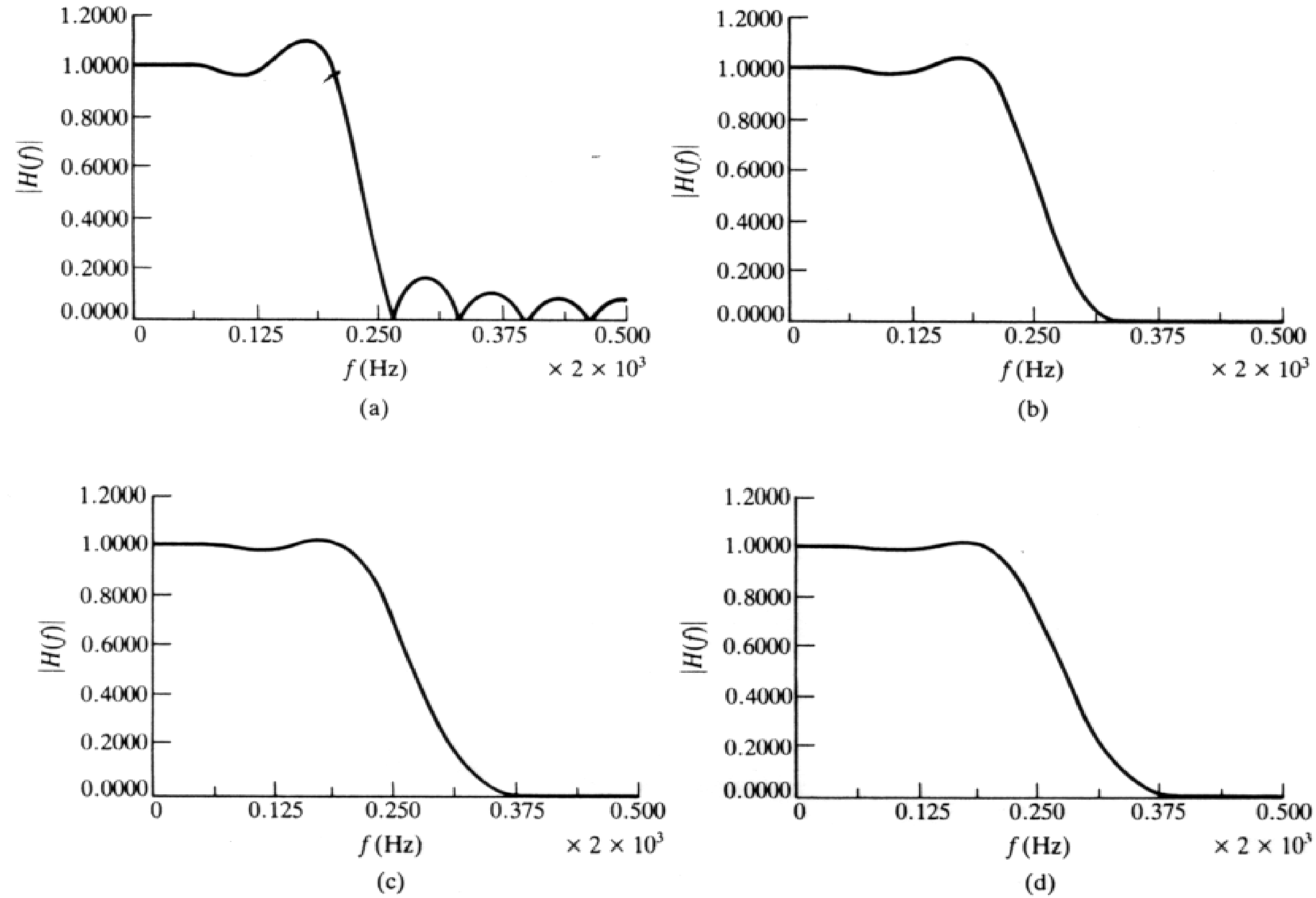


Optimum transition band frequency samples for type I lowpass frequency sampling filters for N = 15 [adapted from Rabiner et al., 1970]

BW	Stopband attenuation (dB)	T_1	T_2	T_3
One transition band frequency sample, N = 15				
1	42.309 322 83	0.433 782 96		
2	41.262 992 86	0.417 938 23		
3	41.253 337 86	0.410 473 63		
4	41.949 077 13	0.404 058 84		
5	44.371 245 38	0.392 681 89		
6	56.014 165 88	0.357 665 25		
Two transition band frequency samples, N = 15				
1	70.605 405 85	0.095 001 22	0.589 954 18	
2	69.261 681 56	0.103 198 24	0.593 571 18	
3	69.919 734 95	0.100 836 18	0.589 432 70	
4	75.511 722 56	0.084 074 93	0.557 153 12	
5	103.460 783 00	0.051 802 06	0.499 174 24	
Three transition band frequency samples, N = 15				
1	94.611 661 91	0.014 550 78	0.184 578 82	0.668 976 13
2	104.998 130 80	0.010 009 77	0.173 607 13	0.659 515 26
3	114.907 193 18	0.008 734 13	0.163 973 10	0.647 112 64
4	157.292 575 84	0.003 787 99	0.123 939 63	0.601 811 54

BW refers to the number of frequency samples in the passband.

Frequency response of frequency sampling filter with (a) no transition band samples; (b) One transition band sample; (c) two transition band samples; (d) three transition Band samples.



7.7.2 Recursive Frequency Sampling Filters

$$h(n) = \frac{1}{N}\sum_{k=0}^{N-1} H(k) r^n e^{j2\pi nk/N} \quad k = 0, 1, \ldots, N-1, r \leq 1$$

$$H(z) = \frac{1}{N}\sum_{k=0}^{N-1} H(k)\left\{\sum_{n=0}^{N-1}[r e^{j(2\pi k/N)} z^{-1}]^n\right\}$$

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n} = \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{k=0}^{N-1} H(k) r^n e^{j2\pi nk/N}\right] z^{-n}$$

$$H(z) = \frac{1 - z^{-N}}{N}\sum_{k=0}^{N-1}\frac{H(k)}{1 - e^{j2\pi k/N} z^{-1}}$$

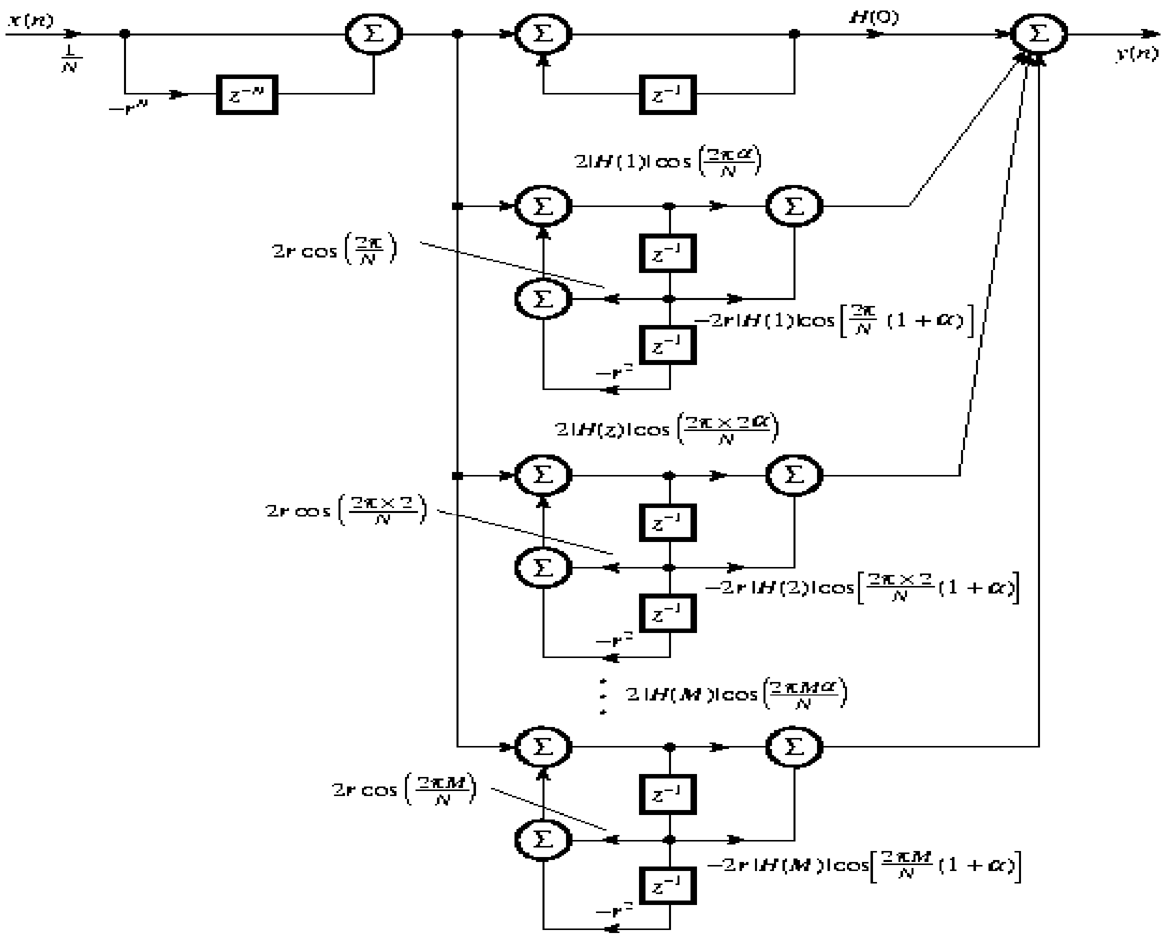
If the symmetry inherent in the frequency response of any FIR filter with real impulse response is use, we have

$$H(z) = \frac{1 - r^N z^{-N}}{N}$$

$$\times \left[\sum_{k=1}^{M}\frac{|H(k)|\{2\cos(2\pi k\alpha/N) - 2r\cos[2\pi k(1+\alpha)/N] z^{-1}\}}{1 - 2r\cos(2\pi k/N) z^{-1} + r^2 z^{-2}} + \frac{H(0)}{1 - z^{-1}}\right]$$

where α=(N-1)/2. For N odd M=(N-2)/2 and for N even M=N/2-1

Realization Diagram for the Frequency Sampling Filter



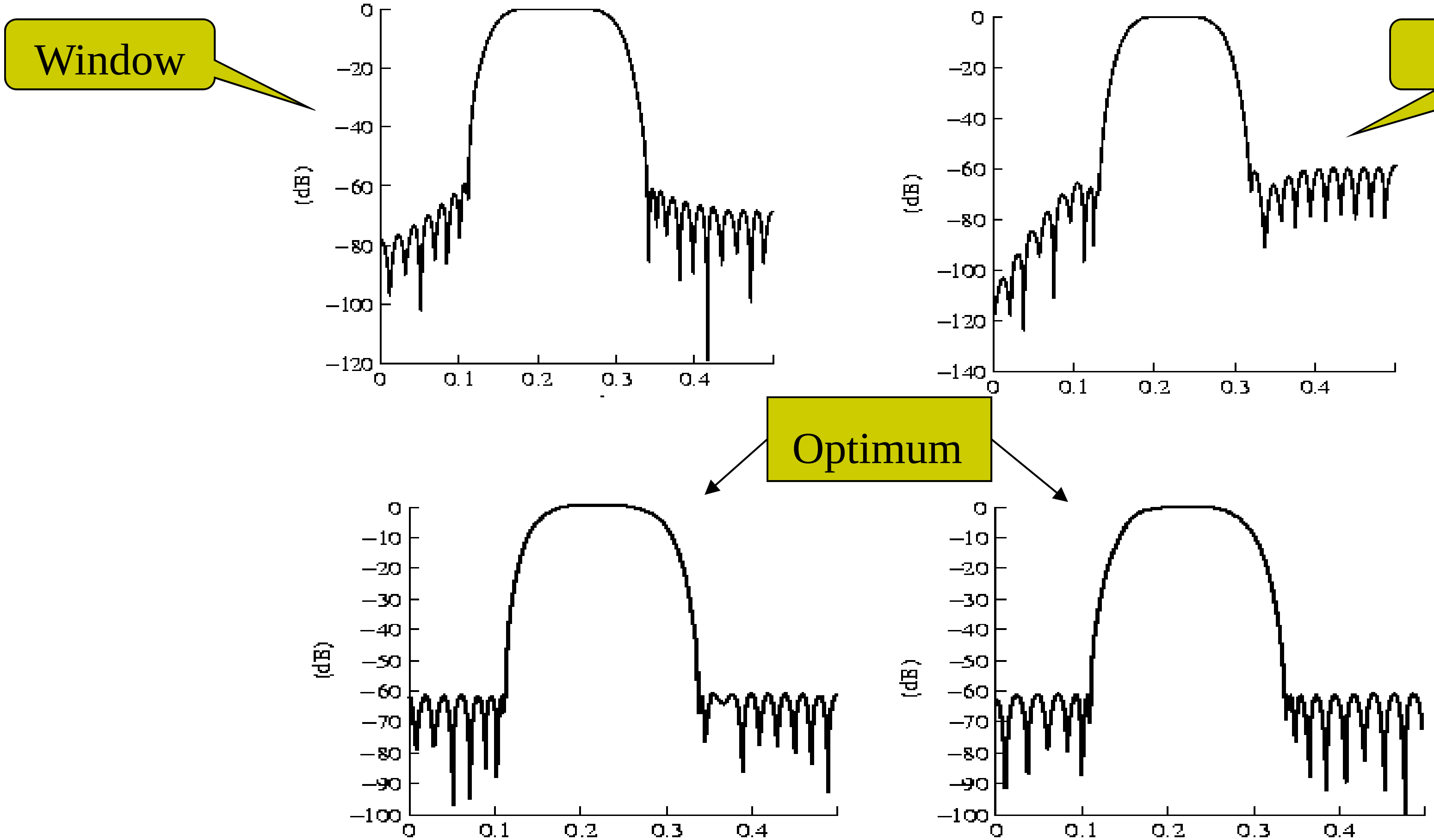
7.7.3 Frequency Sampling Filters with Simple Coefficients

- Using the simple integers (or power of 2) can enforce the FIR filter computational efficiency
- Obtaining perfect zeros and poles cancellation
- Restrictions
  - The locations of the poles of transfer functions are limited.
  - Equivalently, passband only be centered at restricted frequencies.

7.7.4 Summrry of the Frequency Sampling Algorithm

- Step 1: define specifications of desired filter.
- Step 2: select frequency sample type
  - Type I: sampling frequency position is at kF_s/N.
  - Type II: sampling frequency position is at $(k + ½)F_s/N$.
- Step 3: calculate required total number of frequency sample, N, and evaluate the number of frequency sample in transition band, M, and their magnitudes, T_i; $i = 1, 2, \ldots, M$.
- Step 4: evaluate coefficient values of the filter using appropriate formula.
- Step 5: verify the result. If it does not meet requirement, return to step 3 to reselect N and/or M or step 2 to reselect frequency sample type.

7.8 Comparison of Window, Optimum and Frequency Sampling Methods



Advantages and Disadvantages of FIR Design Methods

- Windowing method
  - Most simplify method, and simple understandably conceptual design.
  - Critical frequencies and/or ripples in frequency bands could not manipulated into the desired precision easily.
  - Equally ripple in each frequency band.
- Frequency sampling method
  - Technique may be selected as both recursive and non-recursive.
  - Applicable to both typical and general filter types.
  - Problem to manipulate band edge frequencies and passband ripple into the desired precision.
- Optimum method
  - All of parameters can be manipulated.
  - Coefficient calculation method is easy and efficient.
  - For the same value of M, the result in amplitude is the best.
  - For some filter, i.e. Hilbert transformer, differentiator, this technique is more suitable for in comparable to another method.

7.9 Special FIR filter Design Topics

7.9.1 Half-band FIR Filter

- Features
  - Equal ripples: $\delta_p=\delta_s=\delta$
  - The passband and stopband edge frequency $f_s=F_s/2-f_p$
  - Symmetrical frequency response about a quarter of the sampling frequency

$$H(F_s/4+f)=1-H(F_s/4-f)$$

  - In the unit impulse response, for N odd, every coefficient is zero except $h[(N-1)/2)$

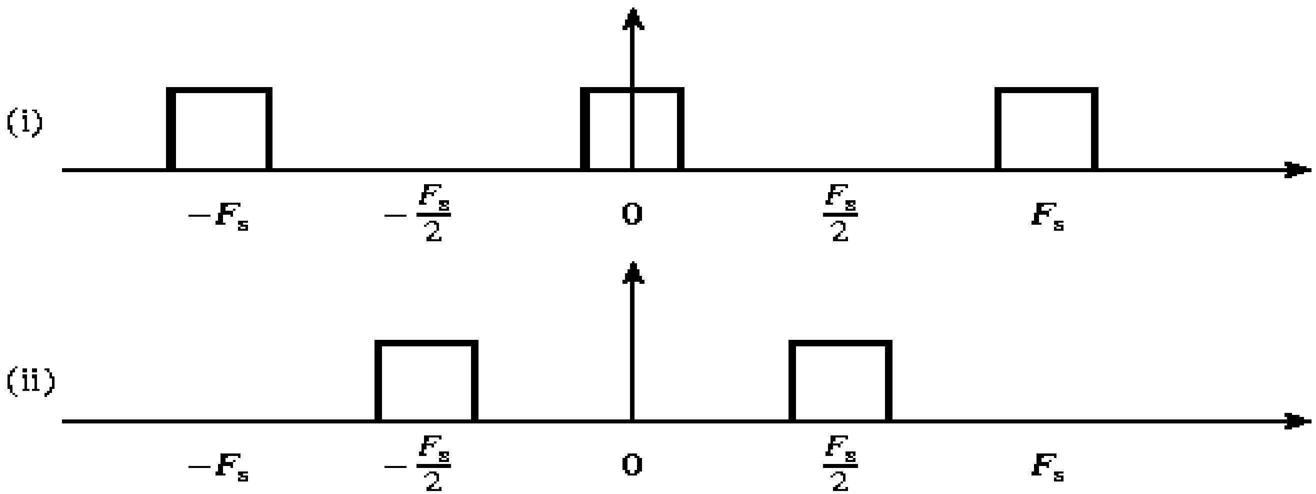
$$h(2n)=\begin{cases} 0, & n=0, 1, \ldots, (N-1)/4 \\ 0.5, & n=(N-1)/2 \end{cases}$$

Half-band FIR Filter Design Method

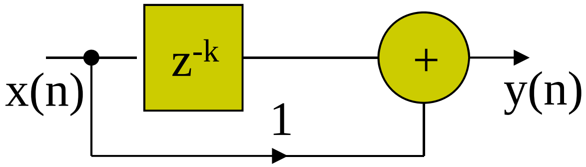
- Using the method as described earlier, such as the window and optimum methods
- With the constrains given as
  - Equal ripples: $\delta_p = \delta_s = \delta$
  - The passband and stopband edge frequency $f_s = F_s/2 - f_p$

7.9.2 Frequency Transformation

- A simple relationship exists for changing a filter from lowpass to an equivalent highpass filter
  - $h_{hp}(n)=(-1)^n h_{lp}(n)$
  - $H_{hp}(f)=H_{lp}(F_s/2-f)$

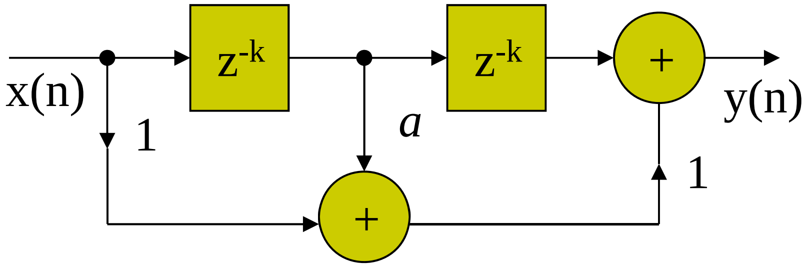


7.9.3 Computationally Efficient FIR Filters



Single adder section

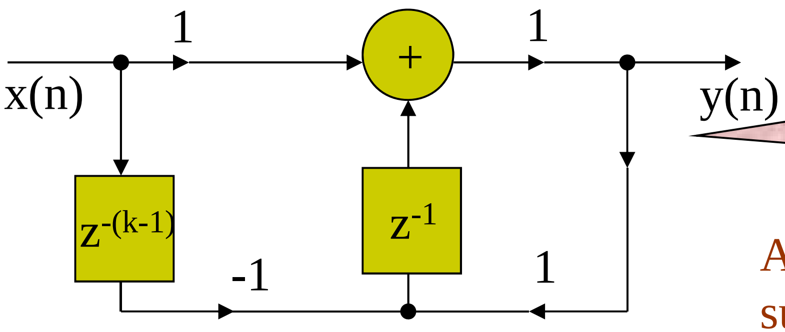
$H(z)=1+z^{-k}$



Two adder section

$H(z)=1+az^{-k}+z^{-2k}$ to avoid an explicit multiplication

$a=\pm 2^{n}$

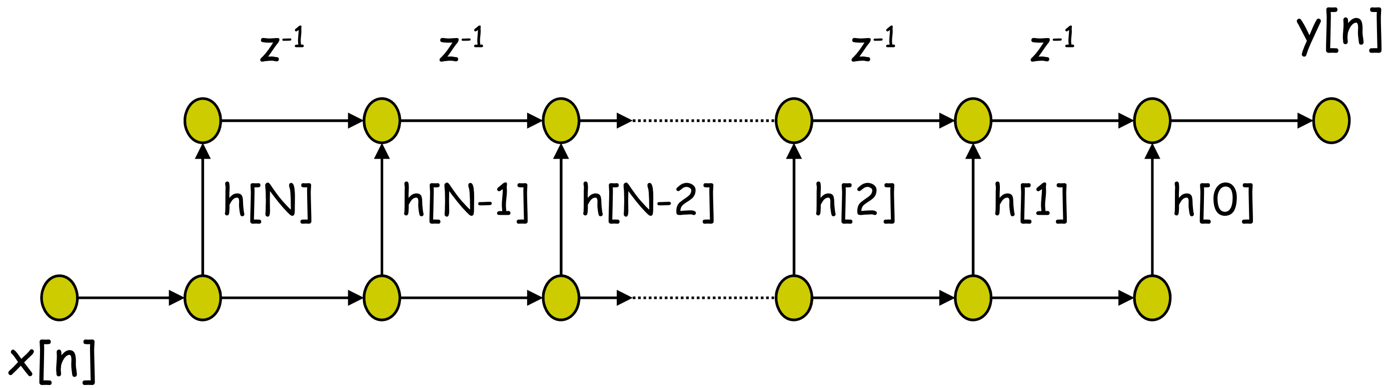


Recursive running sum lowpass section

$H(z)=(1-z^{-k})/(1-z^{-1})=1+z^{-1}+z^{-2}\ldots+z^{-(k-1)}$

A typical filter would consist of 4 to 7 such sections connected in cascade

7.10 Realization Structure for FIR Filters



Transposed Form of FIR

7.10.2 Linear-Phase Structures

- Structures for Linear Phase FIR Systems :
  - $h[N-n] = h[n]$ for $n = 0, 1, ..., N$
    - For N is an even integer : Type I

      $y[n] = S_{k=0}^{(N/2)-1} h[k](x[n-k] + x[n-N+k]) + h[N/2]x[n-N/2]$
    - For N is an odd integer : Type II

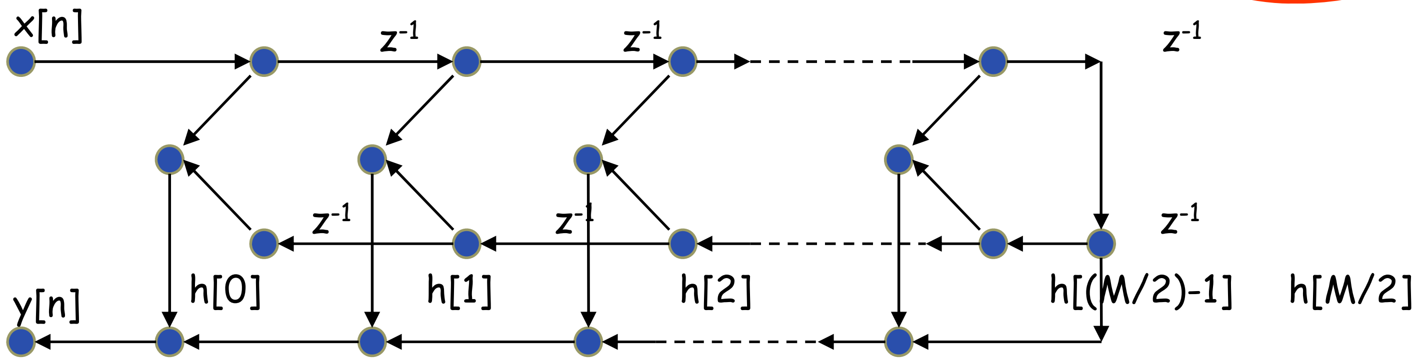
      $y[n] = S_{k=0}^{(N-1)/2} h[k](x[n-k] + x[n-N+k])$
  - $h[N-n] = -h[n]$ for $n = 0, 1, ..., N$
    - For N is an even integer : Type III

      $y[n] = S_{k=0}^{(N/2)-1} h[k](x[n-k] - x[n-N+k])$
    - For N is an odd integer : Type IV

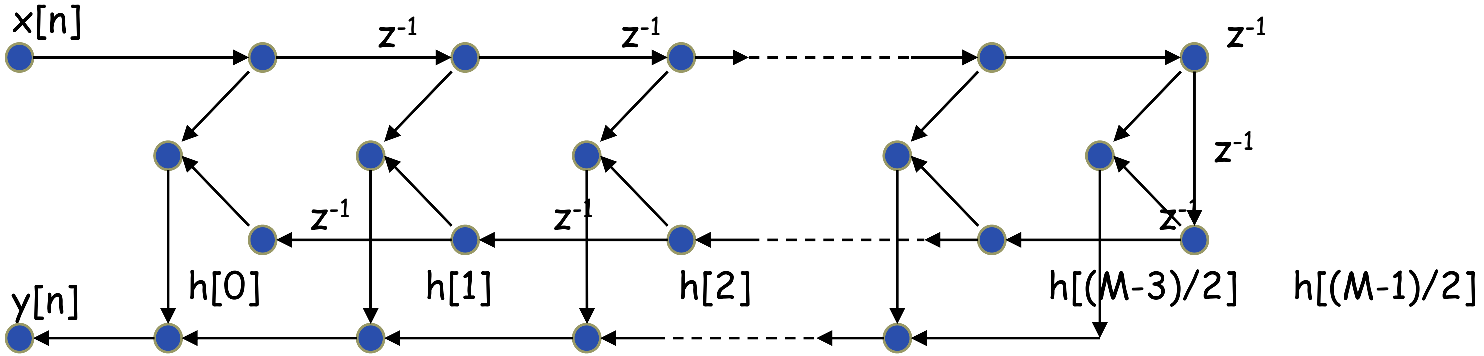
      $y[n] = S_{k=0}^{(N-1)/2} h[k](x[n-k] - x[n-N+k])$

Structures for Linear-Phase FIR Systems

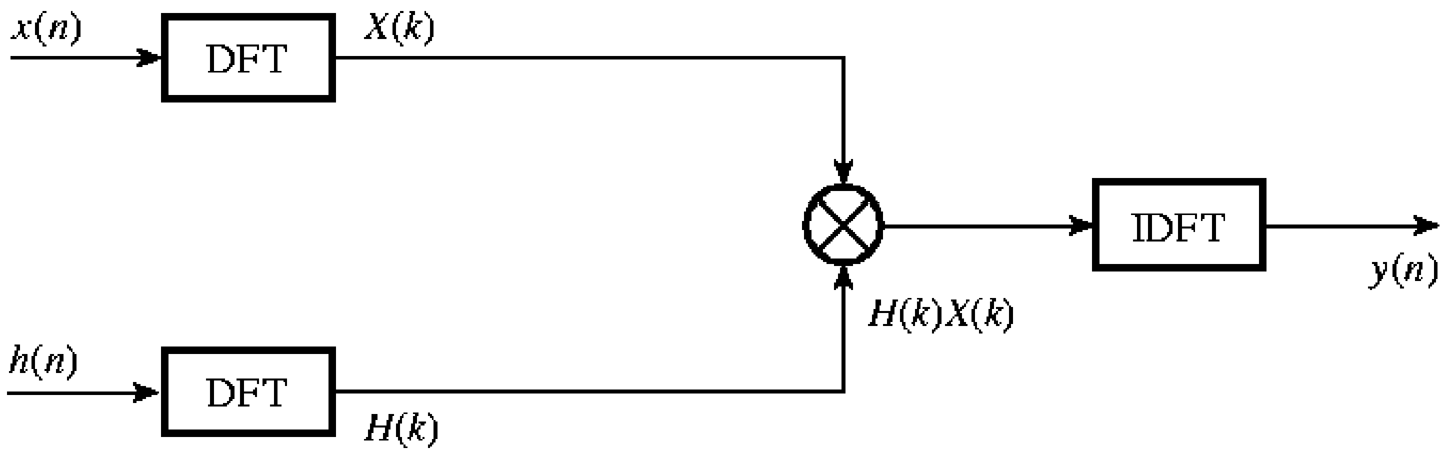
Direct form structure for an FIR linear-phase when M is even.



Direct form structure for an FIR linear-phase when M is odd.

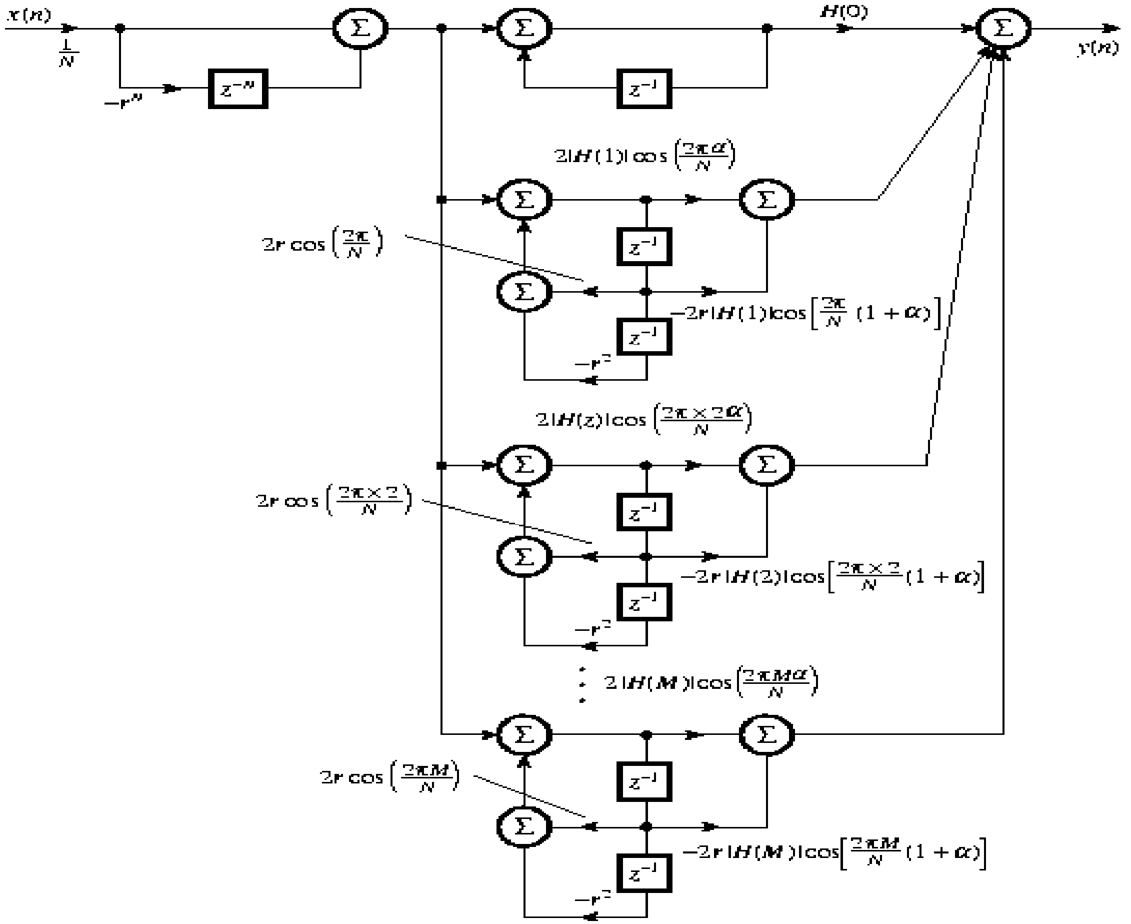


7.10.3 Other Structures

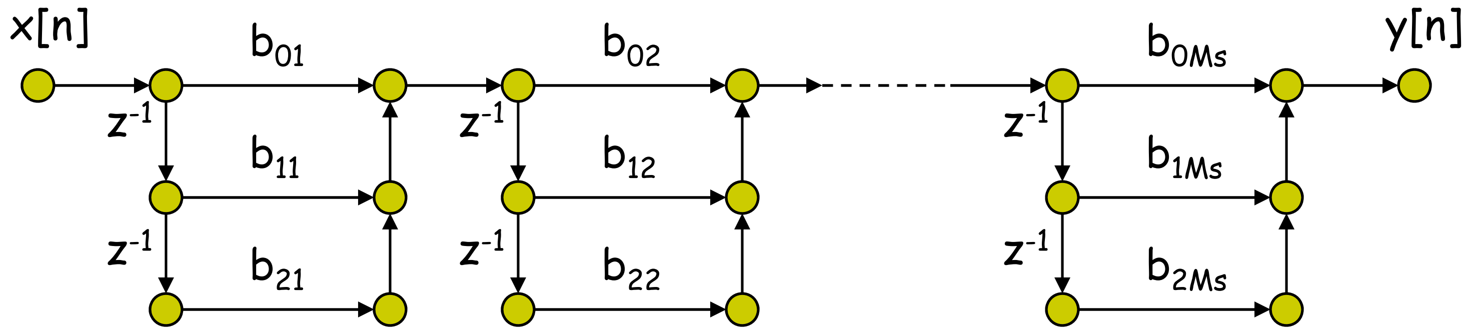


Fast convolution

7.10.3.2 Frequency Sampling Structure



7. 10.3.3 Cascade and Cascade Structure

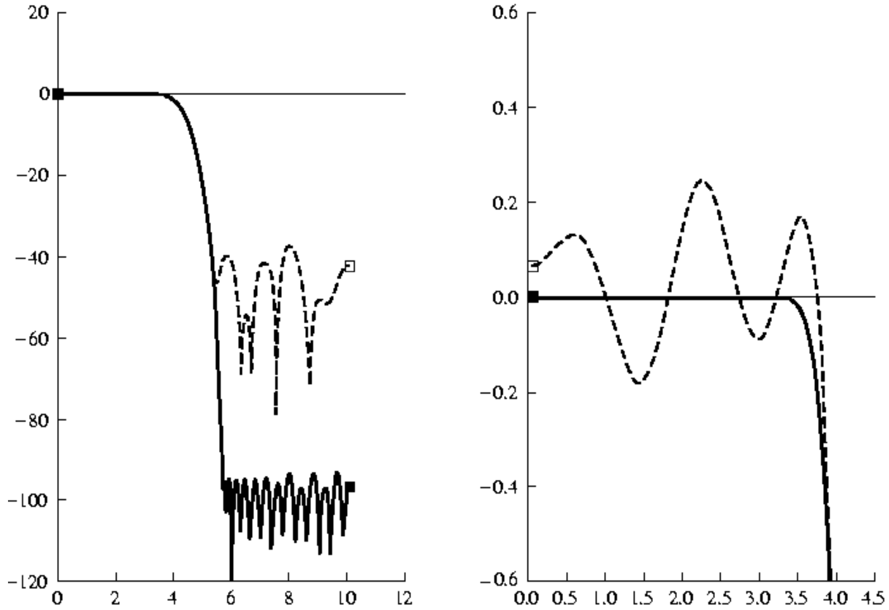


7.10.4 Choosing between Structure

- In general, using tranversal structure
- Unless the specification requirements are satisfied
- Note that other structure features, e.g. the frequency sample structure is suitable for narrowband frequency selective filters
- The fast convolution structure offers significant computational advantages over others, but required the availability of the FFT

7.11 Finite Wordlength Effects in FIR Digital Filters

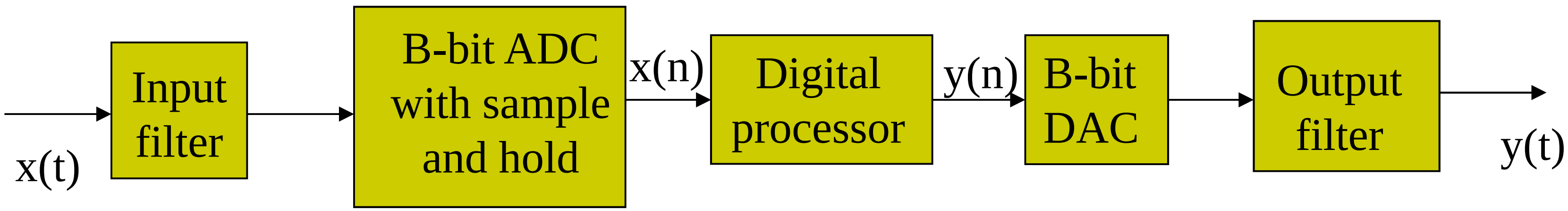
- ADC noise
- Coefficient quantization errors
- Roundoff errors from quantization results of arithmetic operations
- Arithmetic overflow



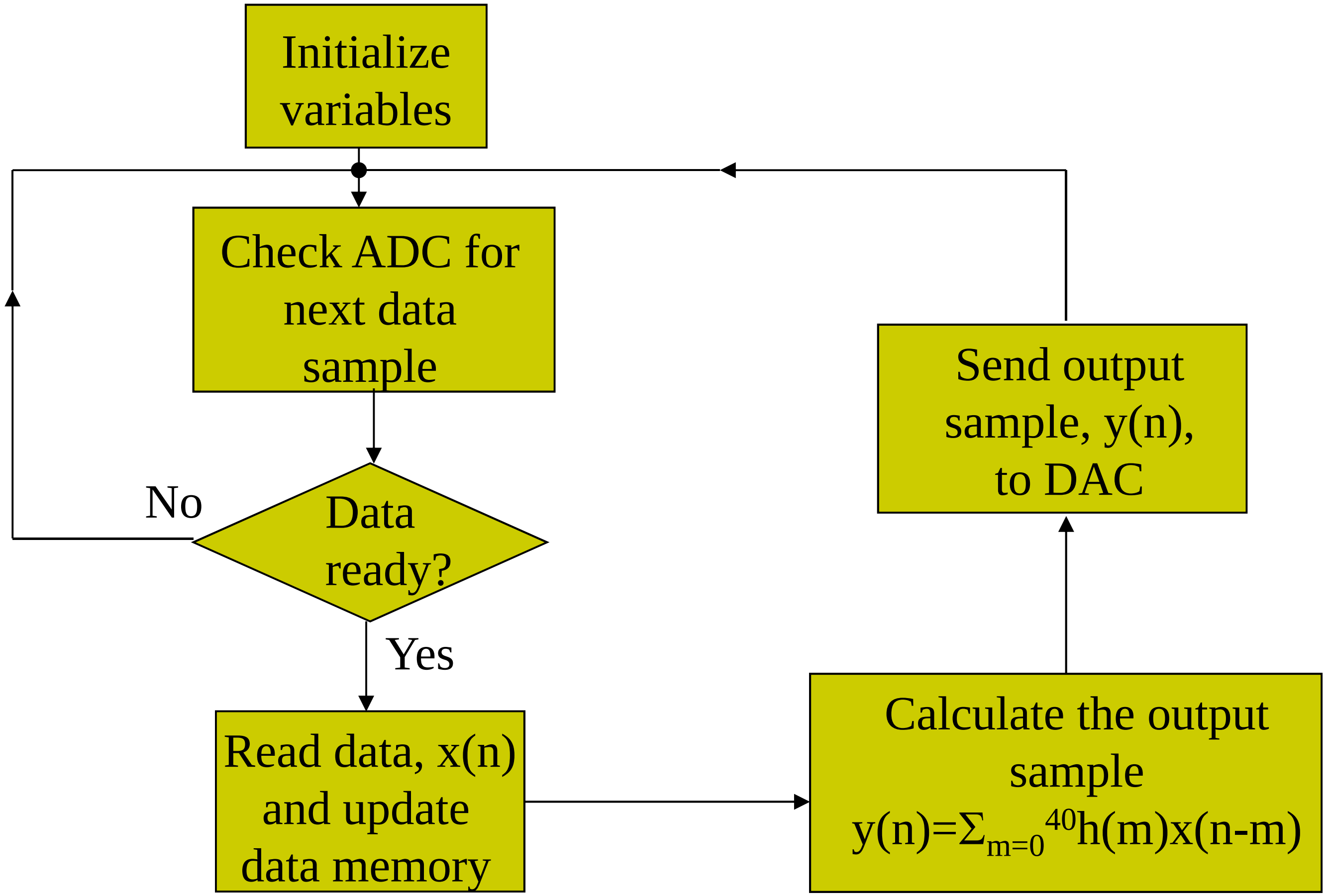
An example for 8 bit roundoff

7.12 FIR Implementation Techniques

- Memory (RAM) to store the present and past input samples, x(n) and x(n-k)
- Memory (RAM or ROM) for storing the filter coefficients, h(k)
- A multiplier (software or hardware)
- Adders or arithmetic logic unit (ALU)



A Simplified Flowchart for a Real-time, Transversal, FIR Filter



7.13 Design Examples

- Using Matlab tools
  - Type FDAtool in Matlab workspace

7.14 Summary

- The five design stages of a digital filter
  - Filter specifications
  - Coefficient calculation
  - Realization
  - Analysis of errors
  - Implementation
- FIR filter design method
  - Window
  - Frequency sampling
  - Optimal methods
- FIR filter structures
  - Transversal
  - Frequency sampling structure
  - Fast convolution