Citadel Statistics Course

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Homework 2

1. The dataset country portfolios contains monthly closing prices for 10 broad based country portfolios. Construct the simple return for each series and report the variance covariance matrix and the correlation matrix.
   a. Which matrix contains more interpretable information regarding the degree of co-movement in each pair of series?
   b. Are certain ones larger than others in a way you might expect?

2. Consider the following joint probability distribution for a trade direction indicator variable. Let X_i denote whether i^{th} trade is a buy or a sell. X_i=1 denotes a buy and X_i=0 denotes a sell. Let X_{i+1} denote a buy sell indicator for the i+1 subsequent trade. Let's suppose that the trade process is Markovian so that $\Pr\left(X_{i+1} \mid X_i, X_{i-1}, \ldots X_0\right) = \Pr\left(X_{i+1} \mid X_i\right)$. Consider the following joint model for X_i and X_{i+1}:

		X_i = 0	X_i = 1
X_{i+1}	0	.3	.2
	1	.2	.3

   a. Are consecutive trade indicators dependent or independent?
   b. Look at the table and describe the relationship between trades.
   c. Explain why X_i and X_{i+1} are *identically* distributed and state the distribution.
   d. What is the probability of a buy?
   e. Given that trade i is a buy, what is the probability that trade i+1 is also a buy?
   f. Given that trade i is a buy, what is the probability that trade i+1 is a sell?
   g. What is the conditional distribution of trade i+1 given trade i is a buy?
   h. What is the conditional distribution of trade i+1 given that trade i is a sell?
   i. Compare the two conditional distributions you calculated to the unconditional distribution. Explain the nature of dependence in plain English.
   j. Consider the sequence of trades 1, 0, 0, 1, 1, 1, 0. Find the probability of this outcome.
   k. Suppose that the current trade is a buy and that you want to know the probability that the next 7 trades are 1, 0, 0, 1, 1, 1, 0, conditional on the current trade being a buy. How does your answer change?

3. Consider the random walk model $y_t = y_{t-1} + x_t$ where x_t is iid with the following distribution:

x	Pr(x)
2	.1
1	.25
0	.30
-1	.25
-2	.1

a. Is y_t an iid process?
b. Given that y_{t-1}=100 find the distribution of y_t.
c. Given that y_{t-1}=100 and y_{t-2}=99, what is the distribution of y_t?
d. Suppose that you are holding a call option at 101 and a put option at 100 (this is called a straddle). This position will deliver a positive payoff when the price rises above 101 ***or*** falls below 100. What is the chance that this straddle position delivers a positive payoff in period *t* if the price in period *t*-1 is 100?
e. Given y_{t-1}=100 find the distribution of y_{t+1}.
f. What is the chance that the straddle position delivers a positive payoff in period t+1 if y_{t-1}=100?

4. Consider the random walk model $y_t = y_{t-1} + x_t$ where x_t is iid Normal with a mean of .01 and a standard deviation of .04. Find the distribution of y_t given y_{t-1}.

5. The dataset TSdata.xls contains two time series, X and Y. Each time series has 100 observations. Your job is to create a good model for each series and give the conditional distribution of the next (101^{st}) value.

6. Consider the two returns X and Y.

		X		
		0	0.05	0.1
Y	0	0.1	0.15	0.05
	0.05	0.15	0.2	0.1
	0.1	0.05	0.1	0.1

a. What are the marginal distributions for X and Y?
b. Are X and Y independent?
c. Are X and Y identically distributed?
d. Find the mean and variance of X and Y.
e. Find the covariance between X and Y.
f. Find the correlation between X and Y.

g. Find the conditional distribution of Y given X=.1.
h. Find the conditional expectation of Y given X=.1 by using the conditional distribution in part g to calculate the expectation of Y. The interpretation here is the expected value of Y when X=.1.
i. How does the conditional expectation of part h. compare to the unconditional expectation for Y calculated in part d. (is it larger or smaller)? Does this make sense? Explain.

7. We have m stocks with each with a variance σ^2 and the correlation between any two stocks is ρ (the variance and the correlations are the same across all stocks). Use the formula for the variance of linear combinations of random variables to find the variance (or standard deviation) of the equally weighted portfolio that invests 1/m fraction of the wealth into each of m assets. Your answer will be expressed as a function of the number of stocks included (m), the correlation (ρ), and the variance (σ^2).

   a. When m gets really big, what is the value does the variance of the portfolio converges to? i.e. what is the smallest variance of the portfolio?
   b. When m gets really big and ρ=1, what is the variance of the portfolio in terms of σ^2?
   c. When m gets really big and ρ=0, what is the variance of the portfolio?

8. We model annual S&P500 returns as a N(.11, $.15^2$) and annual gold returns as N(.06, $.08^2$). The covariance between the S&P500 and gold is .0024. We also assume that S&P500 and gold returns are jointly normal which means that *any linear combination of S&p500 and gold are also Normal*.

   d. Find the distribution of a portfolio (P_1) that invests .3 in gold and .7 in the S&P500.
   e. Find the distribution of a portfolio (P_2) that invests .5 in gold and .5 in the S&P500.
   f. Suppose that the investor's tradeoff between mean and variance is summarized by: $U(P) = E(P) - 5Var(P)$. If the investor is interested in maximizing this utility function, which of the above portfolios would the investor prefer to hold?

9. Consider a model where the annual return on the S&P500 is positive 60% of the time and negative 40% of the time and whether the S&P500 goes up or down is independent from one year to the next (iid). Hence if Y is the number of times that the market goes up in the next 10 years then Y is a Binomial(10,.6).

   a. A contract pays off $10 for each year that the S&P500 return is positive in each of the next 10 years. The payoff is then given by $P = \$10Y$. What is the expected payoff on this contract?
   b. What's the variance of the payoff?

10. Suppose that monthly profits for a startup firm are iid with a mean of $10,000 and a standard deviation of $40,000. The firm will lose financial support from venture capitalists if the cumulative profit after 20 months is negative.

    a. Letting X_i denote the profits of the ith month. Express the total cumulative profits after 20 months as a linear combination of the X_i's. Let's call the cumulative profits Y.
    b. Thinking of 20 as a "large number" what is the distribution of Y?
    c. What, approximately, is the probability that the firm will lose financial support at the end of 20 months?

11. Consider the random walk model $y_t = y_{t-1} + x_t$ where x_t is iid with the following distribution:

x	Pr(x)
2	.1
1	.25
0	.30
-1	.25
-2	.1

    a. What is the mean and variance of x?
    b. Let $w_t = y_t - y_0$ denote the total price change from time period zero to time period t. Find the mean and variance of w_t. Hint: express w_t in terms of the x's.
    c. Find an approximate distribution of w_t.