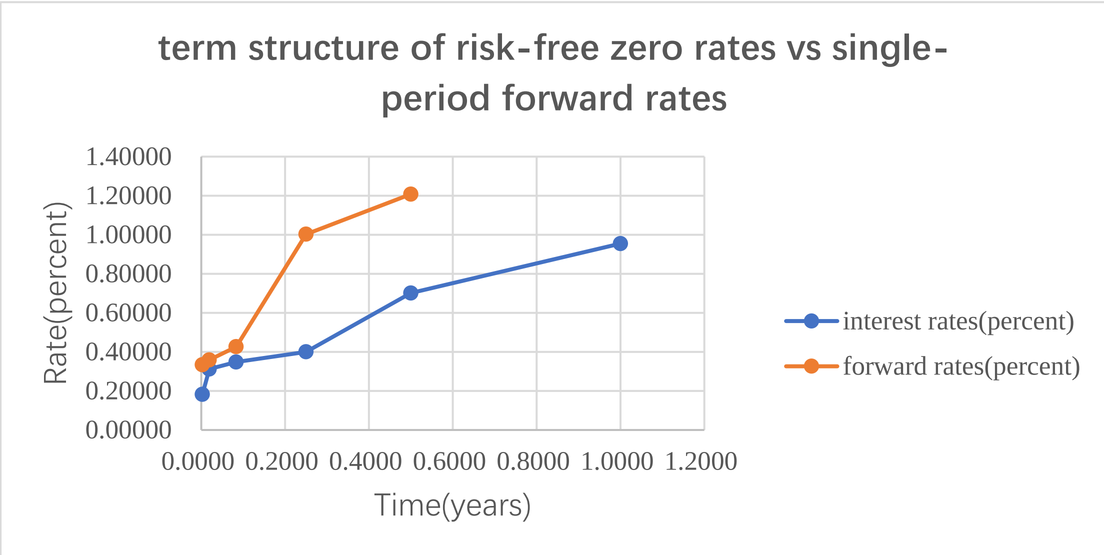
Exercise

1. (a) rate formula: $r = -\frac{1}{T}\ln P_{\mathrm{T}}$

(b) forward rate formula: $r_{t,T} = -\frac{1}{T-t}\ln(\frac{P_T}{P_t})$

maturity(years)	maturity	bond price	interest rates(percent)	forward rates(percent)
0.0027	1D	99.9995	0.18250	0.33353
0.0192	1W	99.9940	0.31201	0.35886
0.0833	1M	99.9710	0.34805	0.42627
0.2500	3M	99.9000	0.40020	1.00226
0.5000	6M	99.6500	0.70123	1.20785
1.0000	1Y	99.0500	0.95454	



c) time-weighted average of forward rates:
0.95454128

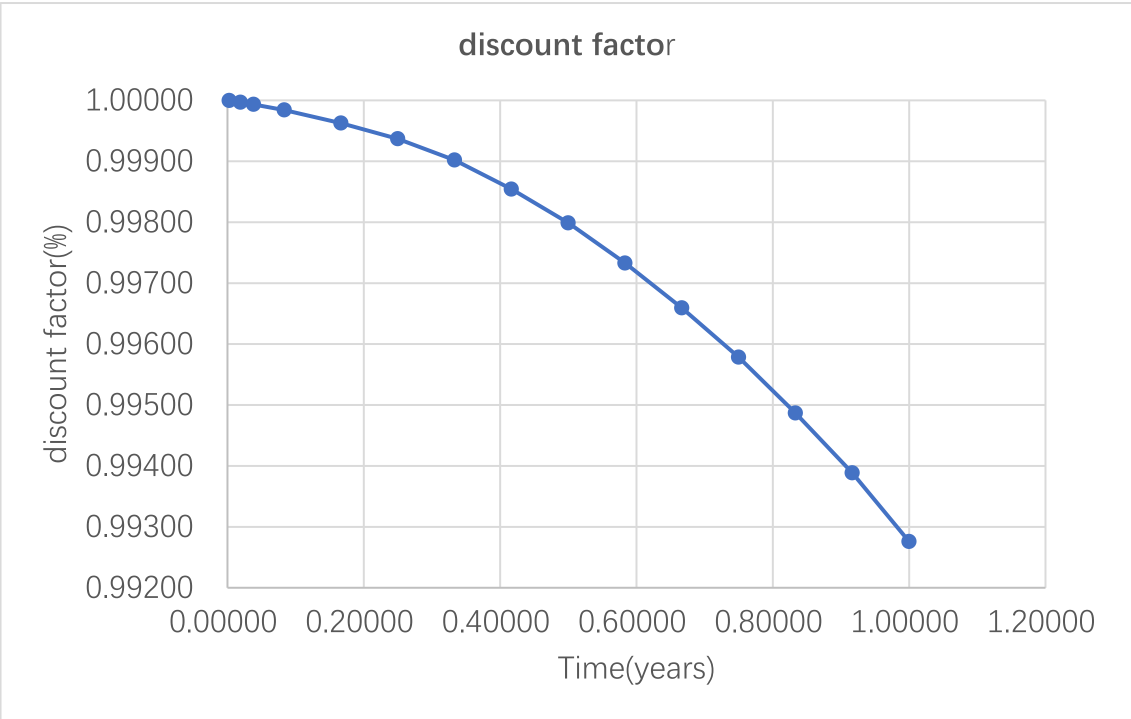
Time-weighted average of forward rates = $\sum \frac{\Delta t}{T} FDT = 0.95454128$, the same as the 1Y zero rate.

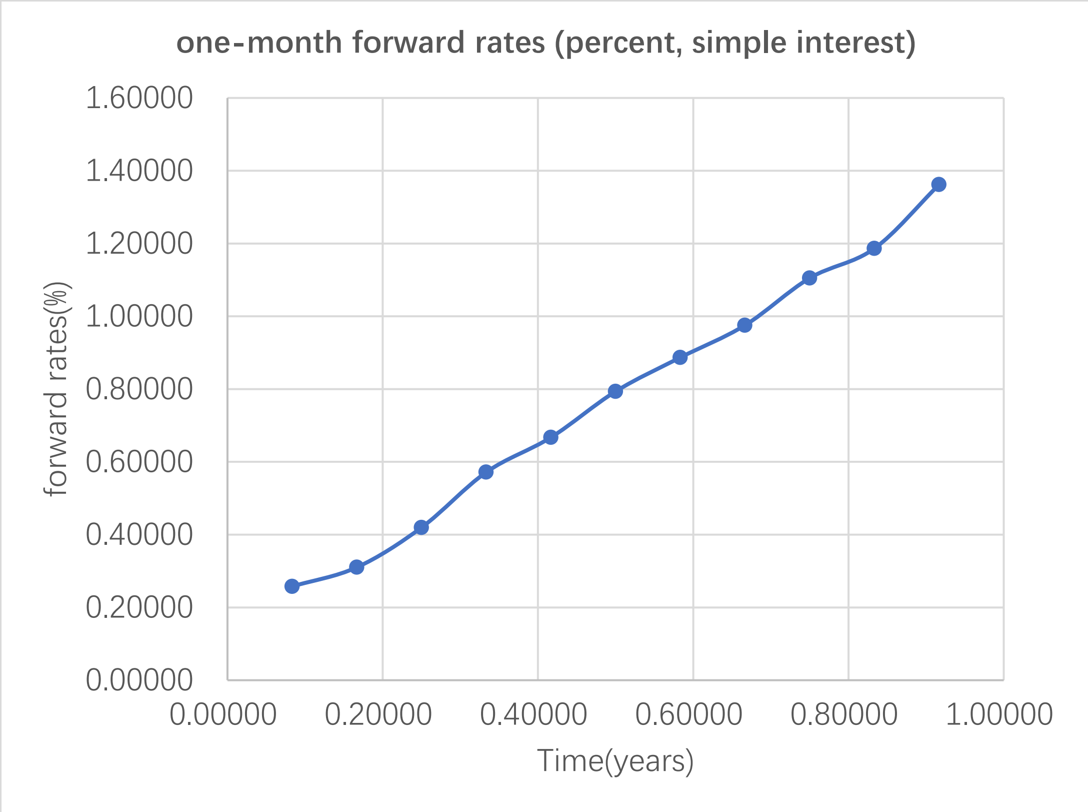
2. (a) simple rates-discount factors formula: $P_T = (1 + r_s T)^{-1}$

(b) simple rates-forward rates formula: $r_{1,2} = \frac{1}{t_2 - t_1}\left(\frac{1 + r_2 t_2}{1 + r_1 t_1} - 1\right)$

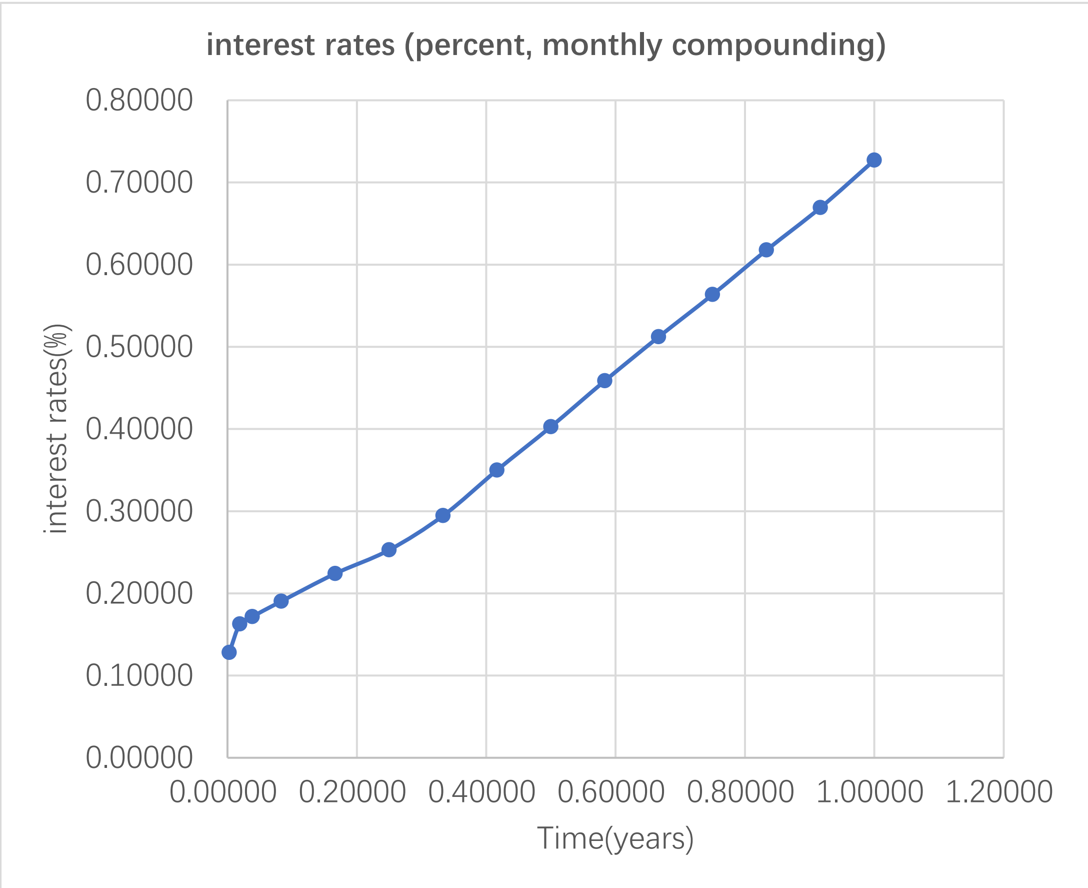
(c) monthly-compounding: $r_m = m(P_T^{-\frac{1}{mT}} - 1)$

Time	Rate(percent)	discount factor	one-month forward rates (percent, simple interest)	interest rates (percent, monthly compounding)
o/n	0.12800	1.00000		0.12801
1w	0.16270	0.99997		0.16271
2w	0.17170	0.99993		0.17171
1m	0.19043	0.99984	0.25779	0.19043
2m	0.22413	0.99963	0.31026	0.22411
3m	0.25288	0.99937	0.41961	0.25283
4m	0.29463	0.99902	0.57157	0.29452
5m	0.35013	0.99854	0.66716	0.34993
6m	0.40313	0.99799	0.79353	0.40279
7m	0.45913	0.99733	0.88676	0.45860
8m	0.51288	0.99659	0.97530	0.51211
9m	0.56463	0.99578	1.10495	0.56357
10m	0.61913	0.99487	1.18633	0.61770
11m	0.67125	0.99388	1.36187	0.66938
12m	0.72950	0.99276		0.72707





one-month forward rates (percent, simple interest)
1.60000
1.40000
1.20000
1.00000
0.80000
0.60000
0.40000
0.20000
0.00000
forward rates(%)
0.00000
0.20000
0.40000
0.60000
0.80000
1.00000
Time(years)



interest rates (percent, monthly compounding)
0.80000
0.70000
0.60000
0.50000
0.40000
0.30000
0.20000
0.10000
0.00000
interest rates(%)
0.00000
0.20000
0.40000
0.60000
0.80000
1.00000
1.20000
Time(years)

3

3. a)

t	discount factor	r_t
0.2500	0.99875	$-\ln P_t / t = 0.00500$
0.5000	0.99576	$-\ln P_t / t = 0.008498$

for $t = \frac{1}{3}$, $i+1 = 0.5$, $i = 0.25$

$$r_t = \frac{r_{i+1} - r_i}{t_{i+1} - t_i}(t - t_i) + r_i = 0.006168$$

$$P_t = e^{-r_t t} = 0.997946$$

b) $$\ln P_t = \frac{\ln P_{t_{i+1}} - \ln P_{t_i}}{t_{i+1} - t_i}(t - t_i) + \ln P_{t_i}$$

$$= \cancel{0.997752} \ 0.990627$$

(c) under a) $i+1 = 1$ $i = 0.5$

$$r_{t=\frac{10}{12}} = \frac{r_{i+1} - r_i}{t_{i+1} - t_i}(t - t_i) + r_i = 0.010834, \quad r_{t=\frac{1}{3}} = 0.006168$$

$$r_{f\,4-6} = \frac{r_{t=\frac{10}{12}} \times \frac{10}{12} - r_{\frac{1}{3}} t_{\frac{1}{3}}}{\frac{10}{12} - \frac{1}{3}} = 0.013944$$

under b). $$r_{t=\frac{10}{12}} = \frac{-\ln P_t}{t} = 0.01130$$

$$\ln P_{t=\frac{1}{3}} = \frac{\ln P_{0.5} - \ln P_{0.25}}{t_{0.5} - t_{0.25}}(t - t_{0.25}) + \ln P_{t_{0.25}} = \cancel{0.006754} \ -0.00225$$

$$r_{t=\frac{1}{3}} = 0.006751$$

$$r_{f\,4-6} = \frac{r_{t=\frac{10}{12}} \times \frac{10}{12} - r_{\frac{1}{3}} t_{\frac{1}{3}}}{\frac{10}{12} - \frac{1}{3}} = 0.01433$$

4.

(a) Given $P_1 = e^{r_1 t_1}, P_3 = e^{r_3 t_3}$, we have

$$r_1^* = r_3^* = \frac{\log P_1 - \log P_3}{t_1 - t_3} = 1.775\%$$

(b) Since $r_t = at + b = 0.175\%t + 1.075\%$,

$$r_t^* = \frac{dr_t \cdot t}{dt} = 2at + b = 0.35\%t + 1.075\%$$

plugging in the values of t_1 and t_3, we have

$$r_1^* = 1.425\%, r_3^* = 2.215\%$$

(c) According to the assumption,

$$r_t^* = \frac{dr_t \cdot t}{dt} = -4.555556\% \times 10^{-3}t^3 + 4.916667 \times 10^{-3}t^2 + 0.02t + 0.002$$

plugging the numbers, we get

$$r_1^* = 2.24\%,\ r_3^* = -1.67\%$$

5,6,7

5. a) $V = \frac{3.5}{e^{r_1 t_1}} + \frac{3.5}{e^{r_2 t_2}} + \frac{3.5+100}{e^{r_3 t_3}} = 103.1072$

b) $r_2 = 0.22$

Linear-interpolation:

$r_{1.5} = (0.015 + 0.022)/2 = 0.0185$

$\frac{C}{e^{r_{0.5} t_{0.5}}} + \frac{C}{e^{r_1 t_1}} + \frac{C}{e^{r_{1.5} t_{1.5}}} + \frac{C+1}{e^{r_2 t_2}} = 1 \Rightarrow C = 0.011011$

par-yield $= 2 \times C = 0.022022$

c) $V = \frac{2.5\%}{e^{r_{0.5} t_{0.5}}} + \frac{102.5\%}{e^{r_1 t_1}} = \frac{2.5\%}{e^{y t_{0.5}}} + \frac{102.5\%}{e^{y t_1}}$

$y = 0.01495$

6. a) $V = 99.875$ face $= 100$

$r_{0.5} = \frac{\ln\left(\frac{face}{V}\right)}{t} = 0.002502$

b) $\frac{2}{e^{r_{0.5} t_{0.5}}} + \frac{102}{e^{r_1 t_1}} = \frac{2}{(1+y)^{0.5}} + \frac{102}{(1+y)^1} \Rightarrow r_1 = 0.005012$

(c) $\frac{1}{e^{r_{0.5} t_{0.5}}} + \frac{1}{e^{r_1 t_1}} + \frac{1}{e^{r_{1.5} t_{1.5}}} + \frac{101}{e^{r_2 t_2}} = \frac{1}{(1+y)^{0.5}} + \frac{1}{(1+y)^1} + \frac{1}{(1+y)^{1.5}} + \frac{101}{(1+y)^2}$

$r_{0.5} = 0.002502$ $r_1 = 0.005012$ $r_{1.5} = \frac{r_1 + r_2}{2}$ 并 $y = 0.0105$

$\Rightarrow r_2 = 0.010513$

$t_{0.5} = t_1 = \cdots = t_5 = y$

7. a) $V = \frac{2}{e^{t_{0.5} r_{0.5}}} + \cdots + \frac{2}{e^{t_{4.5} r_{4.5}}} + \frac{102}{e^{t_5 r_5}} = 106.9329$

b) Duration $= \frac{2}{e^{t_{0.5} y}} \times \frac{t_{0.5}}{V} + \cdots + \frac{102 \times t_5}{e^{t_5 y} V} = 4.597375$, $C = \frac{1}{V}\frac{d^2 V}{dy^2} = 22.2643$

C) $D_m = \frac{D}{(1+\frac{y_m}{2})} = \frac{4.597875}{1+(\frac{0.025}{2})} = 4.5411$

D) $C = \frac{1}{V}\frac{d^2V}{dy^2} \quad e^{-yT} = (1+\frac{y_m}{m})^{-mT} \Rightarrow \frac{dy}{dy_m} = (1+\frac{y_m}{m})^{-1}$

$$\frac{d^2V}{dy_m} = \frac{d(\frac{dV}{dy}\frac{dy}{dy_m})\frac{dy}{dy_m}}{dy} = \left(\frac{d^2V}{dy^2}\frac{dy}{dy_m} + \frac{dV}{dy}\frac{\partial y}{\partial y\,\partial y_m}\right)\frac{dy}{dy_m}$$

$$= \frac{d^2V}{dy^2}\left(\frac{dy}{dy_m}\right)^2 = \frac{1}{V}\frac{d^2V}{d^2y}\left(\frac{dy}{dy_m}\right)^2 = C\left(1+\frac{y_m}{m}\right)^{-2}$$

$$C_m = \frac{22.2643}{(1+\frac{0.025}{2})^2} = 21.71799$$

8.

By assuming piecewise constant forward rates, we can first solve for the 9-Month discount rate from the following equations,

$$\frac{r_3}{4}\times\frac{3}{4} = \frac{r_1}{2}\times\frac{1}{2} + c\left(\frac{3}{4}-\frac{1}{2}\right)$$

$$r_1 = \frac{r_1}{2}\times\frac{1}{2} + c\left(1-\frac{1}{2}\right)$$

Solve these equations, we get $r_{\frac{3}{4}} = \frac{4}{3}(\frac{1}{4}\times r_{\frac{1}{2}} + \frac{1}{2}r_1)$, then according to the relationship between zero rate and discount factor, we get $P_{\frac{3}{4}} = e^{-\frac{3}{4}r_{\frac{3}{4}}} = e^{-(\frac{1}{4}\times r_{\frac{1}{2}}+\frac{1}{2}r_1)}$ = 0.98721.

Then, we move on to get the discount factors at other maturities, assume the constant forward rate between 1Y and 2Y is r, and the coupon rate is c, the price of the 2Y bond is V_2, then

$$V_2 = \sum_{n=1}^{8} c\times\frac{100}{4}disc\left(\frac{n}{4}\right) + 100\times disc\left(\frac{8}{4}\right)$$

where disc(n/4) is the discount factor at n/4 years, for n = 5, 6, 7, 8,

$$\text{disc}\left(\frac{n}{4}\right) = e^{-r1+r\left(\frac{n}{4}-1\right)} = e^{-r_1}\times e^{-\frac{r(n-4)}{4}}$$

where r_1 is the spot zero-rate, from the above equation, we can see that $e^{-\frac{2r}{4}} = (e^{-\frac{r}{4}})^2$, $e^{-\frac{3r}{4}} = (e^{-\frac{r}{4}})^3$, $e^{-\frac{4r}{4}} = (e^{-\frac{r}{4}})^4$, then denote $e^{-\frac{r}{4}}$ as x, we have

$$V_2 = \sum_{n=1}^{4} c \times \frac{100}{4} disc\left(\frac{n}{4}\right) + e^{-r_1}x + e^{-r_1}x^2 + e^{-r_1}x^3 + (e^{-r_1} + 100)x^4$$

which is a forth order polynomial, solve the equation, we get x = 0.976896.

Then, following similar procedure, we can solve for the rest discount factors at the rest maturities, the corresponding equations are list below (Note: In different equations, $e^{-\frac{r}{4}}$ represents different discount values)

$$V_5 = \sum_{n=1}^{8} c \times \frac{100}{4} disc\left(\frac{n}{4}\right) + \sum_{n=1}^{12} (e^{-\frac{r}{4}})^n \times e^{-2r_2} + 100\ (e^{-\frac{r}{4}})^{12} \times e^{-2r_2}$$

$$V_{10} = \sum_{n=1}^{20} c \times \frac{100}{4} disc\left(\frac{n}{4}\right) + \sum_{n=1}^{20} (e^{-\frac{r}{4}})^n \times e^{-5r_5} + 100\ (e^{-\frac{r}{4}})^{20} \times e^{-5r_5}$$

$$V_{30} = \sum_{n=1}^{40} c \times \frac{100}{4} disc\left(\frac{n}{4}\right) + \sum_{n=1}^{80} (e^{-\frac{r}{4}})^n \times e^{-10r_{10}} + 100\,(e^{-\frac{r}{4}})^{80} \times e^{-10r_{10}}$$

Solve all the roots through a forward loop, then we can get the discount factors at 2Y, 5Y, 10Y and 30Y, are

Disc(2Y) = 0.965605,
Disc(5Y) = 0.902578,
Disc(10Y) = 0.790571,
Disc(30Y) = 0.392193,

and all the quarterly discount factors are shown in the following table.

Quarterly Discount Factors in 30 years

	1 Q	**2 Q**	**3 Q**	**4 Q**
0 Y	0.997254	0.993769	0.987208	0.980689
1 Y	0.976896	0.973118	0.969354	**0.965605**
2 Y	0.960189	0.954803	0.949448	0.944122
3 Y	0.938826	0.93356	0.928324	0.923116
4 Y	0.917938	0.912789	0.907669	**0.902578**
5 Y	0.896618	0.890698	0.884816	0.878974
6 Y	0.87317	0.867404	0.861677	0.855987
7 Y	0.850335	0.84472	0.839142	0.833601
8 Y	0.828097	0.822629	0.817197	0.811801
9 Y	0.806441	0.801116	0.795826	**0.790571**
10 Y	0.783674	0.776837	0.770059	0.763341
11 Y	0.756682	0.75008	0.743536	0.73705
12 Y	0.730619	0.724245	0.717927	0.711664

13 Y	0.705455	0.6993	0.693199	0.687152
14 Y	0.681157	0.675214	0.669324	0.663484
15 Y	0.657696	0.651958	0.64627	0.640632
16 Y	0.635043	0.629503	0.624011	0.618567
17 Y	0.61317	0.607821	0.602518	0.597262
18 Y	0.592051	0.586886	0.581766	0.57669
19 Y	0.571659	0.566672	0.561728	0.556827
20 Y	0.551969	0.547154	0.54238	0.537649
21 Y	0.532958	0.528308	0.523699	0.51913
22 Y	0.514601	0.510112	0.505662	0.50125
23 Y	0.496877	0.492542	0.488245	0.483986
24 Y	0.479763	0.475578	0.471429	0.467316
25 Y	0.463239	0.459197	0.455191	0.45122
26 Y	0.447284	0.443381	0.439513	0.435679
27 Y	0.431878	0.42811	0.424375	0.420673
28 Y	0.417003	0.413365	0.409758	0.406184
29 Y	0.40264	0.399127	0.395645	**0.392193**

Application

1. see xlsm document
2.

Application 2

(a) Solution:

① For the case of two deterministic positive cash flows at T_1 and T_2

$$D_i = T_i$$

$$C_i = T_i^2$$

$$D_\pi = \frac{V_1T_1 + V_2T_2}{V_1 + V_2}$$

$$C_\pi = \frac{V_1T_1^2 + V_2T_2^2}{V_1 + V_2}$$

$$\therefore \frac{D_\pi^2}{C_\pi} = \frac{V_1^2T_1^2 + V_2^2T_2^2 + 2V_1V_2T_1T_2}{V_1^2T_1^2 + V_2^2T_2^2 + V_1V_2(T_1^2 + T_2^2)}$$

$\because (T_1 - T_2)^2 \geq 0$ ("=" holds iff $T_1 = T_2$)

$\therefore T_1^2 + T_2^2 \geq 2T_1T_2$

$$\therefore \frac{D_\pi^2}{C_\pi} \leq 1$$

$C_\pi \geq D_\pi^2$, equality holds iff $T_1 = T_2$

② Assume for n cash flows, $C_{\Pi_n} > D_{\Pi_n}^2$

Then for one more cash flown adding to Π_n

$$C_{\Pi_{n+1}} = \left(C_{\Pi_n} + \frac{T_{n+1}^2 V_{n+1}}{\sum_{i=1}^{n} V_i} \right) \cdot \frac{\sum_{i=1}^{n} V_i}{\sum_{i=1}^{n+1} V_i}$$

$$D_{\Pi_{n+1}} = \left(D_{\Pi_n} + \frac{V_{n+1}T_{n+1}}{\sum_{i=1}^{n} V_i} \right) \cdot \frac{\sum_{i=1}^{n} V_i}{\sum_{i=1}^{n+1} V_i}$$

$$C^2_{\Pi_{n+1}} - D_{\Pi_{n+1}} = \frac{\left(\sum_{i=1}^{n} V_i\right)^2 \cdot (C_{\Pi_n} - D_{\Pi_n}^2) + V_{n+1}\left[\sum_{i=1}^{n} V_i (T_i - T_{n+1})^2\right]}{\left(\sum_{i=1}^{n+1} V_i\right)^2} \geq 0 \quad (*)$$

"=" holds iff $T_1 = T_2 = \cdots = T_{n+1}$

$\therefore C_{\Pi_{n+1}} > D_{\Pi_{n+1}}^2$

From ① and ②, We can conclude that $C_\Pi > D_\Pi^2$

(b) The zero-coupon bond which has the same duration as the portfolio π, $D = D_\pi$

Then, its convexity $C = D_\pi^2$

$\because$ The portfolio has convexity C_π

$$C_\pi > D_\pi^2 = C$$

$\therefore$ The zero-coupon bonds offer the least convexity for the portfolio.

(c) No, it can't be extended.

If $V_{n+1} < 0$, for $(*)$, if V_{n+1} is small enough

$(*)$ can be negative

Then, the conclusion of (a) doesn't stand true.

3. see xlsm document