13. Change of Numeraire and Options on Interest Rates

*Forward prices and martingales*
Recall that, in the Black-Scholes view of asset prices, the risk-neutral increments in the spot follow the stochastic differential equation…

$$dS_t = (r-q)S_t dt + \sigma S_t dW_t$$

…with volatility, interest rate, and dividend rate all nonrandom quantities. All the results we have seen can be easily extended to the case of time-varying parameters, but for simplicity of notation we assume here that all three are constant.

We also recall that the arbitrage-free forward price to time $T$ of this asset is given at time $t < T$ by…

$$F_t = S_t e^{(r-q)(T-t)}$$

…and, since this is a function of our stochastic asset price, we can use Ito's Lemma to find the SDE governing its evolution…

$$dF_t = \frac{d}{dt}F_t dt + \frac{d}{dS_t}F_t dS_t + \frac{1}{2}\frac{d^2}{dS_t^2}F_t dS_t dS_t$$

…with…

$$\frac{d}{dt}F_t = -(r-q)S_t e^{(r-q)(T-t)}$$

$$\frac{d}{dS_t}F_t = e^{(r-q)(T-t)}$$

$$\frac{d^2}{dS_t^2}F_t = 0$$

…and so:

$$dF_t = -(r-q)S_t e^{(r-q)(T-t)}dt + (r-q)S_t e^{(r-q)(T-t)}dt + \sigma S_t e^{(r-q)(T-t)}dW_t$$

$$dF_t = \sigma F_t dW_t$$

That is, the forward price is a pure, driftless geometric Brownian motion in the risk-neutral measure. This means that the conditional expectation of $F_t$, given information up to any time $s < t$, is $F_s$.

Stochastic processes that obey this condition are called martingales, and it is this concept that underlies the risk-neutral pricing methods we have used all along, beginning with interest rates. We introduced the idea in connection with the more intuitive notion that, since the payoff of a forward can be statically hedged today, then the arbitrage-free forward price must be the amount owed at $T$ if the cost of the hedge were borrowed today. That is, the forward price of the asset and the "forward price," if you like, of the cash needed to construct its static hedge must be equal.

In an arbitrage-free market, all forward prices are martingales—that is, all have risk-neutral evolutions of the form above. This simpler formulation of asset price dynamics leads to some extremely useful results that permit us to avoid the sometimes difficult question of how the spot price behaves if we can directly determine the forward price of the asset to some time $T$ of interest.

*Options on futures and Black's formula*

Earlier, we made the approximation that futures prices can often be treated in expectation as identical to forward prices. If we accept this approximation, then the above naturally leads to an expression of how we ought to value an option on a futures contract.

Earlier, we saw that the futures price can be expressed in terms of the spot price as…

$$F_t = S_t e^{(r-q)(T-t)}$$

…for some cost of carry $q$ that represents whatever storage costs, convenience yield, etc. that the spot asset may possess over the life of the futures contract. In general, this $q$ is not a value that is directly observable and essentially must be backed out of futures prices. Given this fact, it seems far more sensible to price an option on the contract using the futures price directly, whose volatility is also easier to observe in the market.

Taking τ as the time to expiry in the familiar Black-Scholes equation and the pricing time as time zero, we observe that, for a European call option….

$$V_C = S_0 e^{-q\tau} N(d_1) - K e^{-r\tau} N(d_2) = e^{-r\tau}\left[S_0 e^{(r-q)\tau} N(d_1) - K N(d_2)\right]$$

$$V_C = e^{-r\tau}\left[F N(d_1) - K N(d_2)\right]$$

…where $F$ is the futures price observed now (that is, in our approximation, the asset's forward price to $T$, the expiry of the futures contract). Furthermore…

$$d_1 = \frac{1}{\sigma\sqrt{\tau}}\left[\ln\frac{S_0}{K} + \left(r - q + \frac{\sigma^2}{2}\right)\tau\right] = \frac{1}{\sigma\sqrt{\tau}}\left[\ln\frac{S_0 e^{(r-q)\tau}}{K} + \frac{\sigma^2}{2}\tau\right]$$

$$d_1 = \frac{\ln\frac{F}{K} + \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}$$

It appears that our familiar equation is actually more easily written in terms of the forward than in terms of the spot! This is no coincidence, since the fact that the forward is driftless in the risk-neutral measure makes working with it generally easier.

The resulting equation is referred to as the Black 76 model, or simply Black's formula…

$$V_{call} = P_T\left[F N(d_1) - K N(d_2)\right]$$

$$V_{put} = P_T\left[K N(-d_2) - F N(-d_1)\right]$$

$$d_1 = \frac{\ln\frac{F}{K} + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Here, for notational simplicity $T$ is the time to expiry of the option, and $P_T$ is the present value of a risk-free zero-coupon bond to this maturity. Note that the forward price $F$ is the forward to time $T$ as observed now. In the case of a futures contract, even when the expiry of the futures contract is later than $T$, we are permitted to use today's observed futures price as $F$. Since the futures price itself is a martingale, its expected value is the same at all intervening times between now and the expiry of the contract.

But this formula applies to more than merely futures: Any of the assets we have considered so far can be priced using this approach. By doing so, we conveniently no longer have to concern ourselves while pricing the option with the asset's cash flows between now and option expiry, with estimating interest rates at each time along the way, or indeed with many of the pricing inputs we have examined thus far. As long as we can determine the forward price, the discount factor, and an appropriate volatility of the forward (which, as we have seen, in the Black-Scholes world is the same as the volatility of the spot), we are no longer particularly concerned with the evolution of the spot price itself.

While we have shown that Black's formula is equivalent to the familiar Black-Scholes formula in terms of producing a present value of the option, they are actually two alternate views of the universe that the instrument inhabits. The familiar equation gives a recipe for replicating the option by holding some amount of the spot asset and some amount of the money-market account, which at all moments grows at the risk-free rate.

Black's formula, by contrast, gives a recipe for replicating the option by holding forward contracts and zero-coupon bonds, all with maturity $T$. It operates in what is technically known as the forward measure. This leads to a number of different hedging descriptions of the option: For example, delta calculated with the Black-Scholes equation ("spot delta") does not produce the same result as delta calculated with Black's formula ("forward delta"). This seems sensible, since the asset being used to hedge the option in these two cases is not the same. Nevertheless, as time $T$ approaches, the two models converge and eventually become identical: At $T$, the spot price and the forward to time $T$ collapse into the same quantity; at the same time, a unit zero-coupon bond matures and becomes a unit of the money market asset at that time.

*Forward-start options*

Consider a variant on the vanilla European option whose behavior is as follows: At a future time $t < T$, the strike of the option is set equal to some set multiple $k$ of the asset's spot price at that time; the contract then goes on give its ordinary payoff (put or call; here, we will take a call payoff as an example) at the expiry of the option $T$. Thus, when pricing this option at time zero, the strike is random.

A number of options take this form. In the equity world, employee stock options are often issued in such a way that their strike is set at the money ($k = 1$) at some future time. A variety of OTC option contracts on other sorts of underlying assets are traded with similar properties.

We assume for simplicity that the underlying in this case does not pay dividends. Then, in terms of our familiar risk-neutral pricing formula, the value of this option is:

$$V_{call,fwdstart} = E^*\left[e^{-rT}\left(S_T - kS_t\right)^+\right]$$

We observe that once time $t$ arrives and the strike is set, the payoff becomes our familiar option payoff, which can be priced using the Black-Scholes equation. One approach to pricing this option is thus to make use of what is sometimes called the tower property of conditional expectations:

$$V_{call,fwdstart} = E^*\left[e^{-rT}\left(S_T - kS_t\right)^+\right] = E^*\left[e^{-rt}E_t^*\left[e^{-r(T-t)}\left(S_T - kS_t\right)^+\right]\right]$$

$$V_{call,fwdstart} = E^*\left[e^{-rt}V_{call,t}\right]$$

Here, the expectation inside is a conditional expectation at the future time $t$. We are permitted to add this without any adjustment beyond discounting because, like all assets, the forward price of the call option is a martingale in what we have called the risk-neutral measure (in more thorough treatments of the theory of arbitrage-free pricing, this statement requires only that our measure be an equivalent martingale measure—i.e., one in which all forward prices to all maturities are martingales; this measure is not necessarily unique).

We observe that for the call price $V_{call,t}$, the Black-Scholes equation gives…

$$V_{call,t} = S_t N\left(d_{1,t}\right) - kS_t e^{-r(T-t)} N\left(d_{2,t}\right)$$

$$d_{1,t} = \frac{1}{\sigma\sqrt{T-t}}\left(\ln\frac{S_t}{kS_t} + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right)$$

$$d_{2,t} = d_{1,t} - \sigma\sqrt{T-t}$$

…which can be rewritten…

$$V_{call,t} = S_t\left[N\left(d_{1,t}\right) - ke^{-r(T-t)}N\left(d_{2,t}\right)\right]$$

$$d_{1,t} = \frac{1}{\sigma\sqrt{T-t}}\left(\ln\frac{1}{k} + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right)$$

$$d_{2,t} = d_{1,t} - \sigma\sqrt{T-t}$$

Interestingly, this value is simply some multiple of the value $S_t$, and the value of this multiple itself has no dependence upon $S_t$ at all—only $k$. This multiple is an option denominated in different terms—not in terms of units of the money market account at $t$, which is what we are accustomed to seeing—but instead in terms of units of the asset itself at that time. If we denote this rescaling of the call price…

$$C_{k,t} = N\left(d_1\right) - ke^{-r(T-t)}N\left(d_2\right)$$

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left(\ln\frac{1}{k} + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right)$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

…then the price of the forward-start option is…

$$V_{call,fwdstart} = E^*\left[e^{-rt}S_tC_{k,t}\right] = S_0C_{k,t}$$

…since the expected value of the asset's price at $t$ is simply the forward price, whose present value must of course be equal to the spot price of the asset.

If we think about this result in hedging terms, we encounter an idea that we first saw when discussing forward contracts on assets that pay proportional dividends: There, we saw that although the amount of the dividend may be random, since the proportion is known in advance, we can hold units of the asset itself as a hedge.

A similar thought applies in the case of a forward-start option. While the strike is random before time $t$, the fact that the return on the asset from $t$ to $T$ is independent of its realized return between now and $t$ means that we can hedge the forward-starting option statically by determining the option's value *denominated in units of the asset* and then simply take an offsetting position in the asset to hedge the exposure.

*Change of numeraire*

Let's consider our approach to the previous problem in a slightly different light. Our payoff, as we initially specified it, is…

$$V_{call,fwdstart} = E^*\left[e^{-rT}\left(S_T - kS_t\right)^+\right]$$

…which we can alternatively express as…

$$V_{call,fwdstart} = E^*\left[e^{-r(T-t)}S_te^{-rt}\left(\frac{S_T}{S_t} - k\right)^+\right]$$

By expressing this payoff in units of the (discounted) asset, we are essentially doing the following…

$$V_{call,fwdstart} = E^*\left[S_te^{-rt}\right]E^*\left[\frac{e^{-r(T-t)}S_te^{-rt}\left(\frac{S_T}{S_t} - k\right)^+}{S_te^{-rt}}\right] = S_0E_N^*\left[e^{-r(T-t)}\left(\frac{S_T}{S_t} - k\right)^+\right]$$

…creating an alternate view of the payoff denominated in units of the asset rather than our familiar unit—the money market account. As we have seen, the resulting expectation can be taken far more easily than the one in our initial expression.

This technique—called change of numeraire—can be useful in a great variety of situations. It is beyond the scope of these notes to deal with the theoretical underpinnings of the technique. Intuitively, however, since all traded assets are (discounted) martingales under the usual risk-neutral measure, it seems sensible that we ought to be able to choose any of them as the unit in which we denominate prices.

The measure resulting from this decision is not identical to the basic risk-neutral measure we have used all along, which takes the money-market asset as its numeraire; however, it can be shown that, in this new measure, all assets whose prices were (discounted) martingales under the money-market measure are also martingales under this new measure.

The main requirements for the asset that we take as numeraire is that its value must always be positive, and it must not throw off any cash flows over the period of interest. Under these assumptions, if we choose the asset $N$ to be a new numeraire, then for some arbitrary payoff $V$:

$$E^*\left[V_T\right] = N_0 E_N^*\left[\frac{V_T}{N_T}\right]$$

That is, the value denominated in our money-market numeraire must be the same as the value denominated in the new numeraire (and new measure) we have chosen.

*FRA's revisited and the T-forward measure*

When we introduced the FRA, we showed via a static hedging argument that the present value of a contract on notional $N$ for the rate from future times $t$ to $T$ to pay the fixed strike rate $r_K$ when the forward rate between these times is $r_f$ must be…

$$V_{FRA} = NP_{T,0}\left(r_f - r_K\right)\left(T - t\right)$$

…where $P_{T,0}$ denotes the price today of a zero-coupon bond paying 1 at time $T$. That is, we showed that we can price the contract as if the forward rate is realized. Later, in our more general discussion of forwards, we showed that the FRA is equivalent to a forward contract to time $t$ on a zero-coupon bond maturing at $T$, which agreed with our more general expression of the value of a forward contract on a non-dividend-paying asset.

Let us attempt to formulate the value of this contract using the more general risk-neutral pricing formula we have repeatedly used with options. Under this formula, the value of the contract is…

$$V_{FRA} = E^*\left[NP_T\left(r - r_K\right)\left(T - t\right)\right] = NE^*\left[P_T\left(r - r_K\right)\right]\left(T - t\right)$$

…where $r$ is the rate to maturity $T - t$ observed at future time $t$. At time zero, when we are pricing the contract, the rate $r$ is random, as is the value $P_T$, and clearly the value of $r$ influences the value of $P_T$. Taking this expectation in the money-market numeraire thus proves to be a troublesome proposition. While we have generally treated interest rates as nonrandom (or, at the very least, uncorrelated with the quantity of interest) in our previous pricing efforts, here that assumption is clearly inappropriate.

Suppose, however, that we denominate this payoff by taking the zero-coupon bond with maturity $T$ as our numeraire. By the change-of-numeriare formula we showed above, the valuation expression becomes:

$$V_{FRA} = NE^*\left[P_T\left(r - r_K\right)\right]\left(T - t\right) = NP_{T,0}E_N^*\left[\frac{P_T\left(r - r_K\right)}{P_T}\right]\left(T - t\right) = NP_{T,0}E_N^*\left[r - r_K\right]\left(T - t\right)$$

$$V_{FRA} = NP_{T,0}\left(E_N^*\left[r\right] - r_K\right)\left(T - t\right)$$

All that remains, then, is to determine the expected value of the reset under our new measure.

Recall that, under our risk-neutral measure using the money-market account as our numeraire, all traded assets are martingales. This means that, in particular, the (discounted) values of all zero-coupon bonds are martingales; this is the fact we take advantage of in pricing future cash flows. In particular, as expressed in the money-market numeraire, the zero-coupon bonds that pay 1 at $t$ and at $T$ are (discounted) martingales.

At time $t$, the value of the reset $r$ is given by…

$$1 + r(T - t) = \frac{1}{P_T}$$

$$r = \frac{1}{T - t}\left(\frac{1}{P_T} - 1\right)$$

…and at all times before $t$, the forward rate $r_f$ is calculated as:

$$1 + r_f(T - t) = \frac{P_t}{P_T}$$

$$r_f = \frac{1}{T - t}\left(\frac{P_t}{P_T} - 1\right)$$

Note that when time $t$ arrives, this expression for the forward rate becomes identical to the value we calculate for the reset, since at $t$, $P_t = 1$. Note also that, in the money market numeraire, this expression is random with two sources of variation, since both of the zero-coupon bond prices are random up to $t$.

But in the numeraire we have chosen—where all prices are denominated in terms of the zero-coupon bond maturing at $T$—this expression's random component is a linear function of the price of the bond maturing at $t$. From our discussion above, we know that this price is a martingale, and thus so is the linear function of it!

So, we have an expression that becomes equal to the reset $r$ when $t$ arrives, and this expression is a martingale under our new numeraire at all times before then. Thus, we are justified in concluding that…

$$r_f = E_N^*[r]$$

…and therefore…

$$V_{FRA} = NP_{T,0}\left(r_f - r_K\right)(T - t)$$

…which agrees with the valuation equation we have arrived at by other means.

This is a very exciting result. We have arrived at a method by which a rational function of two asset prices—each of which has its own volatility and whose movements are correlated—can be simplified in a new measure to become a martingale. The measure that we have used here is known as the $T$-forward measure, and in this measure all forward rates *terminating* at $T$ are martingales.

*Interest rate caps and floors*
The technique we have just shown is useful for more, of course, than merely pricing FRA's, which after all we were able to price by other means not requiring it. In particular, this technique makes it clear how options on FRA's ought to be priced.

Among the most common interest rate derivatives are options on future zero rates, which are known as caps and floors. These instruments are actually strips of non-overlapping interest-rate options, which usually make periodic payoffs based on some maturity of LIBOR (most often 3M or 6M).

We specify a contract with the following characteristics:
$T$—maturity
$N$—notional of the contract
$m$—caplet / floorlet frequency
$r_K$—strike rate

This contract contains $mT$ individual options (called caplets or floorlets) that reset at times 0, $1/m$, $2/m$,…, $T-1/m$ and pay at times $1/m$, $2/m$,…, $T$. The payoff of a caplet is…

$$V_{caplet,t+\frac{1}{m}} = \frac{N}{m}\left(r_{t,t+\frac{1}{m}} - r_K\right)^+$$

…where the random reset is the rate observed at $t$ to time $t + 1/m$. The payoff of this option occurs at the end of the caplet period, namely time $t + 1/m$. Similarly, the payoff of a floorlet is:

$$V_{floorlet,t+\frac{1}{m}} = \frac{N}{m}\left(r_K - r_{t,t+\frac{1}{m}}\right)^+$$

That is, a caplet is a call option on a zero rate, while a floorlet is a put option on a zero rate. From a pricing standpoint, it is vitally important that the caplets / floorlets be non-overlapping, and that the payoff of the option occur with "natural" timing—i.e., that the rate be observed at the beginning of the period and paid at the end. If these conditions are met, then each option can be priced separately as, in essence, an option on an FRA.

To simplify the notation, we consider a caplet that resets at time $t$ and pays at time $T$. Under our risk-neutral pricing formula, the present value of this caplet is given by…

$$V_{caplet} = E^*\left[\frac{N}{m}P_T\left(r - r_K\right)^+\right] = \frac{N}{m}E^*\left[P_T\left(r - r_K\right)^+\right]$$

As we saw in risk-neutral pricing of FRA's, the natural measure in which to price this option is the $T$-forward measure. We take the unit zero-coupon bond maturing at $T$ as our numeraire and see that:

$$V_{caplet} = \frac{N}{m} E^*\left[P_T(r - r_K)^+\right] = \frac{N}{m} P_{T,0} E^*\left[\frac{P_T(r - r_K)^+}{P_T}\right]$$

$$V_{caplet} = \frac{N}{m} P_{T,0} E_N^*\left[(r - r_K)^+\right]$$

We have already seen that the forward rate $r_f$ is a martingale in this measure, and that at time $t$ it takes on the value of the reset $r$. As a result, if we assume that the forward rate is lognormally distributed, we can use Black's formula to determine the price of the option in the $T$-forward measure:

$$E_N^*\left[(r - r_K)^+\right] = r_f N(d_1) - r_K N(d_2)$$

$$d_1 = \frac{1}{\sigma_r \sqrt{t}}\left(\ln\frac{r_f}{r_K} + \frac{\sigma_r^2 t}{2}\right)$$

$$d_2 = d_1 - \sigma_r \sqrt{t}$$

To value the full caplet, then, we need only rescale by the notional and accrual, and discount to the present:

$$V_{caplet} = \frac{N}{m} P_{T,0}\left[r_f N(d_1) - r_K N(d_2)\right]$$

$$V_{floorlet} = \frac{N}{m} P_{T,0}\left[r_K N(-d_2) - r_f N(-d_1)\right]$$

$$d_1 = \frac{1}{\sigma_r \sqrt{t}}\left(\ln\frac{r_f}{r_K} + \frac{\sigma_r^2 t}{2}\right)$$

$$d_2 = d_1 - \sigma_r \sqrt{t}$$

Note that caplets and floorlets also have their own version of put-call parity…

$$V_{caplet} - V_{floorlet} = V_{FRA}$$

…and that Greeks such as delta, gamma, vega, and so on can all be calculated (in forward terms) for this option just as they can be for ordinary equity options. The value of the entire cap/floor contract is, then, simply the sum of the values of the caplets and floorlets, since each of these values is independent.

*Market quotes of IR caps/floors*

It may usefully be remarked that the idea of a lognormally distributed forward rate may not be particularly realistic, and indeed other choices are sometimes made. The change-of-numeraire technique is not affected by the choice of method used to determine the value of the option payoff; an equivalent version of Black's formula can be derived, for example, in the case where the rate is normally distributed rather than lognormally distributed.

Given that an interest-rate cap or floor is in fact a strip of option contracts, the standard for quoting these is a bit different that it is for ordinary option contracts. In our above formulation, the volatility $\sigma_r$ is the volatility of a single-period forward rate, which is usually referred to (perhaps confusingly) as the *spot volatility*.

For a 1Y cap contract, which in USD is composed of 4 quarterly caplets, there are in theory 4 spot volatilities needed to price the contract. For a 2Y contract, there are, again in theory, 8 spot volatilities, 4 of which are shared with the 1Y contract.

Since the individual caplets are not traded, however, there is generally no way of determining market-implied values for the spot volatilities. Instead, cap / floor volatilities are generally quoted as *flat volatilities*—that is, the single volatility that, if used to price all of the caplets, matches the traded price of the contract.

With this in mind, we see that we can actually bootstrap a term structure of spot volatilities, if we impose some structure on them, from the market-quoted flat vols. Using the 1Y flat vols, we can determine the dollar price of the first four caplets taken as a unit. Using the 2Y flat vols, we can determine the dollar price of the first 8 taken as a unit. Thus, under some assumption of the relationship between the spot vols in the second year, we can use a nonlinear solution method to determine the spot volatilities in year two that cause the market quotes for the 2Y cap contract to be consistent with the 2Y flat vol and the price of the 1Y cap contract.

It should be noted that, while this approach is theoretically sound, market realities sometimes result in collections of flat volatilities—depending how the market quotes are obtained—that are not arbitrage-free. In this case, there are a variety of possible approximations and assumptions that can be made, but in the end this is one among many illustrations of how market data must be carefully chosen and used when calibrating and pricing other instruments.

*Swaptions*

A swaption is an option on a swap that commences when the option expires. (Options on forward-starting swaps do exist, and can be priced by analogy using the methods below; however, we will not cover these options here.)

We define the contract with the following key attributes:

$N$—the notional of the contract
$t$—expiry of the option and effective date of the swap
$T$—maturity of the swap; the value $T - t$ is called the tenor of the swap being entered into
$r_K$—the fixed strike rate of the underlying swap
$m$—the payment frequency on the fixed leg of the underlying swap

A *payer swaption* confers the right, but not the obligation, to enter a swap at expiry paying the fixed strike rate; correspondingly, a *receiver swaption* confers the right, but not the obligation, to receive the fixed strike rate.

We recall that the present value of a swap paying the fixed rate $r_K$ at a time when the par swap rate to the same maturity with the same frequency is $r$ is given by…

$$V_{swap} = NA_{m,t,T}\left(r - r_K\right)$$

…where $A_{m,t,T}$ is the annuity commencing at future time $t$, ending at future time $T > t$, with fixed payment frequency $m$. This expression arises from the observation that all of the risk in a swap where we pay the fixed rate $r_K$ can be removed by entering into an offsetting swap in which we receive the par swap rate $r$. Since the floating legs in this case offset, all that remains is a strip of payments on the fixed-leg payment dates corresponding to the difference between the contract swap rate and the par swap rate.

If you are long a payer swaption, you pay a premium upfront for the right to pay $r_K$ fixed to receive float on the underlying swap, which commences at $t$. Clearly, you will only do this if the par swap rate at $t$ is greater than the strike rate, since this is the situation in which the value of the swap will be positive. As such, a payer swaption behaves like a call option on the swap rate; correspondingly, a receiver swaption behaves like a put option on the swap rate.

Using our risk-neutral pricing formula, the present value of a payer swaption is…

$$V_{payer} = E^*\left[NA_{m,t,T}\left(r - r_K\right)^+\right] = NE^*\left[A_{m,t,T}\left(r - r_K\right)^+\right]$$

…but, as with caps and floors, this expectation in the money-market numeraire is difficult to imagine taking. The asset $A_{m,t,T}$ actually involves $m(T - t)$ separate discount factors, all of which are random, and the random par swap rate $r$ is likewise a function of all of these discount factors. A full-fledged interest rate model would seem to be required to take account of all of these separate sources of variation.

Fortunately, a simpler approach is possible if we consider the fact that the annuity is a portfolio of zero-coupon bonds, making this a traded market instrument in the senses we have discussed above. Its discounted value is therefore a martingale in the risk-neutral measure. Moreover, until time $t$, this asset has strictly positive value and does not throw off cash flows. Therefore, it serves as an appropriate numeraire.

Therefore, we will attempt to price this option using the following…

$$V_{payer} = NE^*\left[A_{m,t,T}\left(r - r_K\right)^+\right] = NA_{m,t,T,0}E_N^*\left[\frac{A_{m,t,T}\left(r - r_K\right)^+}{A_{m,t,T}}\right]$$

$$V_{payer} = NA_{m,t,T,0}E_N^*\left[\left(r - r_K\right)^+\right]$$

…where $A_{m,t,T,0}$ is the present value of the annuity as seen today, at pricing time.

The expectation looks essentially similar to the one we encountered in the cap/floor case, meaning that if we assume the rate $r$ has a lognormal distribution, we may attempt to use Black's formula to price it. Of course, the assumption of Black's formula is that the random quantity of interest (the forward) is a martingale, which we have not yet established in the case of our chosen numeraire.

As seen today, the reference rate $r$ is a forward par swap rate. This is the rate that satisfies:

$$P_t = rA_{m,t,T} + P_T$$

$$rA_{m,t,T} = P_t - P_T$$

$$r = \frac{P_t}{A_{m,t,T}} - \frac{P_T}{A_{m,t,T}}$$

We note that $P_t$ and $P_T$ are the prices of two traded assets, which means that the price of each divided by the annuity is the price expressed in our chosen numeraire. We have already seen that we can be assured these prices are martingales in the new numeraire; therefore, $r$ is also a martingale, since it is a linear combination of such assets. We can indeed use Black's formula to price the swaption, if we assume that the par swap rate is lognormally distributed.

Thus, we arrive at the expressions for the present values of swaptions…

$$V_{payer} = NA_{m,t,T,0}\left[rN(d_1) - r_K N(d_2)\right]$$

$$V_{receiver} = NA_{m,t,T,0}\left[r_K N(-d_2) - rN(-d_1)\right]$$

$$d_1 = \frac{1}{\sigma_r\sqrt{t}}\left(\ln\frac{r}{r_K} + \frac{\sigma_r^2}{2}t\right)$$

$$d_2 = d_1 - \sigma_r\sqrt{t}$$

…with the volatility here chosen as the volatility of the forward swap rate. Note that, once again, we have a form of put-call parity for these options:

$$V_{payer} - V_{receiver} = V_{swap}$$

As noted above, an equivalent expression can be obtained under the assumption that the swap rate is normally distributed, rather than lognormally distributed. Most market data providers offer swaption volatilities quoted in both forms. The most liquid swaption quotes are ATM, although most providers also offer swaption cubes—i.e., the market-implied volatilities for a variety of combinations of swaption expiry, tenor of the underlying swap, and strike of the swaption.