Class 1

Notes

Interest rates are expressed with continuous compounding unless otherwise specified.

We will not concern ourselves with industry daycount or business-day conventions. To translate maturities into times in years…

For maturities expressed in days, assume a 365-day year (i.e., a 1D maturity corresponds to $T = 1 / 365$).

For maturities expressed in weeks, assume 52 weeks per year.

For maturities expressed in months, assume 12 months per year.

Exercises

1. At a given moment, you observe the following prices for various maturities of risk-free zero-coupon bonds:

maturity	bond price (per 100 face)
1D	99.9995
1W	99.994
1M	99.971
3M	99.9
6M	99.65
1Y	99.05

(a) What is the term structure of risk-free zero rates implied by these bond prices, expressed with continuous compounding?
(b) What is the term structure of single-period forward rates (i.e., the forwards between adjacent maturities), also expressed with continuous compounding?
(c) Verify that the 1Y zero rate is the time-weighted average of the single-period forward rates you calculated.

2. The USD LIBOR fixings for a particular day are as follows:

USD	**Rate (percent)**
o/n	0.12800
1w	0.16270
2w	0.17170
1m	0.19043
2m	0.22413
3m	0.25288
4m	0.29463
5m	0.35013
6m	0.40313
7m	0.45913
8m	0.51288

9m	0.56463
10m	0.61913
11m	0.67125
12m	0.72950

For all parts of this problem, recall that LIBOR is quoted as a rate of simple interest, and make the simplifying assumptions detailed above concerning daycount and business-day conventions. Treat the o/n (overnight) rate as a one-day rate.

(a) What is the term structure of discount factors implied by these fixings?
(b) What are the one-month forward rates, expressed as rates of simple interest, for months 1-11 implied by these fixings?
(c) What is the term structure of interest rates implied by these fixings, with all rates expressed with monthly compounding?

3. Using market data, you extract the following term structure of discount factors:

maturity	discount factor
0.00274	0.99999
0.01923	0.99989
0.08333	0.9995
0.25	0.99875
0.5	0.99576
1	0.98807

(a) Using linear interpolation in the continuously compounded spot rates, determine the discount factor at 4 months.
(b) Using the assumption of piecewise constant forward rates, determine the discount factor at 10 months.
(c) Under each of the interpolation methods used above, what is the six-month forward rate in four months, expressed with continuous compounding?

4. The risk-free zero rate to the maturity 1 year is 1.25%. The risk-free zero rate to the maturity 3 years is 1.6%. Under each of the following interpolation methods, determine the instantaneous forward rate at 1 year (with the limit taken from the right) and 3 years (with the limit taken from the left).

(a) Constant forward rate
(b) Linear zero rates
(c) Cubic spline in the zero rates, where the maturities at 1 and 3 years are connected by a segment with the equation:

$$r(t) = -0.001138889t^3 + 0.001638889t^2 + 0.01t + 0.002$$

Here r is the rate expressed simply as a number—i.e., 1% is 0.01.

5. The term structure of risk-free zero rates is as follows:

t	r (continuous)
0.5	0.011
1	0.015
2	0.022
3	0.024
5	0.029

(a) What is the present value of a risk-free bond with maturity 3 years paying annual coupons of 3.5%? Express your answer per 100 face amount.
(b) What is the 2-year semiannual par yield? Use linear interpolation in the continuously compounded risk-free zero rate.
(c) What is the yield (expressed as a continuously compounded rate) of a 1-year risk-free bond with a 5% coupon rate per year that pays semiannually?

6. The following problems describe risk-free bond prices or yields observed at the same time. All bond prices are expressed per 100 face amount.

(a) A zero-coupon bond with a 6-month maturity trades at 99.875. What is the risk-free zero rate, expressed with continuous compounding, to this bond's maturity?
(b) A bond paying a coupon of 4% per year semiannually with a maturity of 1 year has a yield of 50 basis points, expressed in the bond's natural compounding frequency. What is the risk-free zero rate, expressed with continuous compounding, to this bond's maturity? (A basis point is 1% of 1%, or 0.0001.)
(c) A bond paying a 2% coupon per year semiannually with a maturity of 2 years has a yield of 1.05%, expressed in the bond's natural compounding frequency. Using linear interpolation in the continuously compounded risk-free zero rates, find the risk-free zero rate, expressed with continuous compounding, to this bond's maturity.

7. A bond with a 5-year maturity pays a 4% coupon semiannually. At the moment, it trades with a yield of 2.5%, expressed with continuous compounding.

(a) What is the price of the bond? Express your answer assuming a 100 face amount.
(b) What is the (Macaulay) duration of the bond? What is its convexity?
(c) What is the modified duration of the bond?
(d) Use the discussion in the chapter to derive an expression for a bond's modified convexity, and calculate it for this bond.

8. The following bond prices are observed from the same risk-free issuer in the same currency:

maturity	maturity unit	bond price	coupon	coupon frequency (interest payments / year)
3	months	99.72537778	0	n/a
6	months	99.37694906	0	n/a
1	year	98.06888952	0	n/a
2	years	100.0902937	0.018	4

5	years	100.7173062	0.022	4
10	years	102.3713352	0.026	4
30	years	100.0691529	0.03	4

Bootstrap to determine the term structure of discount factors at each of these maturities. Interpolate by assuming piecewise constant forward rates between the maturities listed above.

Applications

1. *Quartic forward spline*

It has been shown[1] that the method giving the smoothest forward rates matching a set of zero-rate points is produced by a spline in the instantaneous forward rates in which the segments are quartic polynomials. The purpose of this problem is to derive equations pertinent to this method and employ them in a simple example.

We suppose we are given the set of points in the continuously compounded zero rate (t_i, r_i), $i = 0, 1, ..., N$, with $t_0 = 0$ and r_0 thus the initial instantaneous rate. This set of $N + 1$ points defines N segments, each of which is represented by a quartic polynomial. For t in the interval $[t_{i-1}, t_i)$, the instantaneous forward is thus:

$$r^*(t) = Q_i(t) = c_{1,i}t^4 + c_{2,i}t^3 + c_{3,i}t^2 + c_{4,i}t + c_{5,i}$$

To fit a spline we then must determine $5N$ unknown coefficients. As in the simpler case of a cubic spline, we define a set of constraints on the polynomials to produce a system of linear equations that can be solved for these coefficients.

(a) For $i = 1, 2, ..., N - 1$, t_i is an interior node of the curve—i.e., one which is a boundary point of both segments Q_i and Q_{i+1}. We enforce a constraint that our instantaneous forward rates be continuous at these points. Thus, expressed in terms that are amenable to representation in matrix form:

$$Q_i(t_i) - Q_{i+1}(t_i) = 0$$

$$c_{1,i}t_i^4 + c_{2,i}t_i^3 + c_{3,i}t_i^2 + c_{4,i}t_i + c_{5,i} - c_{1,i+1}t_i^4 - c_{2,i+1}t_i^3 - c_{3,i+1}t_i^2 - c_{4,i+1}t_i - c_{5,i+1} = 0$$

We also require that the instantaneous forward rates have 3 continuous derivatives. Write the equations for these additional constraints.

The continuity constraints provide $4(N - 1)$ of the $5N$ required equations.

(b) At each interior node, we additionally require that the *spot curve* arising from the instantaneous forward rates pass through the known points. Recall that, for some $t < T$:

$$P_T = P_t e^{-\int_t^T r^*(s)ds}$$

Show that the spot-curve constraint at the interior node t_i can be expressed as:

[1] For more detail on this method, see the work of Donald van Deventer, a principal exponent of it.

$$\frac{1}{5}c_{1,i}\left(t_i^5-t_{i-1}^5\right)+\frac{1}{4}c_{2,i}\left(t_i^4-t_{i-1}^4\right)+\frac{1}{3}c_{3,i}\left(t_i^3-t_{i-1}^3\right)+\frac{1}{2}c_{4,i}\left(t_i^2-t_{i-1}^2\right)+c_{5,i}\left(t_i-t_{i-1}\right)=r_it_i-r_{i-1}t_{i-1}$$

This constraint provides $N-1$ of the $5N$ required equations. Altogether, interior nodes account for $5(N-1)$ constraints, leaving 5 remaining.

(c) We further require that the curve reproduce the known information at the left and right edges. Because the instantaneous rate at time zero is both the spot rate and the "forward" rate, trivially at the left edge we have:

$$Q_1\left(t_0\right)=r_0$$

$$Q_1\left(0\right)=r_0$$

$$c_{5,1}=r_0$$

Formulate the constraint for the right edge that ensures the *spot curve* passes through the point (t_N, r_N).

With these two constraints added, 3 remain.

(d) There are several possible choices for the 3 remaining constraints. For the sake of this problem, we choose:

$$Q_1''(t_0)=0$$

$$Q_N''(t_N)=0$$

$$Q_N'''(t_N)=0$$

Write these constraints in a form consistent with the previous constraints.

(e) Suppose the term structure of known zero rates is as follows:

t	r
0	0.025
1	0.0265
5	0.0325
10	0.0315

Determine the fifteen unknown coefficients of the quartic forward spline entailed by this data.

(f) Show that for any positive t in the interval $[t_{i-1}, t_i)$, the zero rate at this time can be calculated as:

$$r_t = \frac{1}{t}\left[r_{i-1}t_{i-1} + \frac{1}{5}c_{1,i}\left(t^5 - t_{i-1}^5\right) + \frac{1}{4}c_{2,i}\left(t^4 - t_{i-1}^4\right) + \frac{1}{3}c_{3,i}\left(t^3 - t_{i-1}^3\right) + \frac{1}{2}c_{4,i}\left(t^2 - t_{i-1}^2\right) + c_{5,i}\left(t - t_{i-1}\right) \right]$$

(g) Taking times t from 0 to 10 years spaced 0.1 years apart, graph the zero rate to t and the instantaneous forward rate at t.

2. *A Lower Bound on the Convexity of a Collection of Deterministic Positive Cash Flows*
In the chapter, it was asserted that for any collection of deterministic and strictly positive cash flows with a particular duration, a zero-coupon bond exhibits the least convexity. The goal of this problem is to formalize that assertion.

For the purposes of this problem, we will take advantage of the fact, discussed more thoroughly in a later chapter, that for a portfolio Π consisting of several instruments with values $V_1, V_2, \ldots, V_n$ having durations D_1, D_2, D_n and convexities $C_1, C_2, \ldots, C_n$, we may say that…

$$V_\Pi = \sum_{i=1}^{n} V_i$$

$$D_\Pi = \frac{1}{V_\Pi}\sum_{i=1}^{n} V_i D_i$$

$$C_\Pi = \frac{1}{V_\Pi}\sum_{i=1}^{n} V_i C_i$$

…for certain suitable definitions of duration and convexity. For the moment, we will not concern ourselves with what those definitions are, but will simply use the result.

(a) Show that for an arbitrary portfolio Π consisting of n strictly positive cash flows occurring at different times, the following inequality holds:

$$C_\Pi > D_\Pi^2$$

Suggestion: This can easily be done by induction. Begin with the case of two deterministic positive cash flows at times T_1 and T_2, and show that…

$$C_\Pi \geq D_\Pi^2$$

…with equality holding if and only if $T_1 = T_2$.

Next, assume that we have a portfolio of n cash flows Π_n satisfying…

$$C_{\Pi_n} > D_{\Pi_n}^2$$

…and show that the addition of any deterministic positive cash flow to this portfolio results in a new portfolio Π_{n+1} that satisfies:

$$C_{\Pi_{n+1}} > D_{\Pi_{n+1}}^2$$

(b) Explain why your argument from above in (a) suffices to prove the desired result concerning zero-coupon bonds.
(c) Can the result be extended to portfolios in which some of the cash flows are negative—that is, levered portfolios or portfolios that also include deterministic liabilities? Explain the reasoning behind your answer.

3. *Approximation of Portfolio Yield from Yields of Its Constituent Instruments*
A portfolio consists of two fixed-coupon bonds: The first has maturity 5 years, pays interest semiannually at the rate of 4% per year, and has a yield (expressed with continuous compounding) of 2.5%. The second has maturity 3 years, pays interest annually at the rate of 1.5% per year, and has a yield (expressed with continuous compounding) of 5%.

(a) Compute the prices and durations of these bonds, expressing each price per 100 face amount.
(b) Suppose we have 100 total to invest. We construct a portfolio by choosing weight w to invest in the first bond, and therefore weight $1 - w$ to invest in the second. For w running from 0 to 1 inclusive, spaced 0.025 apart, calculate the quantity (face amount) held of each bond given w.
(c) Using your nonlinear solver of choice, for each of the portfolios above in part b, calculate the yield of the portfolio's cash flows, expressed with continuous compounding.
(d) For each of the portfolios whose yield you calculated above in c, compare your result to the following approximations:

$$\tilde{y}_1 = \frac{V_1 y_1 + V_2 y_2}{V_1 + V_2}$$

$$\tilde{y}_2 = \frac{V_1 D_1 y_1 + V_2 D_2 y_2}{V_1 D_1 + V_2 D_2}$$

...where V is the present value of each bond, D is its duration, and y is its yield. How does each approximation perform in this example? The first—a simple weighted average of yields—is very common for approximating the yield of a portfolio of bonds. Provide an intuitive explanation of why the second is a better approximation.
(e) Consider a portfolio Π of N fixed-coupon bonds, with known V_i, D_i, and y_i for each bond, $i = 1, 2, \ldots, N$. Using the first-order approximation of the bond's price at a given yield...

$$V_i(y) \approx V_i - D_i V_i (y - y_i)$$

...derive the approximation of the portfolio yield:

$$y_\Pi \approx \frac{\sum_{i=1}^{N} V_i D_i y_i}{\sum_{i=1}^{N} V_i D_i}$$