6. Forwards and Futures

Exercises

1. The term structure of risk-free interest rates is as follows:

t	r
3m	0.025
6m	0.027
9m	0.03
1y	0.0285
15m	0.0295
18m	0.03
21m	0.0305
2y	0.031
3y	0.04
4y	0.0465
5y	0.05

All rates are expressed with continuous compounding. Determine the forward price in 6 months of the following assets:

(a) A zero-coupon bond with maturity nine months, per 100 face.
(b) A bond paying a 5% coupon semiannually with maturity 1.25 years, per 100 face.
(c) An equity with spot price 100 paying a 1.2% dividend continuously.
(d) An equity with spot price 300 that is expected to split 3 for 1 in 2 months. (Express the forward as a price per share.)
(e) A foreign currency with spot exchange rate 1.2—expressed as number of units foreign currency per unit domestic currency—in which the 6-month risk-free rate is 2.2%, expressed with continuous compounding. (Express your answer as number of units foreign currency per unit domestic currency.)

2. An equity's spot price S_0 is 100 at the moment. Using other traded market instruments, you have determined that the term structure of zero rates and equity forward prices is as follows:

t	r (continuous)	Ft
1m	0.02	100.0625195
2m	0.022	100.2628448
3m	0.0225	100.3882518
4m	0.0245	100.5934205
5m	0.025	100.8200105
6m	0.026	101.0732181

(a) What is the term structure of dividend rates in each month implied by these values? (Express your answer as the continuously compounded dividend rate from now to 1m, from 1m to 2m, 2m to 3m, and so on.)
(b) Suppose that the equity has an established record of paying its dividend, if any, at the end of each month, relative to the date for which you extracted this data. Under this assumption, what the is amount of the lump-sum cash dividend at the end of each month implied by this data? (Assume that the forward price as expressed here is *after* the payment of the fixed dividend.)

3. Use a two period binomial model and the CRR up and down steps with the following parameters…
$S_0 = 30$
$r = 0.05$
$q = 0$
$T = 0.5$
$\sigma = 0.4$
…to price each of the following options:

(a) A European call option with strike $K = 25$
(b) A European put option with strike $K = 25$

At each node in your lattice, give:
the spot price
the value of the option
the delta of the option

4. Use a two period binomial model and the CRR up and down steps with the following parameters…
$S_0 = 50$
$r = 0.03$
$q = 0.025$
$T = 0.25$
$\sigma = 0.6$
…to price each of the following options:

(a) An American call option with strike $K = 41$
(b) An American put option with strike $K = 52$

At each node in your lattice, give:
the spot price
the value of the option if its holder chooses not to exercise (the continuation value)
the value of the option if its holder chooses to exercise (the intrinsic value)
the value of the option
the delta of the option

5. We wish to price options on an underlying with the following attributes:
$S_0 = 45$
$q = 0.005$

The risk-free rate r is 2.5%, expressed with continuous compounding, to all maturities.

(a) What is the Black-Scholes price of a European call on this asset with expiry 0.25 years, strike price 50, and volatility 35%?
(b) What is the Black-Scholes price of a European put on this asset with expiry 0.25 years, strike price 40, and volatility 45%?
(c) The implied volatility of an option is the volatility that, if used in the appropriate option pricing formula, causes the price returned by the formula to equal the market price. If the market price of a European call option on this asset with strike 50 and expiry 0.5 years is 2.45, then what is the implied volatility of the option?
(d) If the market price of a European put option on this asset with strike 30 and expiry 0.5 years is 0.80, then what is the implied volatility of the option?

6. Use the risk-neutral distribution of the terminal spot price under the Black-Scholes assumptions to derive formulas for the present values of the following derivatives in terms of:
S_0: the spot price
r: the constant continuously compounded risk-free rate
q: the constant continuously compounded dividend rate
T: the expiry of the option
σ: the volatility of the underlying

(a) A cash-or-nothing call, often also called a binary or digital call, with strike price K and payment amount N: This option pays the fixed amount N to the holder of the option if the spot price at expiry is greater than the strike, and nothing otherwise.
(b) An asset-or-nothing call with strike price K: This option delivers the underlying asset to the holder of the option if its spot price at expiry is greater than the strike, and nothing otherwise.
(c) A squared power call option with strike price K: This option delivers the difference between the square of the spot price at expiry and the strike price if that value is positive, and nothing otherwise.
(d) A log-forward contract with strike price K: The payoff of this derivative is the natural log of the ratio between the spot price at expiry and the strike price. In particular, what is the value of this contract if the strike K is equal to the forward price of the asset?

Applications
1. *Put-call symmetry*
The concept of put-call symmetry relates the prices of puts and calls in terms of the ratio of their strike prices. The goal of this problem is to derive the classic put-call symmetry result in the Black-Scholes model.

(a) Show that, for European puts and calls struck at the underlying's forward price (usually called ATMF options) with the same expiry on the same underlying, $V_{call} = V_{put}$. (Note that this does not require Black-Scholes assumptions to show, and is true for any arbitrage-free model.)
(b) Show that, for any positive constant k, if we denote the forward price of the underlying at expiry by F, then the Black-Scholes value of a European call option struck at kF is equal in value to k European put options on the same underlying struck at F / k with the same expiry.
(c) Conclude that, under the Black-Scholes model, the price of a European call on an asset struck at K is equal to the price of K / F European put options on the same asset with the same expiry struck at F^2 / K.

2. *Volatility Skew*
In reality, the Black-Scholes assumption of normally distributed log-returns at constant volatility does not hold. Equities, in particular, tend to exhibit large negative returns more frequently than a normal model would predict, making OTM put options relatively more valuable than OTM call options.

The Black-Scholes model, however, remains useful for quantifying the difference in relative value in an intuitive way. For a given option price, the Black-Scholes model is commonly used to report the *implied volatility* of the option: This is the volatility that, if supplied to the Black-Scholes model, causes the model price to match the traded price. The typical finding from equity markets is that the implied volatilities exhibit a skew: OTM put options have higher implied volatilities than OTM call options.

For the following example, we choose the values:
$S_0 = 100$
$r = 0.01$
$q = 0$
$\sigma = 0.25$
$T = 0.25$

(a) For strikes running from 50 to 200, spaced 5 apart, compute the Black-Scholes value of the corresponding OTMf option (a put option if the strike is less than the forward, a call option otherwise). Implement a VBA function that takes a call / put price and solves for the implied volatility, and verify that your function returns the same volatility, up to estimation error, for all of your options.

(b) A simple way to replicate an equity's tendency to exhibit large negative returns is to augment the geometric Brownian diffusion of the Black-Scholes model with a Poisson jump process. We will consider a version of this model with fixed downward jumps of nonrandom size.

We choose a downward jump size m so that at the time immediately before the jump, if the spot price is S_{t-}, then after the jump the spot price becomes $(1 - m)\, S_{t-}$. We parameterize the Poisson process by choosing an expected wait time in years to the next jump τ and taking the parameter $\lambda = 1 / \tau$.

Under this choice, the expected number of jumps between now and expiry T is λT, making the number of observed jumps Y a Poisson random variable with this as its parameter. In order for our model to agree with the forward price, between jumps the asset must grow *faster* than the risk-free rate by the amount $m\lambda\, dt$, which is the average rate of loss on the equity that the jumps incur. The forward price of the asset under this model is therefore the forward price conditional on the number of jumps, weighted by the probability of observing that number of jumps:

$$F = \sum_{y=0}^{\infty} (1-m)^y S_0 e^{(r-q+m\lambda)T} P(Y=y) = \sum_{y=0}^{\infty} (1-m)^y S_0 e^{(r-q+m\lambda)T} \frac{(\lambda T)^y}{y!} e^{-\lambda T}$$

Using Excel, verify that this formula's result agrees with the forward price in the example from part a when we choose the downward jump size $m = 0.2$ and the expected wait time to the next jump $\tau = 4$ years.

(c) To price an option under this jump diffusion model, we must as above price conditional on the number of jumps and then weight each price by the probability of that number of jumps. However, the drift rate of the asset is no longer $r - q$, as in the usual Black-Scholes case, but instead $r - q + m\lambda$, meaning that we require modified versions of the Black-Scholes formulas.

Suppose that the terminal distribution of some asset S_T is given by:

$$S_T = S_0 e^{\left(r-q-\frac{\sigma^2}{2}+\delta\right)T+\sigma\sqrt{T}Z}$$

...where as usual Z is a standard normal random variable and σ the volatility, with an additional excess drift rate δ. Derive the modified pricing formula for a European call:

$$V_{\delta,call} = e^{-rT} E\left[(S_T - K)^+\right]$$

...under this distribution, and use put-call parity to derive the corresponding formula for a European put.

(d) Use the formulas you derived in c to price the vanilla options in a under the jump-diffusion model using the jump size and expected wait time parameters from part b. That is, for a call option you need to evaluate:

$$V_{call,jump} = \sum_{y=0}^{\infty} V_{\delta,call}\left(spot = (1-m)^y S_0\right) P(Y=y)$$

(e) Use these prices to back out the Black-Scholes implied volatilities of the options, and graph them. Comment on the result.