NAME:

MTH 9814
Sample Final Exam
Fall 2018

Instructions:
For the exam, you must work five questions. The three questions from Part I are all required and are worth 25 points each. You must work two of the three questions from Part II; these are worth 15 points each. Indicate in the provided location at the top of the page which two of the Part II questions you would like graded. Any additional work will not be considered in determining your score.

You are allowed writing utensils, a calculator, one page of notes front and back, and blank scratch paper for preparing your answers. However, any work on scratch paper (or in the blue books provided) will **not be graded**. All solutions must be presented in the test packet itself, no exceptions.

Show the steps you followed to arrive at your answer. A numerical answer with no explanation of its origins will not receive full credit; a numerical answer that is incorrect only because of a mistake in the final calculation will receive substantial credit.

Where rounding is necessary, express your answer to five decimal places.

Required problem

1. The term structure of risk-free zero rates is:

maturity	zero rate (continuous)
6M	0.005
1Y	0.0075
18M	0.01
2Y	0.008

For the following, if interpolation is necessary, assume that the continuously compounded forward rates are constant between maturities.

(a) What is the 6-month forward rate in 6 months, expressed with continuous compounding?

$$P_{t,T} = \frac{P_T}{P_t} = \frac{e^{-r_T T}}{e^{-r_t t}} = e^{r_t t - r_T T} = e^{-r_f (T-t)}$$

So:

$$r_f = \frac{r_T T - r_t t}{T - t} = \frac{0.0075 * 1 - 0.005 * 0.5}{1 - 0.5} = 0.01$$

(b) What is the par swap rate of a 2-year swap where both fixed and floating legs pay semiannually?

For a par swap starting today, receiving fixed:

$$V_{fixed} + V_{float} = N c_{par} A_{m,T} + N P_T - N = 0$$
$$c_{par} = \frac{1 - P_T}{A_{m,T}}$$

Then in this case:

$$P_{2Y} = e^{-0.008*2} \approx 0.98412732$$
$$A_{2,2Y} = 0.5e^{-0.005*0.5} + 0.5e^{-0.0075*1} + 0.5e^{-0.01*1.5} + 0.5e^{-0.008*2} \approx 1.979635218$$
$$c_{par} \approx 0.00802$$

(c) What is the 9-month forward price of a 2-year bond that pays 2% interest per annum with semiannual coupons? Express your answer per 100 face amount of the bond; include the accrued coupon in your answer.

For a coupon-bearing bond, the dirty forward price is:

$$V_{fwd,t} = \frac{1}{P_t}\left[\sum_{i=1}^{n} N \frac{c}{m} P_{t_i} + N P_T\right]$$

...for coupon times $t < t_1 < t_2 < \cdots < t_n = T$

In this case, the discount factor at the forward time requires interpolation, using constant forwards (log-linear):

Required problem

$$lnP_{9M} = \frac{lnP_{1Y} - lnP_{6M}}{1 - 0.5}(0.75 - 0.5) + lnP_{6M} = \frac{-0.0075 * 1 + 0.005 * 0.5}{1 - 0.5} - 0.005 * 0.5$$
$$lnP_{9M} = -0.005$$
$$P_{9M} = e^{-0.005} \approx 0.995012479$$

PV of the coupons after the forward time is....

$$V_{bond} = 100 * \frac{0.02}{2} * \mathrm{e}^{-0.0075*1} + 100 * \frac{0.002}{2} * e^{-0.01*1.5} + 100 * \left(1 + \frac{0.002}{2}\right) * e^{-0.008*2}$$
$$V_{bond} \approx 101.3744993$$

So:

$$V_{fwd,t} = \frac{1}{P_t} V_{bond} \approx 101.88264$$

(d) Asset A has spot price 100 and pays no dividends; asset B has spot price 104 and pays a 0.5% dividend continuously. What is a the fair strike price of a forward contract on the price of B less the price of A in 1 year?

$$V_{fwd,t} = S_{0,B} e^{-q_A t} - S_{0,A} e^{-q_b t} - K e^{-rt}$$

In this case:

$$V_{fwd,1} = 104\, e^{-0.005*1} - 100 - K e^{-0.0075*1} = 0$$
$$K = e^{0.0075}[104 e^{-0.005} - 100] \approx 3.50751$$

Required problem

2. A risky obligor has hazard rate 0.01 to all maturities at a time when the risk-free zero rate is 3% to all maturities, expressed with continuous compounding. Assume that, in the event of default, recovery on the obligor's debt will be 40% of the face amount.

(a) What is the unconditional probability that the obligor defaults within the next five years?

$$P_{default,T} = 1 - S_T = 1 - e^{-\lambda T}$$

In this case:

$$P_{default,5Y} = 1 - S_{5Y} = 1 - \mathrm{e}^{-0.01*5} \approx 0.04877$$

(b) What is the probability that the obligor defaults between years three and five, conditional on survival for two years?

$$P[\tau \in (t_1, t_2) | \tau > s] = \frac{S_{t_1} - S_{t_2}}{S_s} = \frac{e^{-\lambda t_1} - e^{-\lambda t_2}}{e^{-\lambda s}} = e^{\lambda(s-t_1)} - e^{\lambda(s-t_2)}$$

In this case:

$$P = e^{0.01*(2-3)} - e^{0.01*(2-5)} \approx 0.01960$$

(c) What is the fair present value of a zero-coupon bond from the obligor with face value 100 maturing in 3 years?

$$V_{bond} = V_{risky\ cash\ flow} + V_{recovery}$$
$$V_{risky\ cash\ flow} = NP_T S_T = Ne^{-(r+\lambda)T}$$
$$V_{recovery} = NR\int_0^T P_t[-S_t']dt = NR\int_0^T e^{-rt}\lambda e^{-\lambda t}dt = \frac{NR\lambda}{r+\lambda}\left(1 - e^{(-r+\lambda)T}\right)$$

In this case:

$$V_{risky\ cash\ flow} = 100 * e^{-(0.03+0.01)*3} \approx 88.69204367$$
$$V_{recovery} = \frac{100 * 0.4 * 0.01}{0.03 + 0.01}\left(1 - e^{-(0.003+0.001)*3}\right) \approx 1.130795633$$

Then:

$$V_{bond} \approx 89.82284$$

(d) What is the risky par yield for a semiannual bond from this obligor with maturity 2 years? Express your answer with semiannual compounding.

For a risky coupon-bearing bond with recovery:

$$V_{bond} = V_{coupons} + V_{bullet} + V_{recovery}$$
$$V_{coupons} = NcA^*_{m,T}$$

...where A^* denotes a credit-risky annuity.

$$V_{bullet} = Ne^{-(r+\lambda)T}$$
$$V_{recovery} = \frac{NR\lambda}{r+\lambda}\left(1 - e^{-(r+\lambda)T}\right)$$

For par coupon c^*:

$$Nc^*A^*_{m,T} + NP_T S_T + \frac{NR\lambda}{r+\lambda}\left(1 - e^{-(r+\lambda)T}\right) = N$$

Required problem

$$c^* = \frac{1}{A^*_{m,T}}\left[1 - e^{-(r+\lambda)T} - \frac{R\lambda}{r+\lambda}\left(1 - e^{-(r+\lambda)T}\right)\right] = \frac{1}{A^*_{m,T}}\left[\left(1 - \frac{R\lambda}{r+\lambda}\right)\left(1 - e^{-(r+\lambda)T}\right)\right]$$

In this case:

$$A^*_{2,2} = 0.5e^{-(0.03+0.01)*0.5} + 0.5e^{-(0.03+0.01)*1} + 0.5e^{-(0.03+0.01)*1.5} + 0.5e^{-(0.03+0.01)*2}$$

$$A^*_{2,2} \approx 1.902934496$$

$$c^* \approx 0.03636$$

Required problem

3. We consider options written on an underlying with the following properties:
$S_0 = 45$
$q = 0.01$
$\sigma = 0.2$
For all options in the below questions, assume that Black-Scholes assumptions hold and take the values:
$r = 0.025$
$T = 0.25$
$K = 48$
The following approximate calculated values are provided for your use in the following problems:

	d1	d2
value	-0.557885	-0.657885
N(value)	0.288461	0.255306
N'(value)	0.341449	0.321311

(a) What is the price of a European call option with these characteristics?

$$V_{call} = S_0 e^{-qT} N(d_1) - K e^{-rT} N(d_2)$$

In this case:

$$V_{call} \approx 0.77000$$

(b) What is the delta of a European put option with these characteristics?

$$\Delta_{put} = -e^{-qT} N(-d_1) = -e^{-qT}\big(1 - N(d_1)\big)$$
$$\Delta_{put} \approx -0.70976$$

(c) What is the gamma of a European call option with these characteristics?

$$\Gamma_{call} = \frac{1}{S\,\sigma\sqrt{T}} e^{-qT} N'(d_1)$$
$$\Gamma_{call} \approx 0.07569$$

(d) What is the vega of a European put option with these characteristics?

$$vega_{put} = S_0 e^{-qT} N'(d_1)\sqrt{T}$$
$$vega_{put} \approx 7.66342$$

(Note: Result involving K is also acceptable, although if rounded according to instructions, the value of vega determined this way is the same.)

(e) An asset-or-nothing call is a European option that gives the holder the asset if $S_T > K$ at expiry, and nothing otherwise. What is the price of an asset-or-nothing call, under Black-Scholes assumptions, with the above inputs?

$$V_{AoN,T} = \begin{cases} S_T & S_T > K \\ 0 & otherwise \end{cases}$$

Then:

Required problem

$$V_{AoN} = \tilde{E}[e^{-rT}V_T] = e^{-rT}\frac{1}{\sqrt{2\,\pi}}\int_{S_T>K} S_0 e^{\left(r-q-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}z} e^{-\frac{1}{2}z^2}dz$$

Note:

$$S_T > K$$

$$S_0 e^{\left(r-q-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}z} > K$$

$$\left(r-q-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}z > -\ln\left(\frac{S_0}{K}\right)$$

$$z > \frac{-\ln\left(\frac{S_0}{K}\right)-\left(r-q-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = -d_2$$

Then:

$$V_{AoN} = S_0 e^{\left(-q-\frac{\sigma^2}{2}\right)T}\frac{1}{\sqrt{2\,\pi}}\int_{-d_2}^{+\infty} e^{-\frac{1}{2}\left(z^2-2\sigma\sqrt{T}z\right)}dz = S_0 e^{-qT}\frac{1}{\sqrt{2\,\pi}}\int_{-d_2}^{+\infty} e^{-\frac{1}{2}\left(z-\sigma\sqrt{T}\right)^2}dz$$

$$V_{AoN} = S_0 e^{-qT}\frac{1}{\sqrt{2\,\pi}}\int_{-d_2-\sigma\sqrt{T}}^{+\infty} e^{-\frac{1}{2}u^2}du = S_0 e^{-qT}\left(1-N(-d_1)\right) = S_0 e^{-qT}N(d_1)$$

So, in this case:

$$V_{AoN} = 12.94833$$

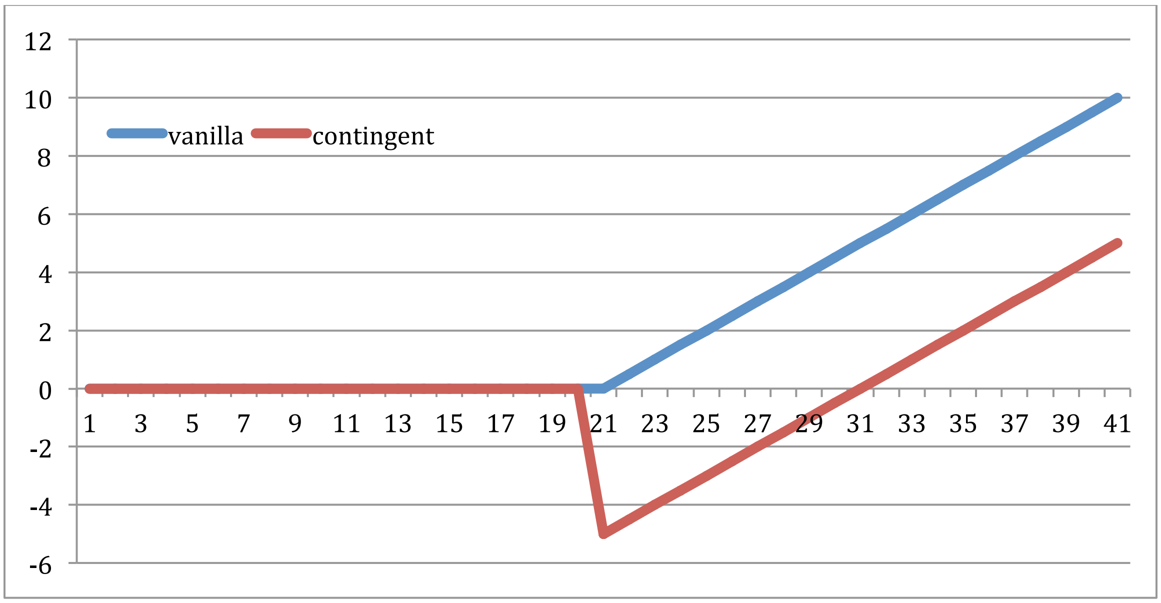
Part II
15 points each

Grade this problem?

4. *European Option With Contingent Premium*
An option with a contingent premium is one where the premium is paid at expiry only if the option expires in the money. The option can be acquired with no initial cash outlay, but exercise of a contingent-premium option is automatic: That is, there are some circumstances where the payoff at expiry is negative, and an additional payment is required.

Let V be the present value of the European vanilla call, and let $V_{C,P}$ be the present value of the same call with contingent premium P. The key values S_0, K, r, q, T, σ are all defined as usual for vanilla options.

(a) On a clearly labeled graph, show the payoff of the vanilla call alongside the payoff of the contingent-premium call with the same attributes.



(b) What is the Black-Scholes valuation formula for a European contingent-premium call? Express your answer in terms of the price of a vanilla European call.

$$V_{contingent\ call,K,P} = AoN_{call,K} - (K+P)CoN_{call,K}$$
$$V_{contingent,call,P} = S_0 e^{-qT} N(d_1) - (K+P)e^{-rT} N(d_2)$$
$$V_{contingent,call,P} = V_{vanilla,K} - Pe^{-rT} N(d_2)$$
$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$
$$d_2 = d_1 - \sigma\sqrt{T}$$

(c) What is the premium P^* of a contingent-premium call that allows the option to be obtained for zero initial cash outlay? Express your answers in terms of F, the forward price of the underlying at expiry.

$$V_{contingent,call,P^*} = 0 = V_{vanilla,K} - P^* e^{-rT} N(d_2)$$
$$P^* e^{-rT} N(d_2) = V_{vanilla,K}$$

Grade this problem?

$$P^* = \frac{S_0 e^{(r-q)T} N(d_1) - K N(d_2)}{N(d_2)}$$
$$P^* = F \frac{N(d_1)}{N(d_2)} - K$$

(d) Find an expression for the delta of a contingent-premium call with premium P.

$$V_{contingent,call,P} = V_{vanilla,K} - P e^{-rT} N(d_2)$$
$$\Delta_{contingent,call,P} = \Delta_{vanilla,K} - P e^{-rT} N'(d_2) \frac{dd_2}{dS}$$
$$\Delta_{contingent,call,P} = e^{-qT} N(d_1) - \frac{P e^{-rT}}{S_0 \sigma \sqrt{T}} N'(d_2)$$

Grade this problem?

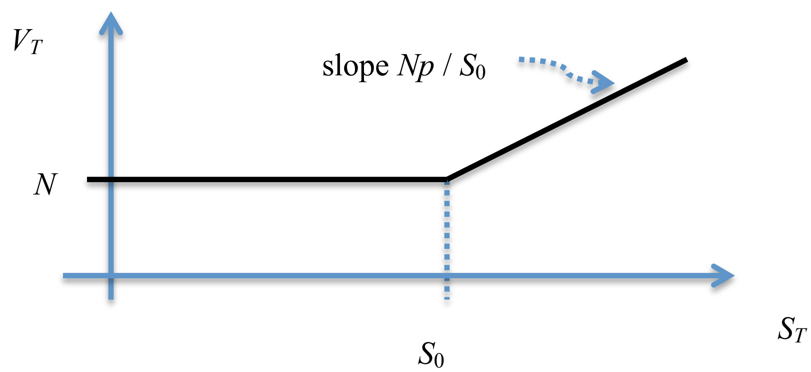
5. *An Equity-Linked Note*
One form of equity-linked note (ELN) that has at times been popular among retail investors outside the US is a principal-protected structure entered at par that works as follows: At inception, the buyer pays the face amount N. At maturity T, the buyer receives repayment of that principal plus an equity-linked coupon whose value is given by...

$$Np\ max\left(\frac{S_T - S_0}{S_0}, 0\right)$$

...where N, S_0, and S_T are defined as usual, and the value p is a participation rate that is set at inception. The buyer thus is guaranteed a return of principal as well as some fraction of any appreciation in the reference asset. The fair participation rate p^* is the one that makes the value of this instrument N at inception.

For the purposes of the following questions, assume the reference asset pays no dividends, and that the continuously compounded risk-free interest rate r is constant to all maturities.

(a) Graph the payoff of this ELN at maturity as a function of the final asset price S_T. Be sure to clearly label how the quantities N, S_0, and p are reflected in your graph.



(b) In the market, suppose you can obtain in any desired quantity zero-coupon bonds maturing at T, the reference asset itself, and European call options on the reference asset with expiry T having any strike price. Using these instruments, construct a static portfolio that replicates the payoff of this ELN at maturity.

Long a zero-coupon bond with face N (price NP_T)
Long call option struck at S_0 on Np^* / S_0 units of underlying asset (price Np^* / S_0 $C_{ATM,T}$)

(c) Use your replicating portfolio to solve for the fair participation rate p^* in terms of the prices of these assets.

$$N = NP_T + \frac{Np^*}{S_0} C_{ATM,T}$$
$$p^* = \frac{S_0(1 - P_T)}{C_{ATM,T}}$$

(d) Use your replicating portfolio to show that, as long as $r > 0$, $0 < p^* < 1$.

Part II
15 points each

Grade this problem?

i:
$r > 0 \to 1 - P_T > 0$
$C_{ATM,T} > 0, S_0 > 0, so$
$p^* > 0$

ii:
By put-call parity:
$C_{ATM,T} - P_{ATM,T} = S_0 - S_0 P_T = S_0(1 - P_T)$
So:
$p^* = \frac{S_0(1 - P_T)}{C_{ATM,T}} = \frac{C_{ATM,T} - P_{ATM,T}}{C_{ATM,T}} = 1 - \frac{P_{ATM,T}}{C_{ATM,T}} < 1$
since both put and call prices are strictly positive.

Grade this problem?

6. *Inflation*

The zero-coupon inflation swap is an OTC derivative where a single payment is exchanged at maturity. The fixed leg pays $N(1+r_K)^T$, where N is the deal notional, r_K is the contract fixed rate, and T is the total tenor (years from inception to maturity) of the deal. The inflation leg pays the indexed notional NI_T/I_0, where I_0 is the inflation index level (CPI) observed at inception, and I_T is the same index observed at maturity. As usual with swaps, the market quote $r_{K,T}$ is the fixed rate for a T-year swap commencing today that can be entered into at zero cost.

An inflation-linked bond is a type of debt issued by many sovereigns. It is similar to a vanilla fixed-coupon bond, except the coupon at time t is paid based on the indexed face amount NI_t/I_0, defined as above, and the final principal payment is the indexed amount NI_T/I_0.

(a) The two-year market quote for a zero-coupon inflation swap today is 1.25%, and the current CPI is 205. What is the expected CPI in 2 years implied by this quote?

$$I_0 = 205, r_{inflation,2} = 0.0125$$

Then:

$$I_2 = I_0\left(1 + r_{inflation,2}\right)^2 \approx 210.15703$$

(b) Four years ago, when CPI was 198, you entered into a 5-year zero-coupon inflation swap, where you pay fixed 0.5% on \$100,000. Today the CPI is 205, the risk-free 1-year interest rate is 2.75% (quoted with annual compounding) and the 1-year market zero-coupon inflation swap rate is 1.5%. What is the value to you of the swap?

At inception:

$$I_{initial} = 198, r_{inflation,K} = 0.005, T = 5, N = 100000$$

On valuation date:

$$I_0 = 205, r_{inflation,1} = 0.015, r = 0.0275$$

swap legs:

$$V_{fixed,T} = 100000(1+0.005)^5 \approx 102525.125312812$$
$$V_{inflation,T} = 100000 * \frac{205(1+0.015)}{198} \approx 105088.383838384$$
$$V_{net,T} = V_{inflation,T} - V_{fixed,T} \approx 2563.25852557144$$

discount:

$$V = V_{net,T}(1+r)^{-1} \approx 2494.6554993396$$

(c) At a time when the interest rate to all maturities is 2% (expressed with annual compounding), the 1-year inflation swap rate is 1.2%, and the 2-year rate is 1.45%. What is the par coupon of an inflation-linked bond paying annual coupons with a 2-year maturity?

Inflation bond with coupon c:

$$V = Nc\left(1 + r_{inflation,1}\right)(1+r)^{-1} + Nc\left(1 + r_{inflation,2}\right)^2(1+r)^{-2} + N\left(1 + r_{inflation,2}\right)^2(1+r)^{-2}$$

Then at par:

Grade this problem?

$$N = Nc\left(1 + r_{inflation,1}\right)(1+r)^{-1} + Nc\left(1 + r_{inflation,2}\right)^2(1+r)^{-2} + N\left(1 + r_{inflation,2}\right)^2(1+r)^{-2}$$

$$1 = c\left[\left(1 + r_{inflation,1}\right)(1+r)^{-1} + \left(1 + r_{inflation,2}\right)^2(1+r)^{-2}\right] + \left(1 + r_{inflation,2}\right)^2(1+r)^{-2}$$

$$c = \frac{\left[1 - \left(1 + r_{inflation,2}\right)^2(1+r)^{-2}\right]}{\left[\left(1 + r_{inflation,1}\right)(1+r)^{-1} + \left(1 + r_{inflation,2}\right)^2(1+r)^{-2}\right]}$$

$$c \approx 0.00542809606974505$$