Intro To Pricing Financial Instruments: Homework 1

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Problem 1

a).

Maturity	Bond Price (per 100 face)	r continuous
1D	99.9995	$1.825 * 10^{-3}$
1W	99.994	$3.129 * 10^{-3}$
1M	99.971	$3.481 * 10^{-3}$
3M	99.9	$4.002 * 10^{-3}$
6M	99.65	$7.012 * 10^{-3}$
1Y	99.05	$9.545 * 10^{-3}$

b).
The forward rates equal $-\frac{\ln P_T - \ln P_t}{T-t}$

Maturity	Forward rates
1D - 1W	$3.334 * 10^{-3}$
1W - 1M	$3.586 * 10^{-3}$
1M - 3M	$4.263 * 10^{-3}$
3M - 6M	0.0100
6M - 1Y	0.0121

c).

$$\left(\frac{1}{52} - \frac{1}{365}\right) * 3.334 * 10^{-3} + \left(\frac{1}{52} - \frac{1}{12}\right) * 3.586 * 10^{-3} + \left(\frac{1}{12} - \frac{1}{4}\right) * 4.263 * 10^{-3} + \left(1 - \frac{1}{4}\right) * 0.0100 + \left(\frac{7}{365} - \frac{1}{2}\right) * 0.121 = 9.545 * 10^{-3} \tag{1}$$

Problem 2

a).

$$disc = \frac{1}{(1 + r_t t)} \tag{2}$$

USD	Rate (percent)	Disc
o/n	0.12800	0.99872
1W	0.16270	0.99978
2W	0.17170	0.99954
1M	0.19043	0.99984
2M	0.22413	0.99963
3M	0.25288	0.99937
4M	0.29463	0.99902
5M	0.35013	0.99854
6M	0.40313	0.99799
7M	0.45913	0.99733
8M	0.51288	0.99659
9M	0.56463	0.99579
10M	0.61913	0.99487
11M	0.67125	0.99389
12M	0.72950	0.99276

For the purpose of this exercise 1W was defined to be $\frac{7}{365}$

b).

For this case, because libor is quoted in simple interest $r_{t_1,t_2} = \frac{1}{t_2-t_1}(\frac{1+r_2t_2}{1+r_1t_1} - 1)$, which can be simplified in this context too $\frac{r_2t_2-r_1t_1}{t_2-t1}$ we get

USD	Forward Rate (percent)
1M - 2M	0.257841
2M - 3M	0.31038
3M - 4M	0.41998
4M - 5M	0.57213
5M - 6M	0.66813
6M - 7M	0.79513
7M - 8M	0.88913
8M - 9M	0.97863
9M - 10M	1.10963
10M - 11M	1.19245
11M - 12M	1.37025

c).

For this case we use $r_m = (1 + tr_t)^{\frac{1}{t}} - 1$ where $t = (1....12)$

USD	Compound Monthly (percent)
1M	0.19043
2M	0.22390
3M	0.25224
4M	0.29333
5M	0.34770
6M	0.39913
7M	0.45300
8M	0.50390
9M	0.55227
10M	0.60253
11M	0.64973
12M	0.70177

Problem 3

a).

First we use the discount factor to get the zero rate of the 3 month and 6 month zero-coupons

$$r_{0.25} = \frac{-ln(0.99875)}{0.25} = .00503 \tag{3}$$

$$r_{0.50} = \frac{-ln(0.99576}{0.5} = 0.008498 \tag{4}$$

Using linear interpolation $r_t = \frac{r_{i+1}-r_i}{t_{i+1}-t_i}(t - t_i) + r_i$
we get $r_{\frac{1}{3}} = 0.006168$

Now convert too discount factor $P_{\frac{1}{3}} = e^{\frac{-r_{\frac{1}{3}}}{3}}$ and get $P_{\frac{1}{3}} = 0.997497$

b). Using piecewise constant forward rates we get

$$ln(P_t) = \frac{ln(P_{t+i}) - ln(P_{t_i})}{t_{i+1} - t_i}(t - t_i) + ln(P_{t_i}) \tag{5}$$

Problem 3 continued on next page...

$$ln(P_t) = \frac{(ln(0.98807) - ln(0.9576)) * 2}{3} + ln(0.99576) \quad (6)$$

So: $ln(P_t) = 0.1663$ and $P_t = 0.99063$

c).

Using the first method we must compute the linear interpolation for the discount factor at 10 months

$$r_{0.5} = 0.008498 \quad (7)$$

$$r_1 = .012 \quad (8)$$

$$r_{10m} = \frac{0.012 - 0.008498}{\frac{1}{2}}(\frac{1}{3}) + .008498 = .010832 \quad (9)$$

$$P_{10m} = e^{-r_{10m}*10m} = 0.99101 \quad (10)$$

and recall $P_{4m} = 0.997497$

Using forward rate $= -\frac{lnP_T - lnP_t}{T-t}$ we get
Forward rate at 10m $= \frac{ln(0.99101) - ln(0.997947)}{\frac{1}{2}} = 0.1395$

Using the second method, we must first interpolate for the 4-month discount factor using:

$$ln(P_t) = \frac{ln(P_{t+i}) - ln(P_{t_i})}{t_{i+1} - t_i}(t - t_i) + ln(P_{t_i}) \quad (11)$$

$$ln(P_t) = \frac{(ln(0.99576) - ln(0.99875))}{3} + ln(0.99875) \quad (12)$$

So: $P_{4m} = 0.997752$
Recall $P_{10m} = 0.99603$
and the forward rate computed as above:

$$fracln(0.99063) - ln(0.99752)\frac{1}{2} = .014335 \quad (13)$$

Problem 4

a). The constant forward rate equals the forward rate which is
$-\frac{-3*.0016-.00125}{2} = .001775$
b).

$$r = \frac{.0016 - .00125}{2}(t-1) + .00125 = .00175t_{+}.01075 \quad (14)$$

The discount factor of t is:

$$e^{-rt} = e^{-(.00175t+.01705)t} = e^{-.00175t^2-.01705t} \quad (15)$$

$$r = -\frac{ln(P_{t+\Delta t}) - ln(P_t)}{\Delta t} = .00175 * 2t + .0175 \quad (16)$$

From this we conclude $r_{1y} = .01425$ and $r_{3y} = .02125$

c).

$$\frac{d}{dt}(r(t)t) = 4(-0.001138889)t^3 + 3(0.00163889)t^2 + .001t + 0.002 \tag{17}$$

From this we conclude $r_{1y} = 0.02236$ and $r_{3y} = -0.01675$

Problem 5

a).

$$100e^{-0.024*3}3.5 * (e^{-0.024*3} + e^{-0.022*2} + e^{-0.015}) = 103.107 \tag{18}$$

b). To compute semi annual par yield we use

$$y = \frac{1 - P_t}{\Sigma_1^n \frac{1}{m} P_{t_i}} \tag{19}$$

where $m = 2$ because it is semi annual
Accordingly $P_{t_1}...P_{t_{11}} = 3.5/2$ and $P_{t_{12}} = 100 + 3.5/2$
To compute the P_{t_i} we must first interpolate to find 18 month spot rate using the formula $r_t = \frac{r_{i+1}-r_i}{t_{i+1}-t_i}(t - t_i) + r_i$ and we find $r_{18m} = 0.185$
we have:

$$y = \frac{1 - P_T}{\Sigma_1^n \frac{1}{m} P_{t_i}} \tag{20}$$

$$P_T = e^{-2*.022)} \tag{21}$$

Plugging this all in we get

$$y = 0.02202 \tag{22}$$

c).

$$V = \Sigma_1^n c_i e^{-yt_i} \tag{23}$$

$$V = 100e^{-.0015} + 2.5e^{-.0011*.5} + 2.5 * e^{-0.015} = 103.46 \tag{24}$$

Plugging V into equation 24 we get $y = .01495$

Problem 6

a).

$$-\frac{ln(0.99875}{.5} = .0025 \tag{25}$$

b).

$$2(1 + .005 * .5)^{-2*.5} + 102(1 + .005 * .5)^{-2} = 103.486 \tag{26}$$

By decomposing the yield into a series of zero coupon bonds, we can now find the value of the first coupon payment using the rate found in part a)

$$PV(continuous) = e^{-.5*.0025} = 1.9975 \tag{27}$$

From here was can determine the present value of the final payment

$$103.4869186 - 1.9975 = 101.4894186 = PV_1 \tag{28}$$

From here it is easy to compute $P_1 = PV_1/CF = 0.9949943$
Now we have:

$$r_1 = -ln(P_1 * 1) = 0.00501827 \tag{29}$$

c).
We can compute the bond price through the yield as before and get

$$V = \Sigma_1^n N\frac{c}{m}(1 + \frac{y_m}{m})^{-mt_i} \tag{30}$$

$$(1 + .0105 * .5)^{-2*.5} + (1 + .0105 * .5)^{-2*1} + (1 + .0105 * .5)^{-2*1.5} + (1 + .0105 * .5)^{-2*2} = 101.87 \tag{31}$$

We can then decompose this into a series of zero-coupon bonds, where we have the continuous rates for $t = .5$ and $t = 1$ from above.
Because we have the continuous present value at $t = .5andt = 1$ from above, we can compute the composite present value of the remaining payments. $101.875322 - .99875 - .9949943 = 99.881577$
Using linear interpolation we also know that $r_{1.5} = \frac{(r_2 - r_1)}{2} + r_1$
This leaves us with:

$$99.881577 = e^{-(\frac{r_2 - .005)}{2} + .005)1.5} + 101e^{-2r_2} \tag{32}$$

using a non linear solver we get $r_2 = 0.01054$

Problem 7

a).

$$P = \Sigma_1^{10} e^{-i*0.5*0.025} * 2 + 100e^{-5*.025} = 106.93 \tag{33}$$

b).

$$D = \frac{\Sigma_1^{10} i * 0.5 * e^{-i*0.5*0.025} * 2 + 5 * 100e^{-5*.025}}{P} = 4.598 \tag{34}$$

$$C = 22.264336 \tag{35}$$

c). d).

Problem 8

This is a placeholder for number 8

Problem 9

Application 2).
a).
To begin, lets begin with two cash flows, as assume by definition that $t_1 > 0$ and $t_2 > 0$ For a single cash flow we know that $D = t$ and that $C = t_2$
Then $D_\Pi = w_1 t_1 + w_2 t_2$ and $C = w_1 t_1^2 + w_2 t_2^2$
where $w_1 = \frac{w_1}{w_1 + w_2}$ and $w_2 = \frac{w_2}{w_1 + w_2}$

We know:

$$(\frac{t_1}{t_2} - 1)^2 >= 0 \tag{36}$$

$$\frac{t_1^2}{t_2^2} - 2\frac{t_1}{t_2} + 1 >= 0 \tag{37}$$

$$\frac{t_1^2}{t_2^2} + 1 >= 2\frac{t_1}{t_2} \tag{38}$$

$$\frac{t_1}{t_2} + \frac{t_2}{t_1} >= 2 \tag{39}$$

$$t_1^2 + t_2^2 >= 2t_1t_2 \tag{40}$$

$$w_1w_2(t_1^2 + t_2^2) >= 2t_1t_2w_1w_2 \tag{41}$$

Next we set $w_1 = 1 - w_2$ and distribute

$$w_1(t_1^2 - w_1t_1^2) + w_2(t_2^2 - w_2t_2^2) >= 2t_1t_2w_1w_2 \tag{42}$$

$$w_1t_1^2 + w_2t_2^2 >= 2t_1t_2w_1w_2 + w_1^2t_1^2 + w_2^2t_2^2 \tag{43}$$

$$C_\Pi >= D_\Pi^2 \tag{44}$$

Now lets assume more than an additional cash flow.

If we had w_1 invested in the previously constructed portfolio, and added w_2 of an additional cash flow, by the same proof above the results will hold.

b).

The equality above only holds when $t_1 = t_2$, which is obvious from the first line. Lets now for any non - zero coupon bond we can construct it in such a fashion that it is comprised of zero-coupon bonds at each cash flow period. In each period, some weight is being returned. Under this construct the convexity of that portfolio must be greater than if all of cash flows were received at expiry by the proof above.

c).

This does not hold for non-negative cash flows. Take for instance a portfolio of completely negative cashflows, as ridiculous as it sound, $D_\Pi^2 > 0$ and $C_\Pi < 0$