3. Floating-Rate Instruments

Forward rate agreement (FRA)

A forward rate agreement is an OTC short-term interest rate derivative, usually written on 3M LIBOR, in which one party agrees to pay a fixed strike rate r_K on some notional N over a future time period from t to T ($t < T$), receiving in return the risk-free floating interest rate r observed at time t to maturity $T - t$. In ordinary FRA's all rates are quoted using simple interest. This makes the terminal value of the instrument to the party paying fixed:

$$V_{FRA,T} = N(r - r_K)(T - t) = N(r - r_K)\Delta t$$

Note that the payment is assumed to occur in arrears—that is, at the terminal time T. (Most FRA's are settled differently, but the settlement amount reflects the present value of this payoff as if it occurred at T.) The value at maturity is the difference in interest rates only applied to the notional; thus, this payoff is generally quite small in comparison with the notional amount.

Pricing an FRA

Suppose we have arranged to receive fixed in an FRA contract, meaning we are effectively short the floating rate r, and we wish to determine the present value of the position. To do this, we imagine a hedging strategy designed to remove the risk associated with paying the floating rate—that is, we want to somehow arrange to receive r paid on the notional N over the period from t to T using traded instruments. In this case, we adopt the usual assumption that risk-free zero-coupon bonds are available to be traded to all maturities, and that we can go short or long these bonds in any amount at the same price.

To receive r, we must buy risk-free bonds with maturity $T - t$ when t arrives. In our initial discussion of forward rates, we saw how we can arrange to do this today: We invest in a zero-coupon bond that pays N at t; to do this, we must pay NP_t now. In order to make the hedge self-financing, we obtain this cash by selling zero-coupon bonds that mature at T. When time t arrives, we will use the cash proceeds of the maturing bond to purchase a bond whose maturity at that time is $T - t$. Finally, at T we will use the payoff from this bond to help repay the bond we initially sold at time zero.

The net final cash flow from this hedge may be positive or negative depending upon the price of the bond with maturity $T - t$ when t arrives. As we have seen, the cash flow required to set up the hedge at time zero is zero, and similarly the strategy we describe here has no net cash flow when t arrives. Thus, this final cash flow is the only cash flow from the hedge.

What is its amount? The bond we initially bought cost NP_t in cash, and at the time the discount factor to maturity was P_T. This means that the amount owed at T is NP_t / P_T. Recall that…

$$P_{t,T} = \frac{P_T}{P_t} = (1 + r_{t,T}\Delta t)^{-1}$$

…where $r_{t,T}$ is the forward rate from t to T as seen at time zero, expressed as a rate of simple interest. Thus the cash flow from the short bond position is:

$$-N\frac{P_t}{P_T} = -N\left(1 + r_{t,T}\Delta t\right)$$

The positive cash flow in our hedge arises from the bond we purchased using the proceeds of the bond that matured at time t. Since we purchased this at the same time the floating rate r from the FRA is observed, we receive this rate on the investment of N, meaning that the final positive cash flow is…

$$N\left(1 + r\Delta t\right)$$

…making the net cash flow at time T of the hedge:

$$V_{T,hedge} = N\left(1 + r\Delta t\right) - N\left(1 + r_{t,T}\Delta t\right) = N\left(r - r_{t,T}\right)\Delta t$$

We have seen that it is possible to obtain this payoff with zero net cash outlay. This is interesting, since the above is the payoff of an FRA with the strike rate set to the forward rate from t to T, and the value today of this instrument must be zero, or else we would not be able to construct it for free.

Consider now the terminal payoff of the portfolio Π formed of our initial FRA position and the hedge:

$$V_{\Pi,T} = V_{FRA,T} + V_{hedge,T} = N\left(r_K - r\right)\Delta t + N\left(r - r_{t,T}\right)\Delta t$$

$$V_{\Pi,T} = N\left(r_K - r_{t,T}\right)\Delta t$$

All of the terms in this expression are known at time zero, meaning that its present value can be determined simply by discounting as we would any other deterministic cash flow:

$$V_{\Pi} = NP_T\left(r_K - r_{t,T}\right)\Delta t$$

Furthermore, in present value terms we know that…

$$V_{\Pi} = V_{FRA} + V_{hedge}$$

…and…

$$V_{hedge} = 0$$

…since we can construct it, as we have mentioned, without any net cash outlay. This means that the present value of the FRA and the present value of the hedged portfolio must be the same, leading us finally to the pricing formula…

$$V_{FRA} = NP_T\left(r_K - r_{t,T}\right)\Delta t$$

…for the party receiving fixed, and obviously the opposite for the party paying fixed. That is, the price obtained if we assume that the floating rate $r = r_{t,T}$ is the arbitrage-free price.

The FRA is not a tremendously common instrument in the market, but the importance of this result should not be judged by that fact. What we have done here is show that we can price a payoff of this form by *assuming that the forward rate is realized*. As it turns out, this is by far not true of all payoffs that depend upon interest rates, but instruments that share this feature are among the most important instruments in fixed income.

Floating-rate note (FRN)
For a fixed-coupon bond, interest rates pose a major risk to the holder. Bondholders often demand an additional premium for assuming this risk; issuers are often willing to pay that premium, since a fixed interest cost is easier to plan for from a cash flow perspective. However, an issuer usually is able to obtain a better rate in the short term by agreeing to pay a floating rate of interest, assuming the interest-rate risk itself instead of transferring it to the holder. The most common means of doing this is called a floating-rate note.

An FRN is specified in a similar way to the fixed-coupon bonds we have already seen. The note has principal amount N, coupon frequency m (usually four for an FRN—i.e., quarterly), and maturity T. The coupon is calculated by consulting the risk-free zero rate quoted as a rate of simple interest (usually, as above with FRA's, a maturity of LIBOR) at the beginning of each coupon period and paying it at the end. Thus, the floating coupons are of the form Nr / m, where r is observed at the beginning of the coupon period for a maturity coinciding with the time the coupon is paid. The FRN may also pay a coupon spread s to the floating rate, thus giving it some characteristics of fixed-rate issuance as well; in this case, the coupons are of the form $N(r + s) / m$. All rates are, as we have seen repeatedly, expressed as rates of simple interest. An FRN, like usual fixed-rate issuance, generally repays its principal in a bullet at the maturity T.

For the moment, we will consider only FRN's paying LIBOR flat—that is, ones where the coupon spread s is zero. Such a bond consists, then, of a floating-rate annuity and a bullet repayment of principal at maturity. For the i^{th} coupon cash flow c_i, we observe the floating rate r_i at t_{i-1} and pay it at t_i, giving the payoff:
$$c_i = Nr_i\Delta t$$

To value the coupon, we observe that this is the floating potion only of an FRA payoff. Since our initial consideration of FRA's imposed no particular constraint on the fixed strike rate r_K, we must be able to price this coupon payment as an FRA with a strike rate of zero. That is, we can determine the fair present value of coupons whose reset has not yet taken place by assuming that today's forward rates are realized.

Value of an FRN paying LIBOR flat
Suppose that we wish to value an FRN before its next floating coupon is reset so that all of its coupons are still truly floating. Since, as we have seen, we may price by assuming that the forward rates are realized, the present value of the cash flows of an FRN is of the form:
$$V = \sum_{i=1}^{n} NP_{t_i} \frac{r_{t_{i-1},t_i}}{m} + NP_T$$

We recall from the above, however, that since $\Delta t = 1/m$, the forward rate can be expressed in terms of discount factors as:

$$1 + \frac{r_{t_{i-1},t_i}}{m} = \frac{P_{t_{i-1}}}{P_{t_i}}$$

$$r_{t_{i-1},t_i} = m\left(\frac{P_{t_{i-1}}}{P_{t_i}} - 1\right)$$

Consider, then, just the last two terms of the valuation equation, recalling that this coupon and the final repayment of principal occur at the same time:

$$\frac{V}{N} = \sum_{i=1}^{n-1} P_{t_i} \frac{r_{t_{i-1},t_i}}{m} + P_T m\left(\frac{P_{t_{n-1}}}{P_T} - 1\right)\frac{1}{m} + P_T$$

$$\frac{V}{N} = \sum_{i=1}^{n-1} P_{t_i} \frac{r_{t_{i-1},t_i}}{m} + \left(P_{t_{n-1}} - P_T\right) + P_T$$

$$\frac{V}{N} = \sum_{i=1}^{n-1} P_{t_i} \frac{r_{t_{i-1},t_i}}{m} + P_{t_{n-1}}$$

Thus, the present value of the terminal payment at T, including both the bullet repayment of principal and the last floating coupon, is the same as the present value of a single payment of the principal one coupon period earlier. This makes perfect sense, since after all the purpose of the floating coupon is to compensate the holder, exactly, for the time value of the principal over each coupon period. There should after all be no preference for either receiving the principal amount at some time t or receiving that same amount at $T > t$ along with the proceeds of a risk-free investment of the principal amount over the intervening period.

Returning to our valuation equation, we see that the sum can be telescoped backwards all the way to the zeroth time point—that is, the time when the next floating-rate reset occurs:

$$\frac{V}{N} = P_{t_0}$$

$$V = NP_{t_0}$$

If we value the instrument the instant before a reset and disregard any cash flows from previous resets, then the discount factor is 1, making this value simply N. If, on the other hand, the FRN starts at some time in the future, then its present value is the same as the face amount N received at the time of the next floating-rate reset. That is, the instrument behaves like a zero-coupon bond that pays at the time of the next coupon reset.

IR DV01 and General Methods to Calculate Duration

We originally introduced the concept of interest rate risk for fixed-coupon bonds in terms of duration; the duration measure, in turn, depends upon a calculation of the instrument's yield. In the case of an FRN, however, it is unclear how the replication argument we originally made to motivate the yield calculation would apply. Since the purpose of duration is to measure interest rate sensitivity, it makes sense to look for a definition that's a bit more general to accommodate FRN's as well as a great number of other instruments whose yields are difficult to define.

A common measure is the interest rate DV01, which is the PL the instrument experiences in the event of a 1bp parallel shift upward in interest rates. This measure requires only a pricing model in order to calculate, and thus it can be found for essentially any instrument. Since it is dollar-denominated, it is easy to aggregate across an entire portfolio. Its shortcoming is that the number by itself is not particularly informative: Fixed-income traders, portfolio managers, and risk managers prefer durations because the measure is free of currency units and thus can easily be compared with other benchmark bonds or portfolios.

Recall that our definition of duration depends primarily upon the derivative of the instrument's value with respect to its yield y, and that furthermore the value of the instrument obeys:

$$V = \sum_{i=1}^{n} N\frac{c}{m}e^{-yt_i} + Ne^{-yT}$$

This holds whether or not the bond is considered risk-free. For the specific case of a risk-free bond, the value V can also be expressed as…

$$V = \sum_{i=1}^{n} N\frac{c}{m}e^{-r_{t_i}t_i} + Ne^{-r_T T}$$

…where the interest rates are the risk-free zero rates to each time expressed with continuous compounding. These interest rates are of course generally different from one another, so we cannot as such calculate a single "derivative" with respect to them all. However, if we imagine applying a parallel shock of size Δr to all of these rates, then…

$$V(\Delta r) = \sum_{i=1}^{n} N\frac{c}{m}e^{-(r_{t_i}+\Delta r)t_i} + Ne^{-(r_T+\Delta r)T}$$

…and we actually can calculate the derivative with respect to this quantity:

$$V'(\Delta r) = -\sum_{i=1}^{n} N\frac{c}{m}t_i e^{-(r_{t_i}+\Delta r)t_i} - NTe^{-(r_T+\Delta r)T}$$

Evaluating this expression at $\Delta r = 0$ gives the expression for a quantity that, without too much notational abuse, we label $dV / d\Delta r$ and will refer to as the dollar duration…

$$V'(0) = \frac{dV}{d\Delta r} = -\sum_{i=1}^{n} Nt_i\frac{c}{m}e^{-r_{t_i}t_i} - NTe^{-r_T T}$$

…which bears a fairly striking resemblance to the result obtained by differentiating with respect to yield:

$$\frac{dV}{dy} = -\sum_{i=1}^{n} Nt_i\frac{c}{m}e^{-yt_i} - NTe^{-yT}$$

The two expressions clearly will generally not give the same value except in the case when the term structure of interest rates is flat, in which case of course the risk-free rate is also equal to y for this risk-free bond. However, the difference is extremely small, and the expression that takes account of the actual term structure of rates could be considered more exact.

The interest rate DV01 is merely the expression we have developed above with Δr = 0.0001. Thus, we can devise a close numerical approximation of the dollar duration using the forward finite difference:

$$\frac{dV}{d\Delta r} \approx \frac{DV01}{0.0001}$$

Even better, we can approximate this quantity using an additional down shock of the same amount using a central finite difference approximation:

$$\frac{dV}{d\Delta r} \approx \frac{V(\Delta r = 0.0001) - V(\Delta r = -0.0001)}{0.0002}$$

This suggests that for any arbitrary instrument whose value depends upon interest rates, the following expression provides a suitable analog for duration:

$$D_{num} = -\frac{1}{V}\frac{V(\Delta r) - V(-\Delta r)}{2\Delta r}$$

Here, we call this quantity the numerical duration. In cases where duration is well defined using the methods we have previously discussed, this gives a result that is extremely close to the value obtained via yield; it also permits the calculation of duration for instruments whose yield is ill defined. Further, it provides the additional flexibility that Δr can be chosen in a manner appropriate for the valuation function V, since some valuation models may be locally a bit jumpy and thus give odd results under shocks that are too small.

Of course, we may define the corresponding expression for numerical convexity also:

$$C_{num} = \frac{1}{V}\frac{V(\Delta r) - 2V_0 + V(-\Delta r)}{\Delta r^2}$$

In the case of a risk-free FRN paying LIBOR flat, it is easy to see how these expressions apply. We have previously seen that a risk-free FRN immediately before its reset has value equal to the bond's face value, irrespective of the term structure of interest rates. Thus, it is plain that a shift to the interest rates has no effect on the instrument's value—as indeed it ought not to, since the purpose of the instrument is to offset interest-rate risk. Thus, the values $V(\Delta r)$ and $V(-\Delta r)$ are precisely the same, yielding a duration of zero, as one would expect since duration is a measure of interest-rate risk. In the case where the next reset occurs at some time in the future, then as we have seen the instrument behaves like a risk-free zero-coupon bond with maturity equal to the next reset; the time to that reset is the duration of the floating portion of the bond.

With fixed-coupon bonds, we decomposed the instrument in some cases as a zero-coupon bond (the bullet repayment of principal) and a fixed coupon strip (the annuity). It seems natural to do a similar thing with an FRN, since its principal payment is also a zero-coupon bond; the floating coupon strip, then, is a floating-rate annuity whose properties are evidently somewhat unusual. After all, we have seen that the duration of the bullet is equal to *T*, the maturity of the bond. Evidently the coupon strip by itself has a duration that enables it, when combined with the zero-coupon bond, to produce an instrument whose duration is zero. In order to determine the characteristics of the floating-rate annuity, we must first consider the question of how we determine the aggregate duration of a collection of instruments.

Aggregate duration

A further advantage of defining the quantity we have called the dollar duration is that it no longer depends upon any determination of yield, and so it is easy to apply to a collection of instruments whose values depend upon the same interest rate curve even when they do not all have the same yield. For a portfolio Π consisting of *n* instruments with values $V_1, V_2, \ldots, V_n$, the portfolio's aggregate duration calculated by the method shown above is given by…

$$D_\Pi = -\frac{1}{V_\Pi}\frac{dV_\Pi}{d\Delta r} = -\frac{\sum_{i=1}^{n}\frac{dV_i}{d\Delta r}}{\sum_{i=1}^{n}V_i}$$

…and since…

$$D_i = -\frac{1}{V_i}\frac{dV_i}{d\Delta r}$$

$$\frac{dV_i}{d\Delta r} = -D_i V_i$$

…we conclude that:

$$D_\Pi = \frac{\sum_{i=1}^{n} D_i V_i}{\sum_{i=1}^{n} V_i}$$

Note that the duration of a risk-free fixed-coupon bond calculated via dollar duration is a special case of this expression, since the durations of the individual cash flows are the times at which they occur, and unlike the traditional duration calculation, the weight accorded each cash flow is its actual present value, rather than its amount discounted by the yield.

In the case of a risk-free FRN paying LIBOR flat immediately before its first reset, as mentioned above we can decompose the instrument as follows…

$$V_{FRN} = N = NA_{float,m,T} + NP_T$$

…into a zero-coupon bond and a floating-rate annuity. This makes it clear that the value of the annuity is…

$$A_{float,m,T} = 1 - P_T$$

…and that its duration can thus be determined:

$$D_{FRN} = 0 = \frac{1}{N}\left(D_{A,float}N(1-P_T) + TNP_T\right)$$

$$D_{A,float} = -T\frac{P_T}{1-P_T}$$

This is an appealingly simple result for an instrument that otherwise appears so complex, particularly since it has no dependency on the reset frequency m. To offset the positive duration T of the bullet, the coupon strip has a large negative duration.

Par yield and the interest rate swap

We initially approached par yield as the coupon rate c^* that satisfies the relation:

$$c^* A_{m,T} + P_T = 1$$

$$c^* = \frac{1-P_T}{A_{m,T}}$$

…where $A_{m,T}$ is the present value of a fixed-rate annuity with maturity T and payment frequency m. The motivation was to find the fair coupon of a fixed-coupon bond—i.e., the coupon rate for which the present value of a bond was also equal to its face value.

Suppose for the sake of argument that one purchased such a bond using cash obtained from borrowing in the market. The borrowing is done, we imagine, so that the terminal repayment of principal from the long fixed-coupon bond occurs at the same time as the repayment of principal owed on the debt. Imagine furthermore that the debt we issue to finance the bond's purchase is an FRN.

By design, the present value of this combination at inception is zero. Its holder receives a fixed rate of interest over the life of the instrument and in return pays a floating rate. Very often, the fixed interest payments occur semiannually, while the floating interest payments occur quarterly. However, as we have already seen, the frequency of payment on the floating leg is immaterial to both the present value and the interest-rate sensitivity of the leg in the context of arbitrage-free pricing.

At maturity, the repayment of principal from the fixed leg exactly offsets the amount owed on the floating leg, resulting in no net principal cash flow. In concept, the instrument reduces to a series of fixed-for-floating interest-rate exchanges, and thus can be seen as an extension of the FRA to multiple periods.

This instrument is a fixed-for-float interest rate swap, the most heavily traded OTC interest rate derivative. While other swap structures based on interest rates are traded, this one is so common that it is often simply referred to as an interest rate swap.

Generally, these contracts are entered into at par, which in the context of a swap means that its value at inception is zero. Suppose, for example, that we wish to determine the fair fixed rate c^* of a swap commencing at t and maturing at T. If we imagine the value of this position to the party receiving fixed on notional N, then the present value of the forward-starting fixed leg is…

$$V_{fixed} = Nc^* A_{m,t,T} + NP_T$$

…while the value of the forward-starting float leg is, as we have seen:

$$V_{float} = -NP_t$$

Thus, we can determine the fair fixed rate as follows:

$$V_{swap,0} = V_{fixed} + V_{float} = 0$$

$$Nc^* A_{m,t,T} + NP_T - NP_t = 0$$

$$c^* = \frac{P_t - P_T}{A_{m,t,T}}$$

In the case where the swap is entered into today, the value P_t is 1, and this reduces to the same expression as we saw above in the context of a risk-free bond. Indeed, for a forward-starting risk-free bond, the expression for par yield matches this exactly.

This is a sensible result; in an arbitrage-free context, it should not matter whether the amount N is paid in cash upfront (as with a bond) or whether it is borrowed at the risk-free rate (as with a swap). In the swap context, this fair fixed rate is usually called the par swap rate. In practice, the only real difference between this and the par yield is not the method of calculation, but the interest rate curve with which it relates. Par yields typically reference sovereign bonds, and lead to one risk-free rate curve for government issuance. Par swap rates typically reference LIBOR, and lead to a different risk-free rate curve for high-quality corporate issuance.

Value and Duration of an IR swap

Suppose we have previously entered into swap contract to receive the fixed rate c and wish to value the position at some later time. We have already seen that this can be accomplished by valuing the fixed and floating legs separately by discounting cash flows, but it is instructive to imagine how it could be accomplished via hedging as well.

If we receive the fixed rate c, then we are short the floating rate. If we imagine that swaps can be freely traded with any frequency to any maturity, then there is a traded instrument with par swap rate c^* that can be used to eliminate the risk associated with floating-rate resets. To do this, we enter into an offsetting swap in which we pay the fixed rate c^* and receive the floating rate.

Since we can enter into this contract at zero cost, the initial swap and the hedged portfolio must have the same present value. Furthermore, the hedged portfolio consists now of only deterministic cash flows, since the floating payments now offset one another. At each fixed payment date, the net payment exchange from our perspective is $N(c - c^*)\Delta t$, where Δt is the time between fixed-leg payments. The present value of this strip of payments is therefore…

$$V_\Pi = V_{receive} = NA_{m,T}\left(c - c^*\right)$$

…and the opposite for the party paying fixed. That is, if swap rates move down over the term of the contract, the party receiving fixed realizes a gain. The party receiving fixed is short the swap rate, while the party paying fixed is long. Expressed in units of the annuity, the value of the swap is linear in this difference in swap rates, a fact that we will make use of much later.

The usefulness of this instrument becomes apparent when we consider the question of its duration. As we have seen, for a portfolio of multiple instruments, we can define the duration as…

$$D_\Pi = -\frac{1}{V_\Pi}\frac{dV_\Pi}{d\Delta r} = \frac{\sum_{i=1}^{n} D_i V_i}{\sum_{i=1}^{n} V_i}$$

…where the numerator in the rightmost expression is the dollar duration of the portfolio. If we treat a swap immediately before its next reset date as a portfolio of a long fixed bond position and a short FRN position, then this expression becomes:

$$\frac{dV_\Pi}{d\Delta r} = D_{fixed}V_{fixed} - D_{float}V_{float} = D_{fixed}V_{fixed} - 0 * N = D_{fixed}V_{fixed}$$

Since the duration of the floating leg is zero, the dollar duration of the swap is simply equal to the dollar duration of a fixed-coupon bond with the same maturity and coupon frequency as the fixed leg of the swap. The net value of the swap, however, is smaller…

$$V_\Pi = V_{fixed} - N$$

…making the final expression of duration:

$$D_{swap} = -\frac{D_{fixed}V_{fixed}}{V_{fixed} - N} = D_{fixed}\frac{V_{fixed}}{N - V_{fixed}}$$

Clearly, since duration is a return-based analytic, it is not entirely appropriate for the description of derivatives. The dollar duration of the swap is equivalent to that of the fixed leg alone, but the leverage inherent in the instrument, which arises from the floating financing leg, gives the instrument a duration that is large in absolute value, and indeed for a par swap is infinite. Furthermore, near par the duration expressed in this manner tends to fluctuate from large positive to large negative depending upon the par swap rate. The leverage inherent in the instrument is indeed its main reason for existence: Whether it is used as a speculative vehicle or as a hedge, the swap offers the interest-rate sensitivity of a bond, which requires a large outlay of cash, at essentially zero cost.

For this reason, it is conventional to report the duration of an interest rate swap as simply the duration of the fixed leg, perhaps with some small adjustment between floating-leg payments to account for the known reset. But the reported duration for these instruments—and, more generally, fixed-income instruments that include leverage—should not be confused with their mathematical durations.

Bootstrapping discount factor curves

In general, for an instrument with sensitivity to interest rates, we may express its value as some function…

$$V_i = f_i\left(P_t(t)\right)$$

…where P_t is a term structure of discount factors. As we have seen, in practical terms this term structure often takes the form of some set of known knot points t_j, P_j and an interpolation function that is used to determine intermediate values.

Of course, this term structure is not known *a priori*, and must be determined from the market prices of traded instruments chosen as "risk-free" reference assets. It is important to understand that this designation of risk-free assets is an exercise in determining reference returns, not a suggestion that the assets themselves are completely without risk.

As has been suggested, it is common to choose different risk-free rates depending upon the instrument being priced. To do this, we consider the instruments to which it is most usefully comparable. For a sovereign bond, these are often other bonds from the same issuer; for a corporate bond, they may be some combination of deposit rates, LIBOR fixings, FRA's, short-term interest rate futures, and swaps.

Once a set of instruments with current prices $V_1, V_2, \ldots, V_N$ are chosen, we select maturities of interest $T_1, T_2, \ldots, T_M$ for the knot points along with the interpolation method that will define $P_t(t)$ at intermediate times. If $M < N$, then in general the resulting nonlinear system of equations has no exact solution, and some method must be chosen to minimize the pricing error across the instruments.

If $M = N$, and the T_i are chosen to coincide with the maturities of the instruments, then generally a curve can be fit that exactly returns the values of all the chosen instruments. The method that must be used to do this depends in part upon the interpolation method chosen in the discount factor curve.

Suppose that we observe the following market data:

data type	market value
overnight rate	0.0018
spot 3M LIBOR	0.0035
3M LIBOR FRA, 3M	0.0038
1Y swap, s/a	0.0065
2Y swap, s/a	0.012
5Y swap, s/a	0.035

We wish to create a discount factor curve that matches these market values. We assume to begin with that we will interpolate in our curve using the method of piecewise constant forwards between knot points. We choose knot points 0, 0.25, 0.5, 1, 2, and 5. We can then consider the instruments one at a time in order of increasing maturity, solving for the discount factor that correctly matches the observed market value at each step. This process is called bootstrapping an interest rate curve.

The initial point, at time zero, has discount factor 1; we associate this with the instantaneous (overnight) rate, but in fact in this use of the data, the instantaneous interest rate is not truly needed.

The spot 3M LIBOR fixing provides the discount factor at time 0.25, since we can generate it directly from the rate. So far, then, our discount factor curve looks like this:

t	df	r (continuous)
0	1	0.0018
0.25	0.999125765	0.00349847

The quote for the 3M LIBOR FRA expiring in 3M is the fair fixed rate on a contract currently trading at zero value. As we have seen, this fair strike rate is equal to the forward rate from 3M to 6M expressed with simple interest, so we can obtain the discount factor at 0.5 using this forward rate in conjunction with the previously obtained discount factor at 0.25:

t	df	r (continuous)
0	1	0.0018
0.25	0.999125765	0.00349847
0.5	0.998177496	0.003648333

The quote for the 1Y s/a swap is the fair fixed rate on a one-year swap with semiannual payments on the fixed leg. As we have seen, in an arbitrage-free world it should not matter with what frequency payments occur on the float leg. (In the market, this is a slight oversimplification; in most currencies, including USD, the standard is for quarterly payments on the float leg, which reset on 3M LIBOR. Float legs resetting monthly on 1M LIBOR or semiannually on 6M LIBOR have slightly different values. We will disregard this difference—called the interest rate basis—throughout our discussions here.)

The fixed leg of the swap has payments at 0.5 and 1 year. If we model the instrument assuming an exchange of the notional at the end, then the float leg by itself has present value equal to the notional, which here we will take to be 1. Then the value of the swap to the party paying the par swap rate r is given by:

$$V_{swap} = 1 - \left(\frac{r}{2} P_{0.5} + \left(1 + \frac{r}{2} \right) P_1 \right)$$

In this case, it is the value P_1 that is our unknown, since r is the par swap rate, meaning that $V = 0$. This reduces to…

$$P_1 = \frac{1 - \frac{r}{2} P_{0.5}}{1 + \frac{r}{2}}$$

…giving the following term structure out to one year:

t	df	r (continuous)
0	1	0.0018
0.25	0.999125765	0.00349847
0.5	0.998177496	0.003648333
1	0.993526961	0.00649408

The 2Y s/a swap has fixed leg cash flows at 0.5, 1, 1.5, and 2. Although the discount factors at the first two times are known, the last two cash flows both have unknown discount factors. Fortunately, our interpolation method provides us a way to express one in terms of the other, meaning that the equation we must solve can be expressed in terms of a single variable, for example the spot interest rate at two years.

The resulting equation, however, is nonlinear, requiring a solver of a sort previously discussed in connection with bond yields. If one of these is chosen, then the resulting term structure becomes:

t	df	r (continuous)
0	1	0.0018
0.25	0.999125765	0.00349847
0.5	0.998177496	0.003648333
1	0.993526961	0.00649408
2	0.976282873	0.012001453

The method for the 5Y swap is, in this case, essentially the same. Here, we have cash flows at time points with unknown discount factors at all of the times 2.5, 3, 3.5, 4, 4.5, and 5. Nevertheless, these can all be expressed thanks to the interpolation method we have chosen as functions of a single variable, and finally the term structure obtained becomes:

t	df	r (continuous)
0	1	0.0018
0.25	0.999125765	0.00349847
0.5	0.998177496	0.003648333
1	0.993526961	0.00649408
2	0.976282873	0.012001453
5	0.837027004	0.035579789

Of course, since the last two rates here depend upon the interpolation method chosen, different results are obtained under a different choice. In the case of another method that imposes no continuity constraints on derivatives of the interest rates—for example, linear spot rates rather than piecewise constant forwards—the same bootstrapping method can be applied with the minor modification of changing the interpolation function.

It is important to note, however, that if a method that does impose such requirements is chosen—a cubic spline in the spot rates, for example—then the term structure can no longer be constructed one instrument at a time. As we have previously seen, a shock applied to one knot point in a cubic spline curve causes changes in interpolated values essentially everywhere across the curve. This means that instruments relying on interpolated discount factors—such as, in this case, the two-year and five-year swaps—must be made to match their market values simultaneously. If handled one at a time, calibration to match the second instrument is apt to cause the first no longer to match its market value.