Consider a general 6x6 tridiagonal nonsingular matrix.

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

What can be said about its ILU(0) factorization (when it exists)?

By definition the zero pattern of L and U (from ILU(0)) will match that of A. More interesting to note though is that the L and U resulting from ILU(0) will be the same as the L and U resulting from a full LU factorization. This is a consequence of the tridiagonal nature of A; a full LU factorization of a tridiagonal matrix is confined to the diagonal and sub-diagonal for L and diagonal and super-diagonal for U. There is no fill-in that exists for the ILU(0) factorization of A to zero, all non-zero terms of the factorization already reside within the non-zero pattern of A.

Now apply the permuation $\pi = [1, 3, 5, 2, 4, 6]$ to the matrix symmetrically (i.e., both rows and columns are permuted).

$$A_{\pi\pi} = \begin{bmatrix} 2 & 0 & 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & -1 & -1 & 0 \\ 0 & 0 & 2 & 0 & -1 & -1 \\ -1 & -1 & 0 & 2 & 0 & 0 \\ 0 & -1 & -1 & 0 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{bmatrix}$$

What is the sparsity pattern of the permuted matrix?

See above. The symmetric permutation permutes the diagonal terms, but keeps them confined to the diagonal. The sub/super-diagonals have been moved elsewhere in the matrix, but the resultant matrix (and sparsity pattern) is still symmetric.

Show the locations of the fill-in elements in the ILU(0) factorization of the permuted matrix.

The ILU(0) factorization has the same zero pattern as $A_{\pi\pi}$, the fill-in elements that would have resulted from a full LU factorization can be collected in a matrix R (here L and U are the ILU(0) factorizations):

$$A = LU - R$$

The fill-in elements of R in the example matrix A chosen are:

$$R = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.5 & 0 \\ 0 & 0 & 0 & -0.5 & 0 & -0.5 \\ 0 & 0 & 0 & 0 & -0.5 & 0 \end{bmatrix}$$