Use this numerical test to investigate the rate of convergence for a couple different values of α. How is the rate impacted by the α in the description of f? Why?

Figure 1 plots the behavior of the convergence order (in the H^0 and H^1 norms) as α varies. Nominally, the order of convergence is as expected for $\alpha > 0$. However for even small negative values of α the order of convergence decays, until for $\alpha = -1$ the accuracy of the solution fails to improve with refinement.

Figure 2 shows several example plots of the solution for various values of α. The most noticeable feature of the change in the solution is the increasing strength and localization of the potential at the center of the forcing function "source". The forcing function acts as a $r^{2\alpha}$ strength source; for example $\alpha = 0.5$ is radially linearly increasing, while for negative α it acts as a nearly-point-like singularity.

The singularity possibly accounts for the decay of the order of convergence in two ways. First, the smoothness of the forcing function and the solution deteriorates as α approaches -1. The Lagrange discretization of the solution has trouble properly representing a non-polynomial-like solution, and the quadrature used to compute integrals involving the RHS will suffer similiar issues. Second, the mesh is refined uniformly. However as α becomes more negative the dominant part of the solution becomes more and more localized. As a result, uniform refinement will result in less gains in accuracy in the solution then it would for solutions that effect most of the domain.

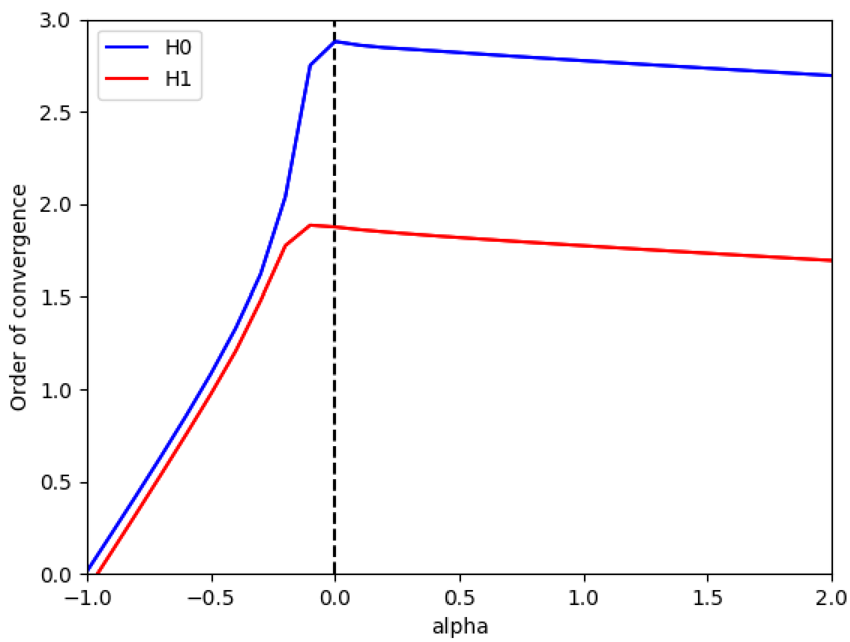

Figure 1: Observed behavior of convergence order as dependent on α, quadratic elements.

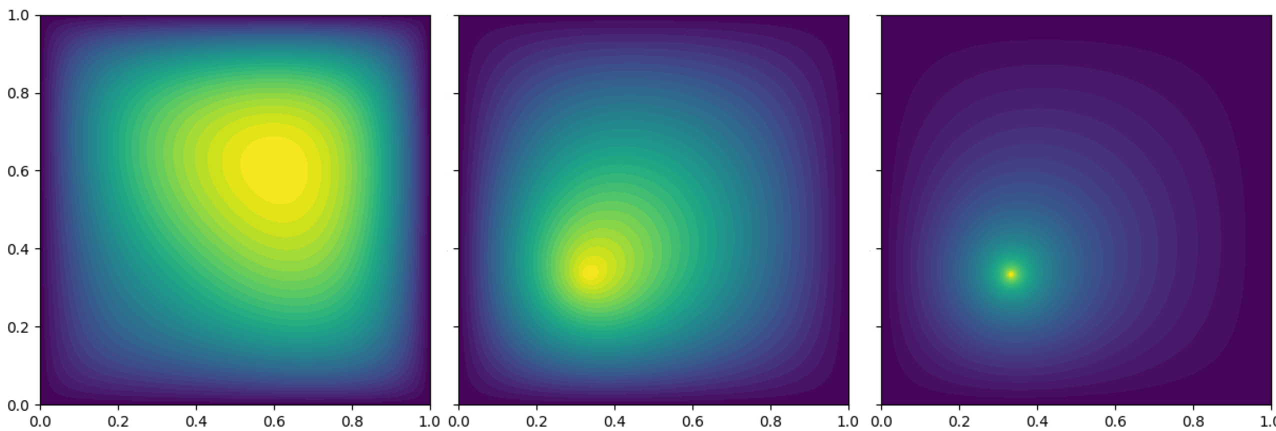

Figure 2: Change in nature of solution as α varies. Left to right: $\alpha = 0.5$, -0.5, -1.