Consider the two-dimensional anisotropic, rotated diffusion problem ... we assume **A** *is the matrix resulting from a Q1 finite element discretization of the problem on a regular mesh.*

Solve for $\epsilon = 0.001$ and $\epsilon = 0.1$ using AMG with "out-of-the-box" Ruge-Stuben coarsening and a problem size of 5 or smaller on the coarsest grid. What convergence factors do you see?

For the case of $\epsilon = 0.1$, AMG performs similar to GMG for isotropic problems, with an excellent asymptotic convergence rate of 0.07 as shown in Figure 1. In the more challenging case of $\epsilon = 0.001$ the more pronounced anisotropy of the problem means that AMG with RS coarsening has a significantly degraded asymptotic convergence factor of 0.69 for the default strength threshold value of $\theta = 0.25$. Figure 2 plots the much lower convergence factor evidenced in the larger residual compared to the $\epsilon = 0.1$ case.

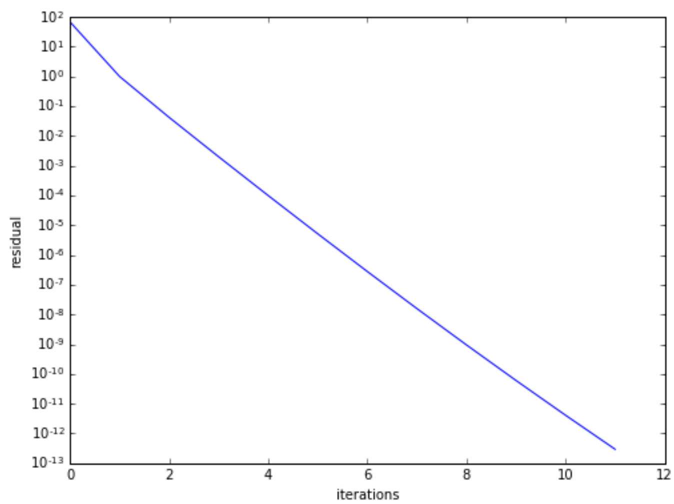

Figure 1: AMG, RS coarsening $\theta = 0.25$(default), $\epsilon = 0.1$

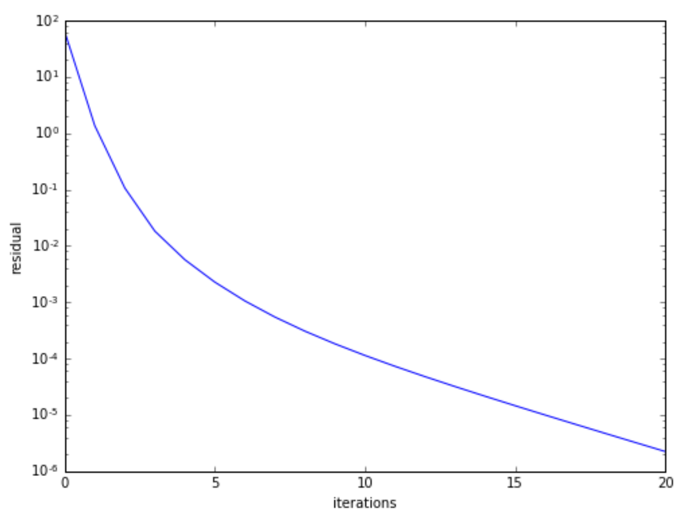

Figure 2: AMG, RS coarsening $\theta = 0.25$(default), $\epsilon = 0.001$,

Now using $\epsilon = 0.001$, try different values for the strength threshold parameter (used to determine the strong connections). How does this change the performance of the solver?

The setup from the previous question was now tested for a range of strength thresholds $0 < \theta < 1$ at 0.01 intervals to determine the optimal value for the smallest residual after 20 iterations. Figure 3 plots the residual for the range of thresholds examined. Compared to the default value of 0.25, the optimal value of 0.46 resulted in a residual nearly three orders of magnitude lower. The optimized convergence factor is now (incidentally) 0.46. Interestingly, the optimal value is rather "brittle" in that increasing by just 0.01 to 0.47 resulted in a huge jump in the residual to a final value even worse than the one resulting from the default value of the threshold.

The performance of the solver is almost unchanged using the optimal value of the strength threshold compared to the default value; Figure 4 compares the tabulated diagnostic data for each case. The optimal threshold leads to an increased number of levels in the hierarchy, but the relative asymptotic work of the optimal case converges to nearly the same value as the default case.

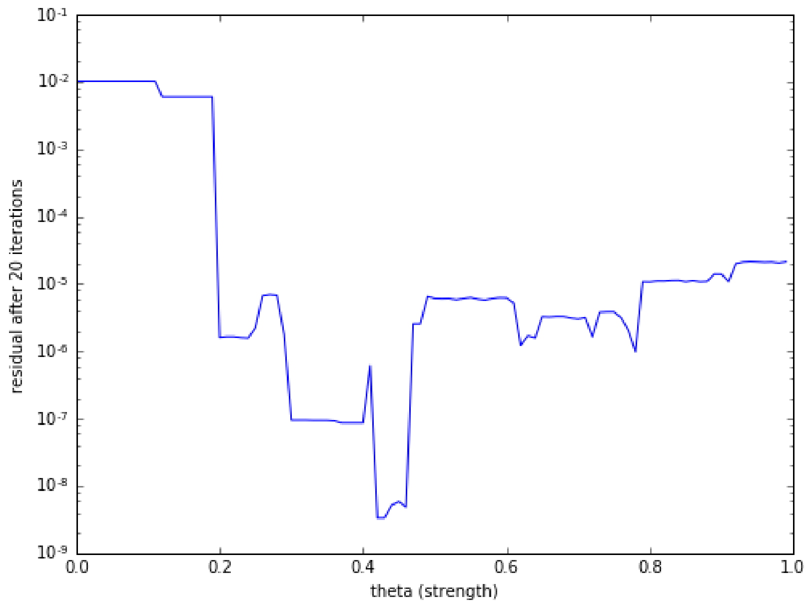

Figure 3: Behavior of residual after 20 iterations for a range of strength threshold values for RS coarsening.

```
             Levels: 8
   Cycle Complexity:  4.447
Operator Complexity:  2.224
    Grid Complexity:  1.707
avg geo conv factor:  0.442
               work: 12.537

level   unknowns     nnz
   0    10000        88804       [44.97%]
   1    5000         73014       [36.97%]
   2    1249         17929       [ 9.08%]
   3    624          14692       [ 7.44%]
   4    148          2520        [ 1.28%]
   5    37           463         [ 0.23%]
   6    9            45          [ 0.02%]
   7    4            14          [ 0.01%]

---Convergence Summary (verbose)------------------
            Factors:
iter       Factor     A-Mean     G-Mean     Work
0
1          0.021      0.021      0.147      5.334
2          0.079      0.050      0.119      4.812
3          0.172      0.091      0.130      5.028
4          0.311      0.146      0.155      5.497
5          0.400      0.196      0.182      6.005
6          0.464      0.241      0.208      6.517
7          0.518      0.281      0.233      7.027
8          0.562      0.316      0.257      7.532
9          0.596      0.347      0.279      8.030
10         0.622      0.374      0.300      8.516
11         0.642      0.399      0.320      8.988
12         0.656      0.420      0.338      9.446
13         0.666      0.439      0.355      9.888
14         0.674      0.456      0.370      10.313
15         0.679      0.471      0.385      10.722
16         0.683      0.484      0.398      11.115
17         0.686      0.496      0.410      11.493
18         0.688      0.507      0.422      11.855
19         0.690      0.516      0.432      12.203
20         0.691      0.525      0.442      12.537
```

```
             Levels: 11
   Cycle Complexity:  6.229
Operator Complexity:  3.114
    Grid Complexity:  1.975
avg geo conv factor:  0.330
               work: 12.931

level   unknowns     nnz
   0    10000        88804       [32.11%]
   1    5000         73014       [26.40%]
   2    2474         53906       [19.49%]
   3    1224         34874       [12.61%]
   4    599          15747       [ 5.69%]
   5    263          7293        [ 2.64%]
   6    126          2342        [ 0.85%]
   7    40           474         [ 0.17%]
   8    13           83          [ 0.03%]
   9    6            28          [ 0.01%]
   10   2            4           [ 0.00%]

---Convergence Summary (verbose)------------------
            Factors:
iter       Factor     A-Mean     G-Mean     Work
0
1          0.021      0.021      0.147      7.468
2          0.078      0.050      0.119      6.734
3          0.153      0.084      0.127      6.938
4          0.254      0.127      0.145      7.439
5          0.318      0.165      0.166      7.978
6          0.360      0.197      0.185      8.503
7          0.390      0.225      0.203      9.000
8          0.411      0.248      0.220      9.465
9          0.425      0.268      0.235      9.895
10         0.434      0.284      0.248      10.291
11         0.440      0.299      0.260      10.656
12         0.444      0.311      0.271      10.992
13         0.448      0.321      0.281      11.302
14         0.450      0.330      0.290      11.589
15         0.452      0.338      0.298      11.854
16         0.453      0.346      0.306      12.100
17         0.454      0.352      0.312      12.329
18         0.456      0.358      0.319      12.543
19         0.456      0.363      0.325      12.743
20         0.457      0.368      0.330      12.931
```

Figure 4: Comparison of performance of default(left) and optimal(right) strength thresholds RS coarsening.

Finally, consider two different types of coarsening: Ruge-Stuben (RS) and Cleary-Luby-Jones-Plassmann (CLJP). Assess the cost of the resulting problem at each level for these coarsening algorithms (you may want to look at the number of non-zeros and size of the problem for each level). Solve the anisotropic, rotated diffusion problem with AMG solvers using RS and CLJP coarsening and compare the performance of the solver in each case.

We can perform a similiar procedure for CLJP coarsening to determine the optimal strength threshold for comparison purposes, with the results plotted in Figure 5. In this case the optimal threshold was 0.36, but notably the optimal value isn't nearly as sensitive as it was in the RS coarsening cases. Additionally the optimal CLJP coarsening results in a much larger residual after 20 iterations compared to RS coarsening as demonstrated in the plots of Figure 6 which shows the evolution of the residual for both coarsening methods. In the optimal threshold CLJP case the asymptotic convergence factor was 0.66 compared to the RS convergence factor of 0.46.

We can now compare the performance of the solver resulting from the two coarsening methods. Figure 7 plots the tabulated data for each method with optimized threshold value. We can see in this case not only does CLJP have a worse residual after a fixed number of iterations and worse convergence factor, but the relative work and sparsity of some coarser levels is worse as well. For instance compare the number of non-zeros for the first several levels; in the CLJP case the number of non-zeros actually remains relatively unchanged across the three levels, despite the number of unknowns decreasing. Level 2 for the CLJP case has nearly 10 times the density of non-zeros as the fine level, to be fair however the density of the RS case at level 2 is 4.4 times that of the fine level (though still not as bad as CLJP). A final performance comparison can be made regarding the asymptotic relative work, with the CLJP case nearly double that of the RS case.

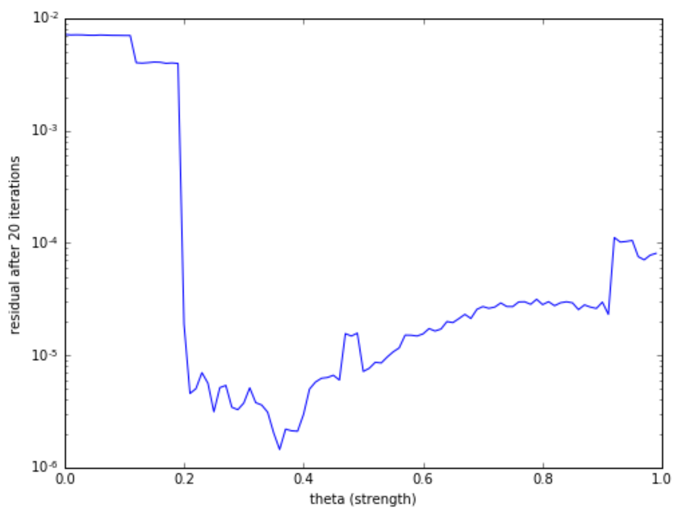

Figure 5: Behavior of residual after 20 iterations for a range of strength threshold values for CLJP coarsening.

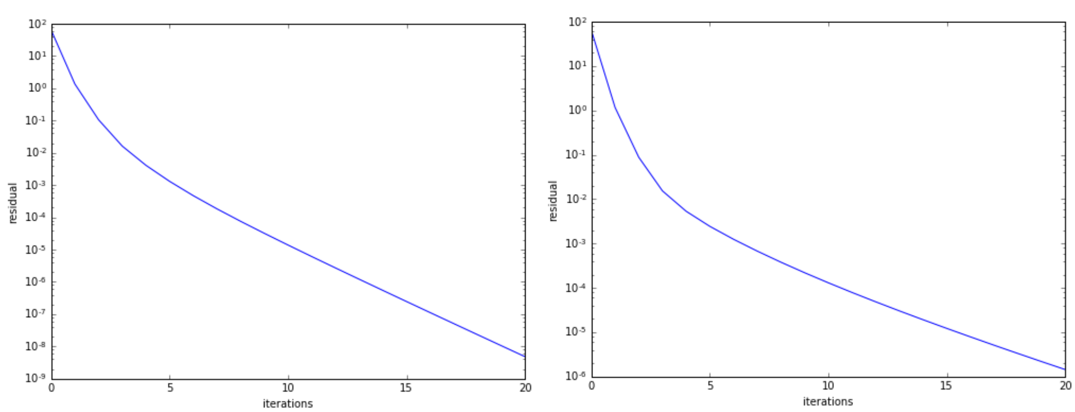

Figure 6: Comparison of evolution of residual for optimal threshold cases for RS(left) and CLJP(right) coarsening.

```
           Levels: 11
   Cycle Complexity:  6.229
Operator Complexity:  3.114
    Grid Complexity:  1.975
avg geo conv factor:  0.330
               work: 12.931

level   unknowns     nnz
   0    10000        88804       [32.11%]
   1    5000         73014       [26.40%]
   2    2474         53906       [19.49%]
   3    1224         34874       [12.61%]
   4    599          15747       [ 5.69%]
   5    263          7293        [ 2.64%]
   6    126          2342        [ 0.85%]
   7    40           474         [ 0.17%]
   8    13           83          [ 0.03%]
   9    6            28          [ 0.01%]
   10   2            4           [ 0.00%]

---Convergence Summary (verbose)-------------------
          Factors:
iter       Factor     A-Mean     G-Mean     Work
0
1          0.021      0.021      0.147      7.468
2          0.078      0.050      0.119      6.734
3          0.153      0.084      0.127      6.938
4          0.254      0.127      0.145      7.439
5          0.318      0.165      0.166      7.978
6          0.360      0.197      0.185      8.503
7          0.390      0.225      0.203      9.000
8          0.411      0.248      0.220      9.465
9          0.425      0.268      0.235      9.895
10         0.434      0.284      0.248      10.291
11         0.440      0.299      0.260      10.656
12         0.444      0.311      0.271      10.992
13         0.448      0.321      0.281      11.302
14         0.450      0.330      0.290      11.589
15         0.452      0.338      0.298      11.854
16         0.453      0.346      0.306      12.100
17         0.454      0.352      0.312      12.329
18         0.456      0.358      0.319      12.543
19         0.456      0.363      0.325      12.743
20         0.457      0.368      0.330      12.931
```

```
           Levels: 14
   Cycle Complexity:  8.093
Operator Complexity:  4.047
    Grid Complexity:  2.298
avg geo conv factor:  0.433
               work: 22.244

level   unknowns     nnz
   0    10000        88804       [24.71%]
   1    6001         87703       [24.41%]
   2    3412         83638       [23.27%]
   3    1732         48938       [13.62%]
   4    894          26424       [ 7.35%]
   5    464          13476       [ 3.75%]
   6    243          6705        [ 1.87%]
   7    116          2482        [ 0.69%]
   8    53           773         [ 0.22%]
   9    26           242         [ 0.07%]
   10   15           91          [ 0.03%]
   11   9            41          [ 0.01%]
   12   6            26          [ 0.01%]
   13   5            19          [ 0.01%]

---Convergence Summary (verbose)-------------------
          Factors:
iter       Factor     A-Mean     G-Mean     Work
0
1          0.018      0.018      0.136      9.331
2          0.075      0.047      0.112      8.498
3          0.175      0.090      0.125      8.956
4          0.348      0.154      0.153      9.934
5          0.457      0.215      0.184      11.003
6          0.511      0.264      0.213      12.042
7          0.541      0.304      0.239      13.024
8          0.562      0.336      0.263      13.951
9          0.579      0.363      0.285      14.827
10         0.593      0.386      0.304      15.660
11         0.605      0.406      0.322      16.453
12         0.616      0.424      0.339      17.209
13         0.624      0.439      0.354      17.933
14         0.632      0.453      0.368      18.626
15         0.638      0.465      0.381      19.290
16         0.644      0.476      0.393      19.927
17         0.648      0.486      0.404      20.540
18         0.653      0.496      0.414      21.129
19         0.657      0.504      0.424      21.697
20         0.660      0.512      0.433      22.244
```

Figure 7: Comparison of cost at each level for the two optimal threshold coarsening methods, RS(left) and CLJP(right).