1. Consider the following elliptic PDE

$$-\nabla \cdot (p(\vec{x}) \nabla u) = q(\vec{x}) \quad \text{for} \quad \vec{x} \in \Omega \subset \mathbb{R}^2,$$

$$u(\vec{x}) = 0 \quad \text{when} \quad \vec{x} \in \partial\Omega,$$

with $p(\vec{x}) \geq m > 0$.

(a) Write the weak formulation of this problem in the form,

$$\text{Find } u \in V, \text{ such that}$$
$$a(u, v) = G(v), \text{ for all } v \in V.$$

> What is $a(\cdot, \cdot)$? What is $G(\cdot)$? What is V? Why are u and v in the same space? Is $a(\cdot, \cdot)$ symmetric?

(b) Define the energy functional,

$$J(u) = \frac{1}{2} a(u, u) - G(u),$$

and show that the weak formulation above is equivalent to finding the "critical" or "stationary" point of J, by computing

$$\frac{\mathrm{d}}{\mathrm{d}\tau} J(u + \tau v)\Big|_{\tau=0} = 0.$$

> What is the second directional-derivative of J?
>
> $$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} J(u + \tau v)\Big|_{\tau=0}$$
>
> What can we conclude about the stationary point and what should we assume about $a(\cdot, \cdot)$?

(c) Suppose the domain is a disjoint union of M triangles, $E_1, E_2, \cdots, E_M$, with shared vertices. Further suppose that there are a total of N vertices with $\vec{x} = (x, y)$ coordinates given by $\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_N$.

> Show that the bilinear form, $a(\cdot, \cdot)$, and linear functional, $G(v)$, derived in the previous part can be written as a sum of integrals over each triangle (element).

(d) Consider approximating the solution to the weak formulation by restricting the solution space to a subspace spanned by N basis functions, $\{\phi_1, \cdots, \phi_N\}$. Write the (approximate) solution as a linear combination of these basis functions,

$$u(\vec{x}) = \sum_{i=1}^{N} u_i \phi_i,$$

where $\phi_i \in V$ and $\mathrm{span}(\phi_1, \cdots, \phi_N) \subset V$.

Show that the minimizer of the energy, $J(u)$, over the subspace $\mathrm{span}(\phi_1, \cdots, \phi_N)$, solves

$$\sum_{j=1}^{N} u_j a(\phi_j, \phi_i) = G(\phi_i), \quad \forall\, i = 1 \ldots N.$$

(e) Suppose we modified the PDE to include nonhomogeneous Dirichlet boundary conditions,

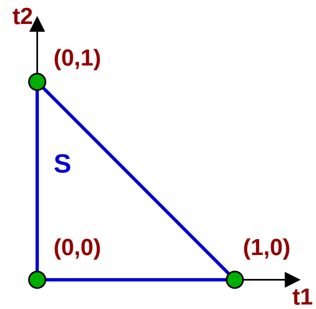
$$-\nabla \cdot (p(\vec{x}) \nabla u) = q(\vec{x}) \quad \text{for} \quad \vec{x} \in \Omega \subset \mathbb{R}^2,$$

$$u(\vec{x}) = g(\vec{x}) \quad \text{when} \quad \vec{x} \in \partial\Omega,$$

with $p(\vec{x}) \geq m > 0$. Let $\hat{u} \in V$ and $u := u^0 + \hat{u}$ with u^0 satisfying the boundary conditions.

What problem does $\hat{u}$ solve?

2. Here we derive the analytical formulae necessary for implementation of the finite element method using piecewise linear finite elements on a triangulated domain.



(a) Suppose $\lambda_1(\alpha, \beta)$ is the piecewise linear nodal basis function on the standard element (triangle), with $\lambda_1(0,0) = 1$ and $\lambda_1(1,0) = \lambda_1(0,1) = 0$. Show that when restricted to the standard triangle with vertices $(0,0), (1,0), (0,1) \in \mathbb{R}^2$, this basis function can be written as

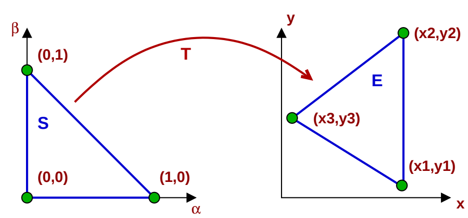
$$\lambda_1(\alpha, \beta) = 1 - \alpha - \beta$$

and conclude that the gradient of λ_1 is

$$\nabla \lambda_1 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}.$$

Derive a similar result for λ_2 which satisfies $\lambda_2(1,0) = 1$ and $\lambda_2(0,0) = \lambda_2(0,1) = 0$.

Derive a similar result for λ_3 which satisfies $\lambda_3(0,1) = 1$ and $\lambda_3(0,0) = \lambda_3(1,0) = 0$.



(b) Given three vertices of a general triangle, $\vec{x}_1, \vec{x}_2, \vec{x}_3 \in \mathbb{R}^2$,

write down an affine transformation, T, that maps the vertices of the "standard triangle" $(0,0), (1,0), (0,1) \in \mathbb{R}^2$ to the "general triangle".

Show that the Jacobian (derivative) of this transformation is

$$J_T = \left[\begin{array}{cc} (\vec{x}_2 - \vec{x}_1) & (\vec{x}_3 - \vec{x}_1) \end{array} \right] \in \mathbb{R}^{2\times 2},$$

where $\vec{x} = (x, y)$.

(c) Suppose ϕ_r is the piecewise linear nodal basis function for a general triangulation with $\phi_r(\vec{x}_r) = 1$ and $\phi_r(\vec{x}_s) = 0$ if $r \neq s$.

Using the results of the prior two parts show that when restricted to the "general triangle" that ϕ_r can be written as a composition of λ_r and T^{-1},

$$\phi_r(\vec{x}) = \lambda_r\left(T^{-1}(\vec{x})\right)$$

and the gradient of ϕ_r is

$$\nabla\phi_r|_{\vec{x}} = J_T^{-T}\,\nabla\lambda_r$$

where $\nabla\lambda_r$ is constant.

(d) Let $\vec{x} = (x, y) \in \mathbb{R}^2$, $\vec{\alpha} = (\alpha, \beta) \in \mathbb{R}^2$, E the region defined by the general triangle, S the region defined by the standard triangle, T the affine transformation defined above, and $|J_T|$ the determinant of the Jacobian of the transformation. Using the change of variables formula

$$\iint_E h(\vec{x})\, d\vec{x} = \iint_S h(T(\vec{\alpha}))\, |J_T|\, d\vec{\alpha}$$

,

show that

$$\langle p(\vec{x})\, \nabla\phi_s, \nabla\phi_r\rangle_E = \left(J_T^{-T}\nabla\lambda_r\right)^T \left(J_T^{-T}\nabla\lambda_s\right)\, |J_T| \iint_S p(T(\vec{\alpha}))\, d\vec{\alpha}$$

and

$$\langle q(\vec{x}), \phi_r\rangle_E = |J_T| \iint_S q(T(\vec{\alpha}))\, \lambda_r(\vec{\alpha})\, d\vec{\alpha}$$