Consider the leapfrog scheme given by:

$$v_{k,\ell+1} = v_{k,\ell-1} - \frac{ah_t}{h_x}(v_{k+1,\ell} - v_{k-1,\ell}) \tag{1}$$

Show that the leapfrog scheme is consistent with $u_t + au_x = 0$.

A PDE of the form $\mathcal{L}u = f$ with finite difference scheme $\mathcal{L}_{\Delta t \Delta x} v = f$ (where u is the PDE solution and v the FD solution) is consistent if for a smooth function $\phi(x,t)$:

$$\mathcal{L}\phi - \mathcal{L}_{\Delta t \Delta x}\phi \to 0 \text{ as } \Delta t, \Delta x \to 0 \tag{2}$$

For the given PDE we have:

$$\mathcal{L}\phi = \phi_t + a\phi_x \tag{3}$$

and the FD leapfrog scheme:

$$\mathcal{L}_{\Delta t \Delta x}\phi = \frac{\phi_{k,l+1} - \phi_{k,l-1}}{2\Delta t} + a\frac{\phi_{k+1,l} - \phi_{k-1,l}}{2\Delta x} \tag{4}$$

We can expand each ϕ term with Taylor expansions:

$$\phi_{k+1,l} = \phi_{k,l} + \Delta x \phi_x + \frac{1}{2}\Delta x^2 \phi_{xx} + \frac{1}{6}\Delta x^3 \phi_{xxx} + \mathcal{O}(\Delta x^4) \tag{5}$$

$$\phi_{k-1,l} = \phi_{k,l} - \Delta x \phi_x + \frac{1}{2}\Delta x^2 \phi_{xx} - \frac{1}{6}\Delta x^3 \phi_{xxx} + \mathcal{O}(\Delta x^4) \tag{6}$$

$$\phi_{k,l+1} = \phi_{k,l} + \Delta t \phi_t + \frac{1}{2}\Delta t^2 \phi_{tt} + \frac{1}{6}\Delta t^3 \phi_{ttt} + \mathcal{O}(\Delta t^4) \tag{7}$$

$$\phi_{k,l-1} = \phi_{k,l} - \Delta t \phi_t + \frac{1}{2}\Delta t^2 \phi_{tt} - \frac{1}{6}\Delta t^3 \phi_{ttt} + \mathcal{O}(\Delta t^4) \tag{8}$$

Substituting into Eqn. 4, adding and canceling terms from the expansions yields:

$$\mathcal{L}_{\Delta t \Delta x}\phi = \phi_t + a\phi_x + \frac{1}{6}\Delta t^2 \phi_{ttt} + a\frac{1}{6}\Delta x^2 \phi_{xxx} + \mathcal{O}(\Delta x^3) + \mathcal{O}(\Delta t^3) \tag{9}$$

so:

$$\mathcal{L}\phi - \mathcal{L}_{\Delta t \Delta x}\phi = \frac{1}{6}\Delta t^2 \phi_{ttt} + a\frac{1}{6}\Delta x^2 \phi_{xxx} + \mathcal{O}(\Delta x^3) + \mathcal{O}(\Delta t^3) \to 0 \text{ as } \Delta t, \Delta x \to 0 \tag{10}$$

and therefore the scheme is consistent.