Design an algorithm to solve the linear least squares problem... Write an additional version of your algorithm that also makes use of a a row interpolative decomposition $Q \approx PQ_{(j,:)}$

We can compute the solution to the least squares problem $Ax \approx b$ if we have the SVD of A by:

$$x = V\Sigma^{+}U^{T}b$$

where Σ^{+} has the reciprocals of the singular values.

For our case we have the special form $QB \approx b$, so:
Step 1: Rearrange as $B \approx Q^{T}b$
Step 2: Compute SVD of B
Step 3: Compute x from SVD terms

We can do better than this if we use the row ID of Q. We know that this subset is also appropriate for A, $A \approx PA_{(j,:)}$. So our steps become:
Step 1: Compute $\bar{Q}\bar{R}$ of $A_{(j,:)}$
Step 2: Compute SVD of $P\bar{R}$
Step 3: Used SVD to compute x as shown above

Find (and write down) the asymptotic complexity of each step in both of your algorithm in terms of m, n, k as well as the overall complexity of the algorithm.

In the full-rank case $k = n$, what is the asymptotic complexity of a solution procedure based on Householder QR? How do your algorithms compare in this case?

In order of steps:
Alg 1: $mk, kn^2, mk + k^2$ Overall: m^2k
Alg 2: $2n^2m, nk^2, mk^2$ Overall: mk^2

Suppose the low-rank projection matrix Q is not given, what is the asymptotic complexity of finding it, using the (non-adaptive) range finder, assuming A is given as a dense matrix? How does this compare to the complexity of the above methods?

We will need to multiply k columns of a particular Ω by A, or in other words cost mnk. This cost is OK for the slower method that doesn't use the ID (makes the overall asymptotic cost no worse). But for the faster mk^2 method with ID it now becomes the dominant cost.

In order to improve matters in this situation, consider replacing the matrix Ω with Ω' ...

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If A is a square, low-rank matrix, show that exactly one of the following is true:
-The linear system $(I - A)x = b$ has a solution x.
-The linear system $(I - A)Ty = 0$ has a solution y with $y^{T}b \neq 0$.
Hint: Show that (A) if and only if not (B).