Take Fourier transform of $\frac{1}{\sqrt{r^2+a^2}}$:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\sqrt{r^2+a^2}}e^{-2\pi i\vec{k}\cdot\vec{r}}\,dx\,dy\,dz = \int_0^{\infty}\int_0^{\pi}\int_0^{2\pi}\frac{e^{-2\pi ikr\cos(\theta)}}{\sqrt{r^2+a^2}}\,r^2\,\sin(\theta)\,d\phi\,d\theta\,dr$$

$$= 2\pi\int_0^{\infty}\int_0^{\pi}\frac{e^{-2\pi ikr\cos(\theta)}}{\sqrt{r^2+a^2}}\,r^2\,\sin(\theta)\,d\theta\,dr$$

$$= 2\pi\int_0^{\infty}\frac{r^2}{\sqrt{r^2+a^2}}\int_0^{\pi}e^{-2\pi ikr\cos(\theta)}\sin(\theta)\,d\theta\,dr$$

$$= 2\pi\int_0^{\infty}\frac{r^2}{\sqrt{r^2+a^2}}\left(\frac{\sin(2\pi kr)}{\pi kr}\right)\,dr$$

$$= \frac{2}{k}\int_0^{\infty}\frac{r}{\sqrt{r^2+a^2}}\,\sin(2\pi kr)\,dr$$

We now find that the integral fails to converge, since $\frac{r}{\sqrt{r^2+a^2}} \to 1$ as $r \to \infty$, and so $\frac{r}{\sqrt{r^2+a^2}}\sin(2\pi kr)$ is undamped. However, we expect that a value for the integral exists since one exists for $\int_0^{\infty} sin(2\pi kr)\,dr = 1/2\pi k$, the "physicist's point charge" example. For this example they introduce a screening term $e^{-br}, b > 0$ in the integral which artificially damps the integrand and forces it to converge, then they take the limit as $b \to 0$. Apply same principle here:

$$\frac{2}{k}\int_0^{\infty}\frac{r}{\sqrt{r^2+a^2}}\,\sin(2\pi kr)\,dr = \lim_{b\to 0}\frac{2}{k}\int_0^{\infty}\frac{r}{\sqrt{r^2+a^2}}\,e^{-br}\sin(2\pi kr)\,dr$$

$$= \lim_{b\to 0}\frac{2}{k}\int_0^{\infty}\frac{r}{\sqrt{r^2+a^2}}\,\frac{(e^{(b-2\pi ki)r}-e^{-(b+2\pi ki)r})}{2i}\,dr$$

$$= \frac{2}{\pi k^2}-\frac{\pi a}{k}\,\mathrm{Im}\,Y_1(i2\pi ak)$$

If we realize that $\lim_{a\to 0} aY_1(i2\pi ak) = i/\pi^2 k$, we see that the limit of the Fourier transform for $a \to 0$ is simply $1/\pi k^2$, and we recover the Fourier transform of $1/r$ (i.e $1/\sqrt{r^2+a^2}$ for $a = 0$).

An alternate form:

While the previous solution seems to reduce to the expected case for $a = 0$, Mathematica/Fourier transform tables are unable to verify that the inverse Fourier transform recovers $1/\sqrt{r^2+a^2}$. I managed to find another solution form of the Fourier transform that is readily verifiable, however it requires a leap of faith at one step that could do with some additional rigor.