Use the method of undetermined coefficients to determine the nodes and weight for a three-point Chebyshev quadrature rule on the interval $[-1, 1]$.

A three point quadrature rule must satisfy the following fo r the method of undetermined coefficients:

$$\begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} a-b \\ (a^2-b^2)/2 \\ (a^3-b^3)/3 \end{bmatrix}$$

For a constant set of weights $w_1 = w_2 = w_3$ the constant part must then satisfy:

$$w_1 + w_2 + w_3 = 3w = 1 - (-1) \to w = 2/3$$

We now have two equations and three unknowns to satisfy for x_n, so we add the additional criteria that the nodes will be symmetric in the interval; that is $x_1 = -x_3$. The two equations to satisfy are now:

$$2/3\,(x_1 + x_2 - x_1) = 0 \to x_2 = 0$$

$$2/3\,(x_1^2 + x_2^2 + x_1^2) = 2/3 \to 2/3\,(2x_1^2 + 0^2) = 2/3 \to x_1 = 0.5^{1/2} \approx 0.707$$

so $x = \{-0.707, 0, 0.707\}$, $w = \{2/3, 2/3, 2/3\}$

What is the degree of the resulting rule?

We can determine the degree by seeing the highest order polynomial it exactly integrates. Consider a lower order test case:

$$f(x) = 4x^3 + 4x^2$$

$$\int f(x) = x^4 + 4/3x^3 \to \int_{-1}^{1} f(x) = 8/3 \approx 2.666666...$$

If we apply our quadrature rule, we get

$$w \sum_i f(x_i) = 2.666666666666667$$

which agrees within machine precision of the exact answer. Looking at the lowest order example polynomial we don't integrate exactly:

$$f(x) = 5x^4 + 4x^2$$

$$\int f(x) = x^5 + 4/3x^3 \to \int_{-1}^{1} f(x) = 14/3 \approx 4.666666...$$

If we apply our quadrature rule, we get

$$w \sum_i f(x_i) = 4.3333333333333339$$

indicating that this quadrature is not exact for 4th order polynomials; therefore the degree of the resulting rule is 3rd order.