Determine the critical points of each of the following functions and characterize each as a minimum, maximum, or saddle point. Also determine whether each function has a global minimum or maximum on $\mathbb{R}^2$.

Equation 1: $f(x, y) = x^2 - 4xy + y^2$

Gradient:

$$\nabla f(x,y) = \begin{bmatrix} 2x - 4y \\ 2y - 4x \end{bmatrix}$$

Hessian:

$$\mathbf{H}(x,y) = \begin{bmatrix} 2 & -4 \\ -4 & 2 \end{bmatrix}.$$

Critical point $(0, 0)$, saddle point, $\mathbf{H}(0, 0)$ is indefinite.
No global minimum or maximum.

Equation 2: $f(x, y) = x^4 - 4xy + y^4$

Gradient:

$$\nabla f(x,y) = \begin{bmatrix} 4x^3 - 4y \\ -4x + 4y^3 \end{bmatrix}$$

Hessian:

$$\mathbf{H}(x,y) = \begin{bmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{bmatrix}.$$

Critical point $(0, 0)$, saddle point, $\mathbf{H}(0, 0)$ is indefinite.
Critical point $(1, 1)$, local minimum, $\mathbf{H}(1, 1)$ is positive definite.
Critical point $(-1, -1)$, local minimum, $\mathbf{H}(-1, -1)$ is positive definite.
Both local minima are global minima. No global maximum.

Equation 3: $f(x, y) = 2x^3 - 3x^2 - 6xy(x - y - 1)$

Gradient:

$$\nabla f(x,y) = \begin{bmatrix} 6x^2 - 12xy + 6y^2 - 6x + 6y \\ -6x^2 + 12xy + 6x \end{bmatrix}$$

Hessian:

$$\mathbf{H}(x,y) = \begin{bmatrix} 12x - 12y - 6 & -12x + 12y + 6 \\ -12x + 12y + 6 & 12x \end{bmatrix}.$$

Critical point $(0, 0)$, saddle point, $\mathbf{H}(0, 0)$ is indefinite.
Critical point $(1, 0)$, local minimum, $\mathbf{H}(1, 0)$ is positive definite.
Critical point $(0, -1)$, saddle point, $\mathbf{H}(0, -1)$ is indefinite.
Critical point $(-1, -1)$, local maximum, $\mathbf{H}(-1, -1)$ is negative definite.
No global minimum or maximum.

Equation 4: $f(x, y) = (x - y)^4 + x^2 - y^2 - 2x + 2y + 1$

Gradient:

$$\nabla f(x, y) = \begin{bmatrix} 4(x-y)^3 + 2x - 2 \\ -4(x-y)^3 - 2y + 2 \end{bmatrix}$$

Hessian:

$$\mathbf{H}(x, y) = \begin{bmatrix} 12(x-y)^2 + 2 & -12(x-y)^2 \\ -12(x-y)^2 & 12(x-y)^2 - 2 \end{bmatrix}.$$

Critical point $(1, 1)$, saddle point, $\mathbf{H}(1, 1)$ is indefinite.
No global minimum or maximum.