Show that the combined operator $G(F\phi)(x)$ can be recast in the same form as the above two, i.e.

$$(G \circ F)\phi(x) := \int_a^x k_3(x,y)\phi(y)\,dy.$$

Given two integral operators

$$F\phi(z) := \int_a^z k_1(z,y)\phi(y)\,dy$$

$$G\phi(x) := \int_a^x k_2(x,z)\phi(z)\,dz$$

we can form the composition:

$$(G \circ F)\phi(x) = \int_a^x k_2(x,z) \int_a^z k_1(z,y)\phi(y)dy\;dz.$$

We are free to move $k_2(x,z)$ inside the inner integral since it is not a function of z.

$$(G \circ F)\phi(x) = \int_a^x \int_a^z k_2(x,z)\,k_1(z,y)\phi(y)\,dy\;dz.$$

We can then exchange the order of integration to obtain

$$(G \circ F)\phi(x) = \int_a^z \int_y^x k_2(x,z)\,k_1(z,y)\phi(y)\,dz\;dy.$$

We can then remove $\phi(z)$ from the inner integral since it does not depend on y

$$(G \circ F)\phi(x) = \int_a^z \phi(y) \int_y^x k_2(x,z)\,k_1(z,y)\,dz\;dy. \tag{1}$$

We can now see that the inner integral has the form of $f(x,y)$ thanks to the integration in z.

Give an expression for the kernel k_3.

If we think of this inner integral as another possible kernel function

$$k_3(x,y) = \int_y^x k_2(x,z)\,k_1(z,y)\,dz$$

and make the appropriate substitution in (1) we obtain

$$(G \circ F)\phi(x) = \int_a^x k_3(x,y)\phi(y)\;dy.$$