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Linear Algebra

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Exercise 1

Let $E = \mathbb{R}^3$ and let $u\ v\ w$ be 3 vectors

$u = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ $v = \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix}$ $w = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$

Ques (1) show that $B = (u, v, w)$ is a basis of E

Consider $B = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 4 & 4 \\ -2 & -1 & 2 \end{bmatrix}$ & we know B is a Basis of $\mathbb{R}^3 = E$

if $\rightarrow$ span $(B) = \mathbb{R}^3$ $\forall_a \in E$, $\exists\ (\overline{\lambda_1, \lambda_2, \lambda_3}) \in \mathbb{R}^3$

$a = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3$

$B \rightarrow$ linear Independent

check Independency

Solve $\Longrightarrow$ Consider the linear Combination as

$\lambda_1 \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + \lambda_2 \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} + \lambda_3 \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} = 0 \longrightarrow$

$\lambda_1 + 2\lambda_2 + \lambda_3 = 0$
$\lambda_1 + 4\lambda_2 + 4\lambda_3 = 0$
$-2\lambda_1 - \lambda_2 + 2\lambda_3 = 0$

Isolate λ_1 From eq (1)

$\boxed{\lambda_1 = -2\lambda_2 - \lambda_3} \longrightarrow$

Put λ_1 in eq (2) $\Longrightarrow$ $-2\lambda_2 - \lambda_3 + 4\lambda_2 + 4\lambda_3 = 0$

$\boxed{\lambda_2 = \frac{-3\lambda_3}{2}}$

Put λ_2 in eq (3) $\Longrightarrow$ $\boxed{\lambda_3 = 0}$

For, $\lambda_3 = 0$

$\lambda_2 = \frac{-3(0)}{2}$

$\longrightarrow \boxed{\lambda_2 = 0}$

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$\lambda_1 = -2\lambda_2 - \lambda_3 \longrightarrow$ For $\lambda_2 \lambda_3 = 0$

$\lambda_1 = 2(0) - 0)$

$\boxed{\lambda_1 = 0}$

It follows the equation solution is $\boxed{\lambda_1 = \lambda_2 = \lambda_3 = 0}$

$\Downarrow$

linearly Independent

As B has 3 linearly Independent vectors
it is Basis of $\mathbb{R}^3$

Ques (2)

let $a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ what are the coordinates of 'a' in basis?

Assume the cordinates are $\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} \longrightarrow$ is a vector

let $B = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 4 & 4 \\ -2 & -1 & 2 \end{bmatrix} \longrightarrow \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 1 & 4 & 4 \\ -2 & -1 & 2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}}_{\text{Matrix Multiply}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Longrightarrow$

$\lambda_1 + 2\lambda_2 + \lambda_3 = 1$

$\lambda_1 + 4\lambda_2 + 4\lambda_3 = 1$

$-2\lambda_1 - \lambda_2 + 2\lambda_3 = 1$

$3\times3 \quad 3\times1$

Solve the above equations to get coordinates of a in basis

Eliminate "λ_1" in eq (1) $\longrightarrow \boxed{\lambda_1 = 1 - 2\lambda_2 - \lambda_3}$

Put λ_1 in eq (3) $\longrightarrow -2(1 - 2\lambda_2 - 2\lambda_3) - \lambda_2 + 2\lambda_3 = 1$

Eliminate (λ_2) $\quad -2\lambda_2 - \lambda_3 + 4\lambda_2 + 4 = 1$

$3\lambda_3 + 2\lambda_2 = 0$

$\boxed{\lambda_2 = \frac{-3\lambda_3}{2}}$

let Put λ_1, λ_2 in eq (1)

$$\rightarrow 2\left(1 - 2\left(\frac{-3\lambda_3}{2}\right) - \lambda_3\right) - \left(\frac{-3\lambda_3}{2}\right) + \lambda_3 = 1$$

$$-\lambda_3/2 - 2 = 1 \rightarrow \boxed{\lambda_3 = -6}$$

Put λ_3 in λ_2 $\longrightarrow$ $\lambda_2 = \frac{-3(-6)}{2}$

$$\boxed{\lambda_2 = 9}$$

put $\lambda_2 \lambda_3$ in λ_1 $\longrightarrow$ $\lambda_1 = 1 - 2\lambda_2 - \lambda_3$

$$= 1 - 2(9) - (-6)$$

$$\boxed{\lambda_1 = -11}$$

The cordinates of a in the basis: $\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} -11 \\ 9 \\ -6 \end{bmatrix}$

Ques (3) let $a = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $\rightarrow$ arbitary vector of E. What are the cordinates of A in basis B

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 4 & 4 \\ -2 & -1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \longrightarrow \begin{array}{ll} a + 2b + c = x & \longrightarrow \text{eq (1)} \\ a + 4b + 4c = y & \longrightarrow \text{eq (2)} \\ -2a - b + 2c = z & \longrightarrow \text{eq (3)} \end{array}$$

Eliminate a from eq (1) $\boxed{a = -2b - c + x}$

Put a in eq (2) $x - 2b - c + 4b + 4c = y$

Eliminate b $\longrightarrow$ $\boxed{b = \frac{y - 3c - x}{2}}$

Put b in eq (3) $\quad -4x + 2y - 9_2 + x/_2 - 4/_2 = z$

$-8x + 4y - c + x - y = 2z$

$\boxed{c = -7x + 3y - 2z}$

Put c in b $\longrightarrow$ $b = \dfrac{-8y + 21x + 6z - x}{2}$

$\boxed{b = 10x - 4y + 3z}$

Put b, c in a $\longrightarrow$ $a = -2(10x - 4y + 3z) - (-2z + 3y - 7x) + x$

$\boxed{a = -12x + 5y - 4z}$

$$\left.\begin{array}{l} a = -12x + 5y - 4z \\ b = 10x - 4y + 3z \\ c = -7x + 3y - 2z \end{array}\right] \longrightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \underbrace{-12x + 5y - 4z}_{\lambda_1} \underbrace{\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}}_{e_1} + \underbrace{10x - 4y + 3z}_{\lambda_2} \underbrace{\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}}_{e_2} + \underbrace{-7x + 3y - 2z}_{\lambda_3} \underbrace{\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}}_{e_3}$$

So the cordinates of a in basis B is

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} -12x + 5y - 4z \\ 10x - 4y + 3z \\ -7x + 3y - 2z \end{bmatrix}$$

Exercise - 2

let $f: \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map defined by its matrix in the standard basis of $\mathbb{R}^3$

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 5 & -3 \\ 4 & 1 & -1 \end{bmatrix} \longrightarrow \text{let call } A \text{ as this matrix.}$$

Ques ① → To find the basis of ker(A)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \ker(A) \Longleftrightarrow \begin{aligned} x - 2y + z &= 0 \\ 2x + 5y - 3z &= 0 \quad \longleftarrow R_2 = R_2 - 2R_1 \\ 4x + y - z &= 0 \quad \longleftarrow R_3 = R_3 - 4R_1 \end{aligned}$$

(row reduction method)

$$\begin{bmatrix} 1 & -2 & 1 \\ 2 & 5 & -3 \\ 4 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/9 \\ 0 & 1 & -5/9 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \text{be the Pivot values}$$

Lets solve the matrix equation

$$\begin{bmatrix} 1 & 0 & -1/9 \\ 0 & 1 & -5/9 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Assume as "t" $\longrightarrow$ $x = t/9, \quad y = 5t/9, \quad z = t$

There fore

The Basis of ker (A) $= \begin{bmatrix} t/9 \\ 5t/9 \\ t \end{bmatrix} = \begin{bmatrix} 1/9 \\ 5/9 \\ 1 \end{bmatrix} t$

Ques ② Deduce the dimension of Im(A)

By Rank Theorem

$\hookrightarrow$ $\dim(E) = \dim(\ker(A)) + \dim(\mathrm{Im}(A))$

To calculate num of linearly independent rows

$$\xrightarrow[\text{from}]{\text{eliminate}} \begin{bmatrix} 1 & -2 & 1 \\ 2 & 5 & -3 \\ 4 & 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 9 & -5 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \text{rank is } 2 \text{ for } \dim(r)$$

$\boxed{\dim(\ker(A)) = 1} \longrightarrow$ By Rank Theorem

So $\longrightarrow$ $\underbrace{2}_{\dim(r)} = \underbrace{1}_{\dim(\ker(A))} + \dim(\operatorname{Im}(A))$

$\longrightarrow$ Hence $\boxed{\dim(\operatorname{Im}(A)) = 1}$

Ques ③ Find the Basis of $\operatorname{Im}(A)$

We need to find the vectors that are linearly independent

Reduce the Matrix by Reduced Row Echelon Form (RREF)

$$\begin{bmatrix} 1 & -2 & 1 \\ 2 & 5 & -3 \\ 4 & 1 & -1 \end{bmatrix} \xrightarrow{-2R_1 + R_2} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 9 & -5 \\ 4 & 1 & -1 \end{bmatrix} \xrightarrow{-4R_1 + R_3} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 9 & -5 \\ 0 & 9 & -5 \end{bmatrix} \xrightarrow{-R_2 + R_3} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 9 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

After Proof, the matrix have 2 Pivots

To obtain the basis of a column space, we use

$$\alpha \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + \beta \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix} \longrightarrow \alpha, \beta \text{ are real numbers}$$

are vectors in column ① & ②

Given span of $\operatorname{Im}(A)$

Hence $\operatorname{Im}(A) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} -2 \\ 5 \\ 1 \end{pmatrix} \right\}$

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Ques 4 Is (A) Bijective?

Matrix A is Bijective iff $ker(f) = \{0, P\}$ "x" is unique
$Im(f) = $ if $\hat{}$ x $\forall$ y exists

In the given Problem
$\dim(ker(A)) = 1$ ⟶ Rank Theorem

Also $\dim(Im(A)) = 1$

For a matrix A ⟶ $m \times n$:

- If matrix has Full Rank (Rank A = min(m, n), A is
  - ⟶ Bijective if $m = n$ = Rank A
  - ⟶ Injective if $m \neq n$ - Rank A
  - ⟶ Surjective if $n \neq m$ = rank A

Hence Rank (A) $\neq 3$
Rank (A) = 2 & $\dim(ker(A) \neq 0)$

So "A" is **NOT BIJECTIVE**

Ques 5 We want to solve the system

$$x - 2y + z = 3$$
$$2x + 5y - 3z = -4$$
$$4x + y - z = 2$$

(a) Particular Solution

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 2 & 5 & -3 & -4 \\ 4 & 1 & -1 & 2 \end{array}\right]$$

RREF
$$\begin{bmatrix} 1 & 0 & -1/9 & 7/9 \\ 0 & 1 & -5/9 & -10/9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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Set all free vectors to 0

$$\boxed{\begin{aligned} x &= 7/9 \\ y &= -10/9 \\ z &= 0 \end{aligned}}$$

Hence Particular Solution is $\begin{bmatrix} 7/9 \\ -10/9 \\ 0 \end{bmatrix}$

(b)

In RREF → write equations

$$x - 1/9\, z = 7/9$$

$$y - 5/9\, z = -10/9$$

$$x = 7/9 + 1/9\, z$$

$$y = -10/9 + 5/9\, z$$

$$\longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7/9 + 1/9\, z \\ -10/9 + 5/9\, z \\ z \end{bmatrix}$$

General Solution → set of all solutions

$$\begin{bmatrix} 7/9 \\ -10/9 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1/9 \\ 5/9 \\ 1 \end{bmatrix}$$

Exercise - 3

Let use define the vector $\boxed{a \in \mathbb{R}^{50}}$ → 50 components
→ A, s, D are samples

$u \rightarrow 1$ → provided.

Ques ① To start, we search with an exact relation

Proof $\boxed{d = \lambda_0 u + \lambda_1 a + \lambda_2 s}$ → given

Steps.
$$d_1 = \lambda_0 + \lambda_1 a_1 + \lambda_2 s_1$$
$$d_2 = \lambda_0 + \lambda_1 a_2 + \lambda_2 s_2$$
$$d_3 = \lambda_0 + \lambda_1 a_3 + \lambda_2 s_3$$
$$\vdots$$
$$d_{50} = \lambda_0 + \lambda_1 a_{50} + \lambda_2 s_{50}$$

There is a system of linear equations having 3 unknown $(N = 3)$ and $(M = 50)$ → independent equations

- check values in the excel sheet

Since $M > N$ ie no of equations > no of unknowns
So there is No solution

Ques ②

ⓐ Given
$$MSE = \frac{1}{50} \sum_{k=1}^{50} (d_k - \lambda_0 - \lambda_1 a_k - \lambda_2 s_k)^2$$

The best values of $(\lambda_0 \lambda_1 \lambda_2)$ are those that minimize the expression $(d_k - \lambda_0 - \lambda_1 a_k - \lambda_2 s_k)$ over all the values of

a_k & s_k, $k \longrightarrow 1$ to 50

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This hapens when the value of $\lambda_0 + \lambda_1 q_k + \lambda_2 s_k$ is absent to d_k at each value of k

↓

For a specific set of $\lambda_0\ \lambda_1\ \lambda_2$

The closet value of $d = [d_1\ d_2\ \cdots\ d_{50}]$ to $F = (u, q, s)$ can be found by taking the orthogonal projection of d onto F.

Orthogonal projection of D onto F

$$d_F = \frac{d\,u}{\|u\|^2} u + \frac{d\,a}{\|a\|^2} a + \frac{d_s}{\|s\|^2} s$$

Proof → F is the subspace in $\mathbb{R}^{50}$

let $d_F \rightarrow$ orthogonal projection of d onto F

$d_r \rightarrow$ vector such that $d_r = d - d_F$

$d_0 \rightarrow$ vector such that $d_0 = d - f$

where $f \rightarrow$ any vector in subspace F

let $v_1 = d_F - f \implies d_0 = d_r + v_1$

since $\boxed{d_r \perp F} \rightarrow d_r \cdot v_1 = 0$

$$\|d_0\|^2 = d_0 \cdot d_0$$

$$= (d_r + v_1) \cdot (d_r + v_1)$$

$$= d_r \cdot d_r + d_r \cdot v_1 + v_1 \cdot d_r + v_1 \cdot v_1$$

$$= \|d_r\|^2 + \|v_1\|^2 \geq \|d_r\|^2$$

$$\rightarrow \boxed{\|d - d_F\| = \min_{f \in F} \|d - f\|}$$

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