Lecture notes

Subject

Optimal Control

automation & control systems group

University of Novi Sad, Faculty of Technical Sciences, Automation and Control Systems Group

Agenda

- Pontrayagin’s vs. Bellmans’ theory
- Pontryagin's maximum principle
- Hamilton–Jacobi–Bellman (HJB) equation
- Linear regulators (LQR)-HJB approach
- Linear regulators Pontryagin approach

L.S.Pontrjagin

1958. year

R.E.Bellman

1957. year

“If you don’t do the best with what you happen to have got, you’ll never do the best you might have done with what you should have”

$$\dot{x} = f\left(t, \mathbf{x}, \mathbf{u}\right)$$

$$I = \int_{t_0}^{t_1} F\left(t, \mathbf{x}, \mathbf{u}\right) dt$$

$$\mathbf{x}\left(t_0\right) = \mathbf{x}_0$$

L.S.Pontryagin – Pontryagin's maximum principle

1958.

$$H(t,\mathbf{x},\mathbf{u},\mathbf{p}) = F(t,\mathbf{x},\mathbf{u}) + \sum_{i=1}^{n} p_i f_i(t,\mathbf{x},\mathbf{u})$$

$$\min_u H(t,\mathbf{x},\mathbf{u},\mathbf{p}) \qquad \frac{\partial H}{\partial u_k} = 0;\ k = 1,\dots,m$$

Two Point Boundary Value Problem

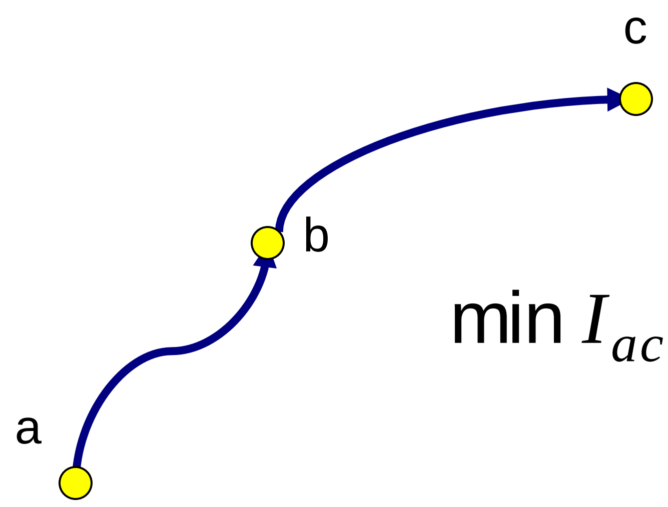
$$\dot{x}_i = \frac{\partial H}{\partial p_i} = f_i(t,\mathbf{x},\mathbf{p}) \qquad x_i(t_0) = x_{i0}$$

$$\dot{p}_i = -\frac{\partial H}{\partial x_i} = -\frac{\partial F}{\partial x_i} - p_i \frac{\partial f_i}{\partial x_i} \qquad p_i(t_1) = 0$$

$$u_{kopt} = u_{kopt}(t)$$ Open loop solution

R.E.Bellman – **Dynamic Programming Theory**

1957.



Every part of an optimal process is an optimal as well

$$\min I_{ac} = I_{ab} + I_{bc}$$

Hamilton–Jacobi–Bellman (HJB) equation

$$\frac{\partial S}{\partial t} + H^*\left(t, \mathbf{x}, \frac{\partial S}{\partial \mathbf{x}}\right) = 0$$

$$u_{kopt} = \Theta_k\left(t, \mathbf{x}\right)$$ Closed loop solution

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$$\dot{x}_i = f_i(t, x_i, u_k) \qquad i = 1, \ldots, n \qquad k = 1, \ldots, m$$

$$I = \int_0^T F[t, x_i(t), u_k(t)]dt \qquad 0 \le t \le T \qquad x_i(0) = \alpha_i$$

$$\bar{I} = \int_0^T \left\{ F[t, x_i(t), u_k(t)] - \sum_{i=1}^{n} p_i(t)[\dot{x}_i - f_i(t, x_i, u_k)] \right\} dt$$

$$\boxed{H = F[t, x_i(t), u_k(t)] + \sum_{i=1}^{n} p_i(t) f_i(t, x_i, u_k)}$$

$$\bar{I} = \int_0^T \left\{ H[t, x_i(t), u_k(t), p_i(t)] - \sum_{i=1}^{n} p_i(t)\dot{x}_i \right\} dt$$

$$\bar{I}=\int_0^T\left\{H\left[t,x_i(t),u_k(t),p_i(t)\right]-\sum_{i=1}^{n}p_i(t)\dot{x}_i\right\}dt$$

$$\delta\bar{I}=\int_0^T\left\{\sum_{i=1}^{n}\left[\frac{\partial H}{\partial x_i}\delta x_i+\frac{\partial H}{\partial p_i}\delta p_i-\delta p_i\dot{x}_i-p_i\delta\dot{x}_i\right]+\sum_{k=1}^{m}\frac{\partial H}{\partial u_k}\delta u_k\right\}dt$$

$$-p_i\delta\dot{x}_i=-\frac{d}{dt}\left(p_i\delta x_i\right)+\dot{p}_i\delta x_i$$

$$\delta\bar{I}=-\sum_{i=1}^{n}p_i\delta x_i\Big|_0^T+\int_0^T\left\{\sum_{i=1}^{n}\left[\left(\frac{\partial H}{\partial x_i}+\dot{p}_i\right)\delta x_i+\left(\frac{\partial H}{\partial p_i}-\dot{x}_i\right)\delta p_i\right]+\sum_{k=1}^{m}\frac{\partial H}{\partial u_k}\delta u_k\right\}dt$$

$$\delta \bar{I} = -\sum_{i=1}^{n} p_i \delta x_i \Big|_0^T + \int_0^T \left\{ \sum_{i=1}^{n} \left[\left(\frac{\partial H}{\partial x_i} + \dot{p}_i \right) \delta x_i + \left(\frac{\partial H}{\partial p_i} - \dot{x}_i \right) \delta p_i \right] + \sum_{k=1}^{m} \frac{\partial H}{\partial u_k} \delta u_k \right\} dt = 0$$

$$k = 1, \ldots, m$$

$$i = 1, \ldots, n$$

$$\dot{x}_i = \frac{\partial H}{\partial p_i} = f_i(t, x_i, u_k)$$

$$x_i(0) = \alpha_i$$

$$-p_i(T)\delta x_i(T) + p_i(0)\delta x_i(0) = 0$$

$$\dot{p}_i = -\frac{\partial H}{\partial x_i}$$

$$p_i(T) = 0$$

$$\frac{\partial H}{\partial u_k} = 0$$

$$\frac{\partial^2 H}{\partial u_k^2} \geq 0$$

$$\frac{\partial^2 H}{\partial u_k^2} \leq 0$$

$$\dot{x}_i = \frac{\partial H}{\partial p_i} = f_i\left(t, x_i, u_k\right) \qquad \dot{p}_i = -\frac{\partial H}{\partial x_i} \qquad \frac{\partial H}{\partial u_k} = 0$$

$$\frac{dH}{dt} = \frac{\partial H}{\partial x_i}\dot{x}_i + \frac{\partial H}{\partial p_i}\dot{p}_i + \frac{\partial H}{\partial u_k}\dot{u}_k + \frac{\partial H}{\partial t}$$

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}$$

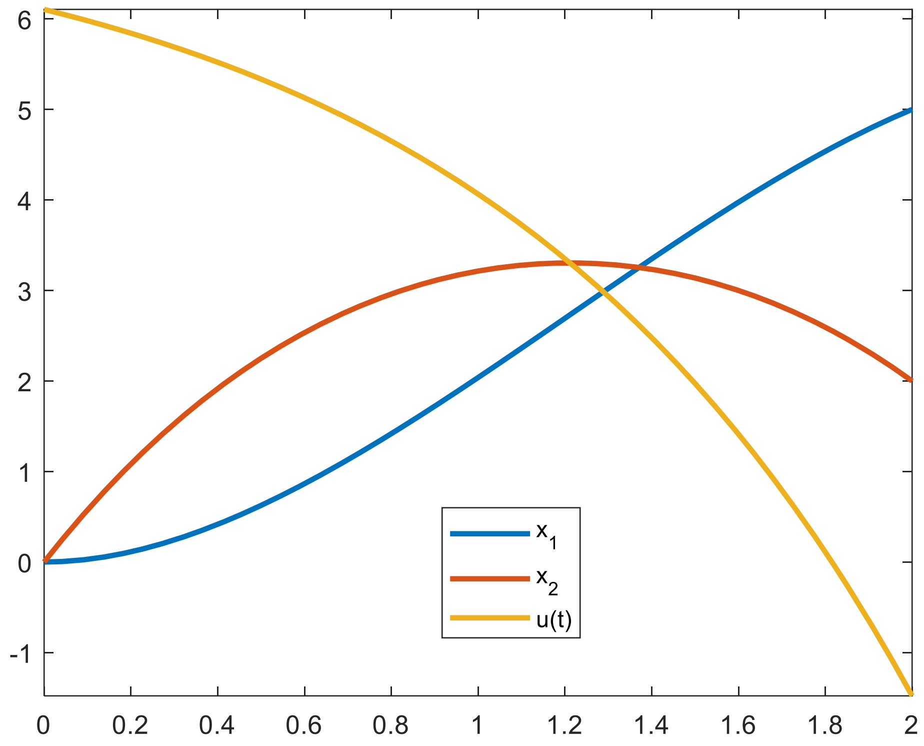
$$H = const$$

Example 1

$$\dot{x}_1 = x_2(\mathrm{t})$$
$$\dot{x}_1 = -x_2(\mathrm{t}) + u(t)$$

$$I(x,u) = \frac{1}{2}\int_0^2 u^2 dt$$

$$x_1(0) = x_2(0)=0$$
$$x_1(2) = 5$$
$$x_2(2)=2$$



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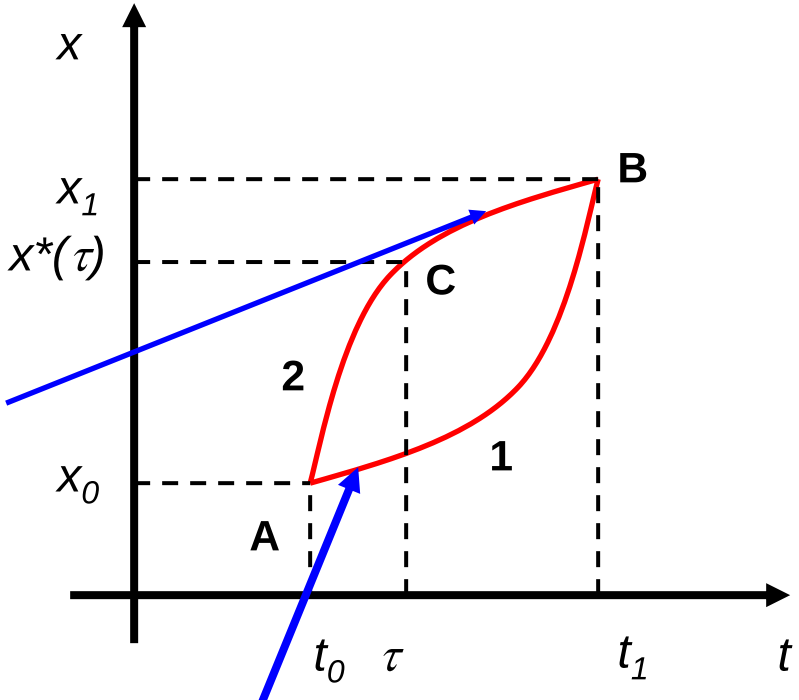
$$\dot{x} = f(t, x, u) \qquad x(t_0) = x_0$$

$$I = \int_{t_0}^{t_1} F(t, x, u)\,dt$$



$$\int_{t_0}^{\tau} F(t, x, u)\,dt + S\left[\tau, x^*(\tau)\right]$$

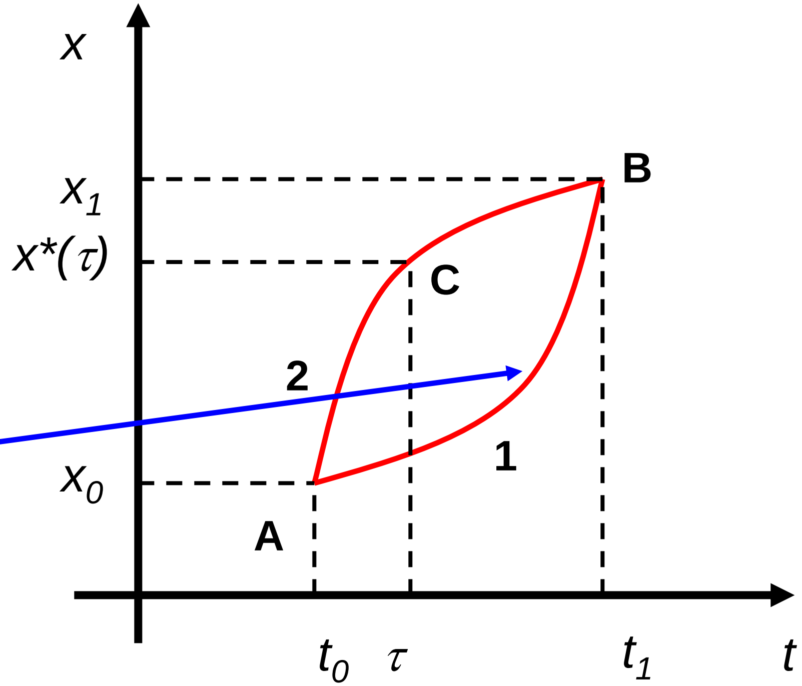
$$t_0 \le t \le \tau \qquad u_{opt}^* = u_{opt}^*(t)$$

$$S(t_0, x_0) = \int_{t_0}^{t_1} F\left(t, x_{opt}(t), u_{opt}(t)\right)dt$$

$$S(t_0, x_0) = \min_u \left\{ \int_{t_0}^{\tau} F(t,x,u)dt + S\left[\tau, x^*(\tau)\right] \right\}, \quad t_0 \le t \le \tau$$

$$t_0 + dt = \tau$$

$$x_0 + dx = x^*$$



$$S(t_0, x_0) = \int_{t_0}^{t_1} F\left(t, x_{opt}(t), u_{opt}(t)\right)dt$$

$$x^*(\tau) = x^*(t_0 + dt) \approx x_0 + \frac{dx}{dt}dt = x_0 + f(t, x_0, u_0)dt \quad \Leftarrow \quad \dot{x} = f(t,x,u)$$

$$S\left[\tau, x^*(\tau)\right] \approx S(t_0, x_0) + \frac{\partial S(t_0, x_0)}{\partial x} f(t, x_0, u_0)dt + \frac{\partial S(t_0, x_0)}{\partial t}dt$$

$$\int_{t_0}^{\tau} F(t,x,u)dt = \int_{t_0}^{t_0+dt} F(t,x,u)dt \approx F(t, x_0, u_0)dt$$

$$S(t_0,x_0)=\min_{u_0}\{F(t,x_0,u_0)dt+S(t_0,x_0)+$$

$$+\frac{\partial S(t_0,x_0)}{\partial x}f(t,x_0,u_0)dt+\frac{\partial S(t_0,x_0)}{\partial t}dt\}$$

$$\frac{\partial S(t_0,x_0)}{\partial t}+\min_{u_0}\left\{F(t,x_0,u_0)+\frac{\partial S(t_0,x_0)}{\partial x}f(t,x_0,u_0)\right\}=0$$

General form without index “0”

$$\frac{\partial S(t,x)}{\partial t}+\min_{u}\left\{F(t,x,u)+\frac{\partial S(t,x)}{\partial x}f(t,x,u)\right\}=0$$

General form–multivaribil system

$$\frac{\partial S(t,x)}{\partial t}+\min_{u}\left\{F(t,\mathbf{x},\mathbf{u})+\sum_{i=1}^{n}\frac{\partial S}{\partial x_i}f_i(t,\mathbf{x},\mathbf{u})\right\}=0$$

$$\frac{\partial S(t,x)}{\partial t} + \min_u \left\{ F(t,\mathbf{x},\mathbf{u}) + \sum_{i=1}^{n} \frac{\partial S}{\partial x_i} f_i(t,\mathbf{x},\mathbf{u}) \right\} = 0$$

$$p_i = \frac{\partial S(t,\mathbf{x})}{\partial x}; \qquad i = 1,\dots,n$$

$$H(t,\mathbf{x},\mathbf{u},\mathbf{p}) = F(t,\mathbf{x},\mathbf{u}) + \sum_{i=1}^{n} p_i f_i(t,\mathbf{x},\mathbf{u})$$

$$\frac{\partial S}{\partial t} + \min_u H\left(t,\mathbf{x},\mathbf{u},\frac{\partial S}{\partial \mathbf{x}}\right) = 0 \qquad \frac{\partial H}{\partial u_k} = 0;\ \mathrm{k} = 1,\dots,\mathrm{m} \qquad u_{kopt} = \Theta_k\left(t,\mathbf{x},\frac{\partial S}{\partial \mathbf{x}}\right) = 0$$

$$\frac{\partial S}{\partial t} + H^*\left(t,\mathbf{x},\frac{\partial S}{\partial \mathbf{x}}\right) = 0$$

$$S = S(t,\mathbf{x})$$

$$u_{kopt} = \Theta_k(t,\mathbf{x})$$

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$$\dot{x} = ax + bu \qquad x(t_0) = x_0 \qquad t_0 \le t \le t_1$$

$$I = \frac{1}{2}\Psi x^2(t_1) + \frac{1}{2}\int_{t_0}^{t_1}\left[qx^2 + ru^2\right]dt$$

$$H = \frac{1}{2}\left(qx^2 + ru^2\right) + p(ax + bu)$$

$$\frac{\partial H}{\partial u} = 0 \qquad u_{opt} = -br^{-1}p(t)$$

$$H^* = \frac{1}{2}qx^2 + pax - \frac{1}{2}b^2r^{-1}p^2$$

$$\boxed{\begin{aligned} \dot{x} &= \frac{\partial H^*}{\partial p} = ax - b^2r^{-1}p(t) \\ \dot{p} &= -\frac{\partial H^*}{\partial p} = -qx - p(t)a \end{aligned}}$$

$$\dot{x} = \frac{\partial H^*}{\partial p} = ax - b^2 r^{-1} p(t)$$

$$p(t_1) = \Psi x(t_1)$$

$$\boxed{p(t) = P(t)x(t)}$$

$$\dot{p} = -\frac{\partial H^*}{\partial x} = -qx - p(t)a$$

$$\dot{p}(t) = \dot{P}(t)x(t) + P(t)\dot{x}(t)$$

$$\left(\dot{P} + 2aP - b^2 r^{-1} P^2 + q\right)x = 0$$

$$\dot{P} + 2aP - b^2 r^{-1} P^2 + q = 0 \qquad P(t_1) = \Psi$$

$$u_{opt} = -br^{-1}P(t)x(t)$$

$$u_{opt} = K(t)x(t)$$

$$K(t) = -br^{-1}P(t)$$

$$\dot{x} = \frac{\partial H^*}{\partial p} = ax - b^2 r^{-1} p(t)$$

$$p(t_1) = \Psi x(t_1)$$

$$\boxed{p(t) = P(t)x(t)}$$

$$\dot{p} = -\frac{\partial H^*}{\partial x} = -qx - p(t)a$$

$$\dot{p}(t) = \dot{P}(t)x(t) + P(t)\dot{x}(t)$$

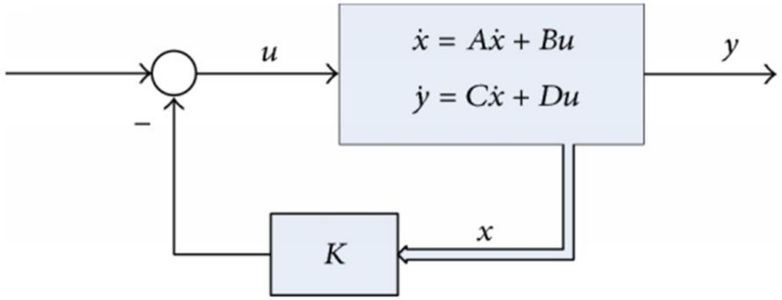
$$\left(\dot{P} + 2aP - b^2 r^{-1} P^2 + q\right)x = 0$$

General form–multivariable system

$$\dot{P} + A^T P + PA - PBR^{-1}B^T P + Q = 0 \qquad P(t_1) = \Psi$$

$$u_{opt} = K(t)x(t)$$

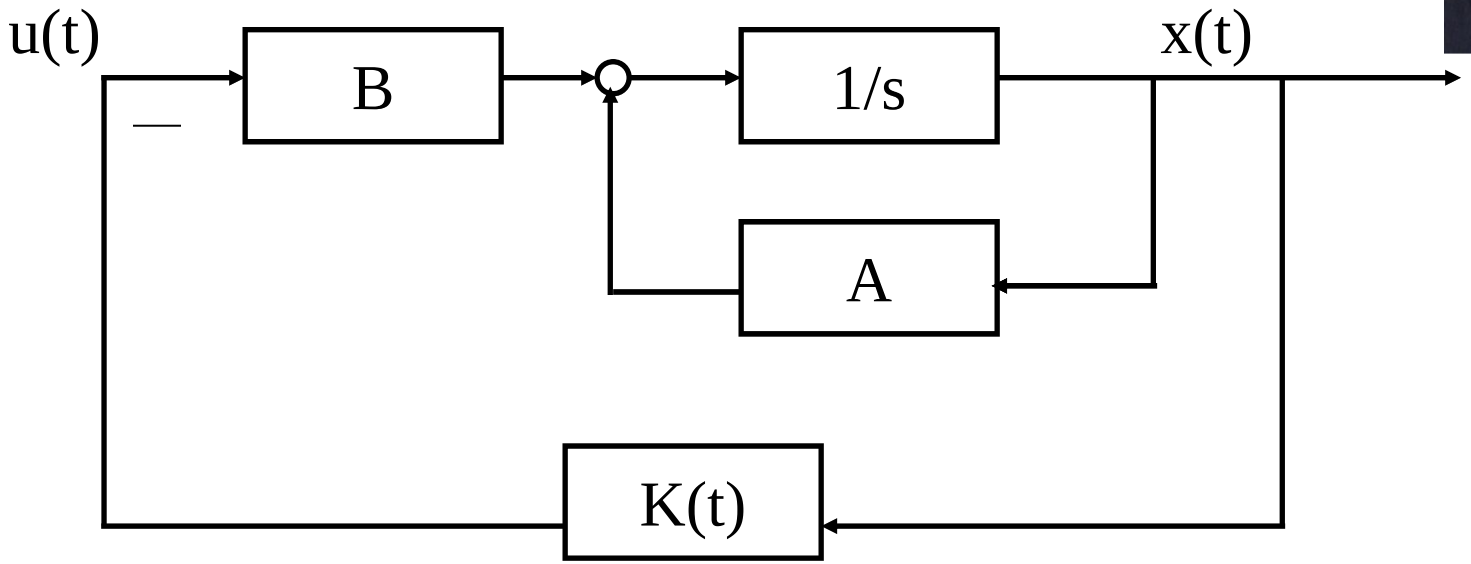
$$K(t) = -R^{-1}B^T P(t)$$



$$\dot{\mathbf{P}} + \mathbf{A}^T\mathbf{P} + \mathbf{PA} - \mathbf{PBR}^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{Q} = 0$$

$$\mathbf{u}_{opt} = \mathbf{K}(t)\mathbf{x}(t)$$

$$\mathbf{K}(t) = -\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}(t)$$



Example 1

$$\dot{x}(t) = ax(t) + u(t)$$

$$I = \frac{1}{2}\Psi x^2(T) + \int_0^T \frac{1}{4}u^2(t)dt$$

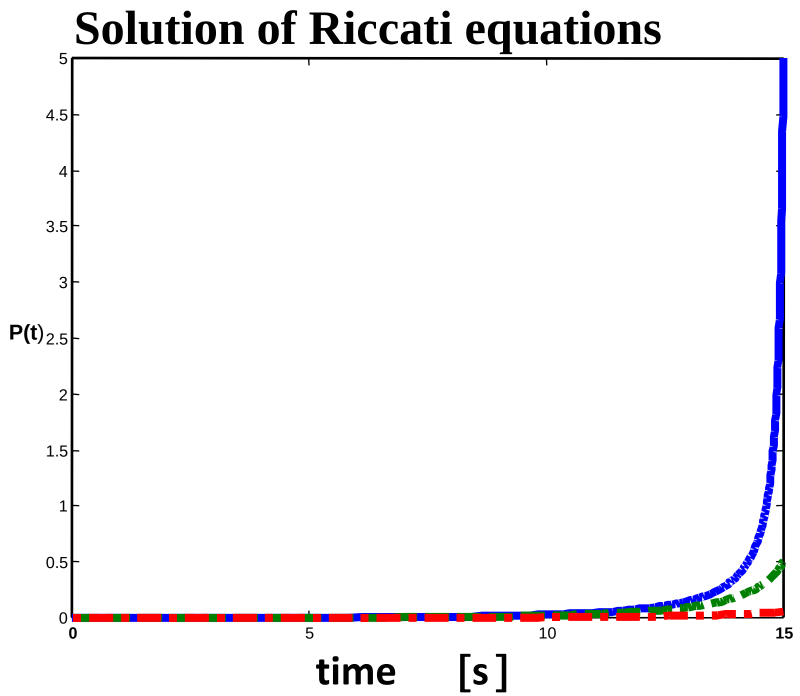
$$H^* = \tfrac{1}{4}u^2 + p(ax + u)$$

$$u_{opt} = -2p \longrightarrow u_{opt}(t) = -2P(t)x(t)$$

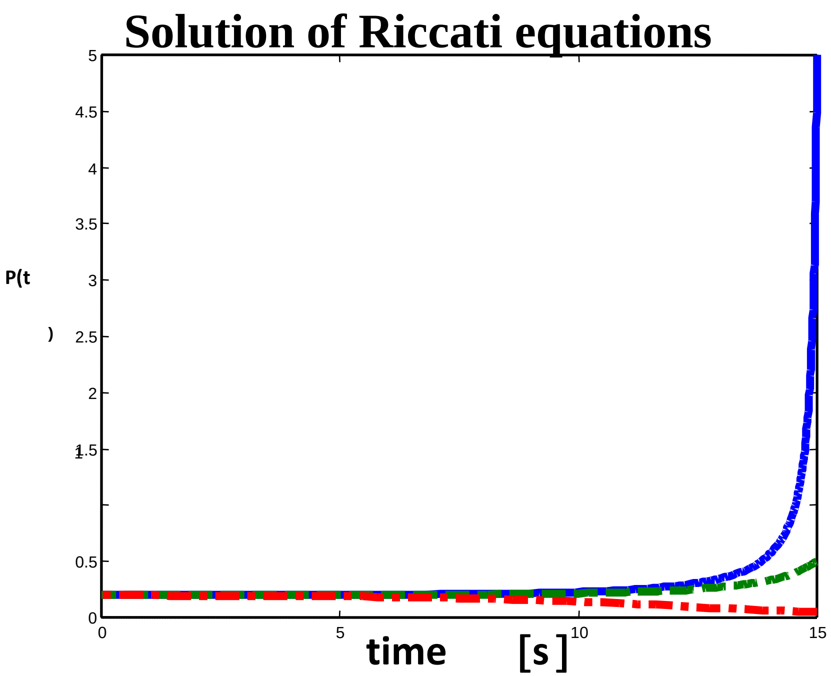
$$\dot{P}(t) = 2P^2(t) - 2aP(t)$$

$$P(T) = \Psi$$

Example 1

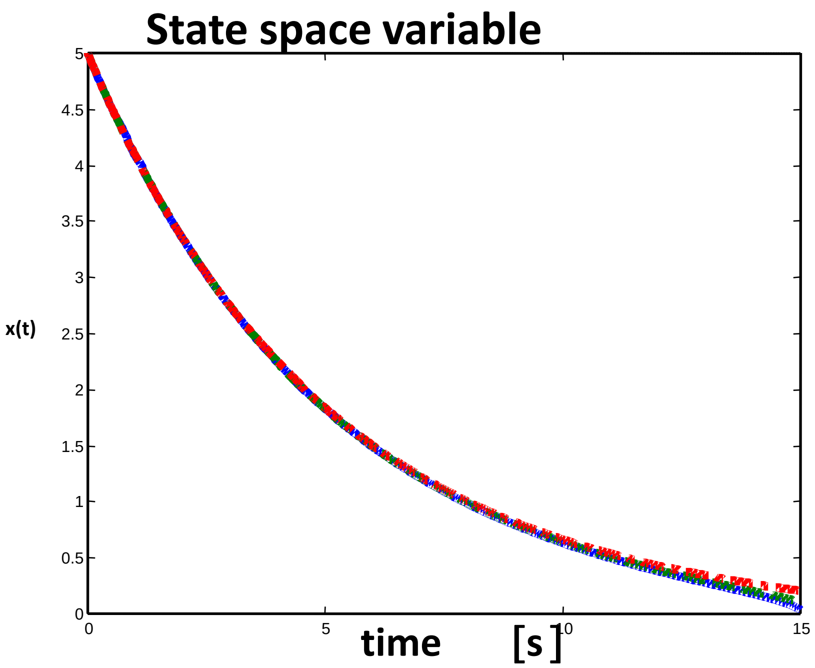


H=5
H=0.5
H=0.05
a=-0.2,T=15s

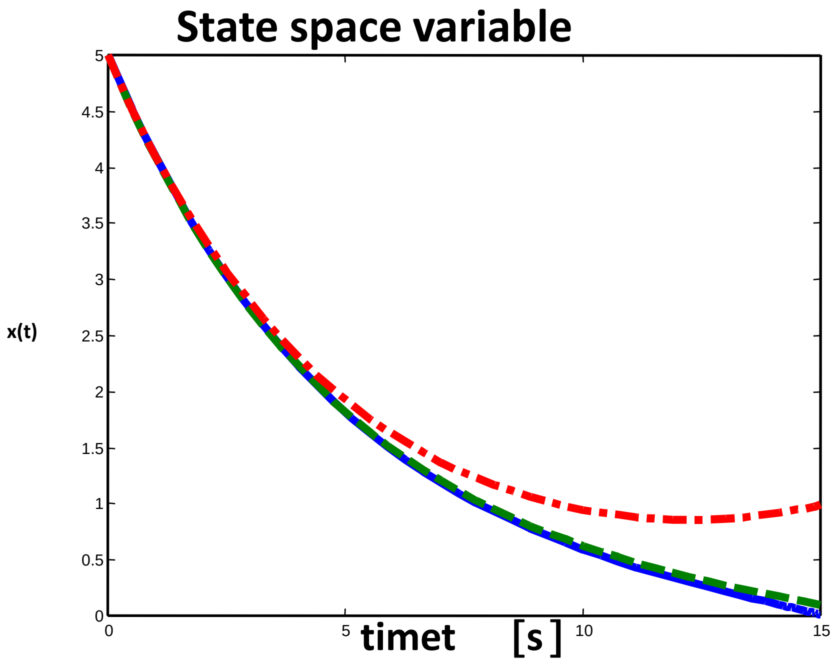


H=5
H=0.5
H=0.05
a=0.2,T=15s

Example 1

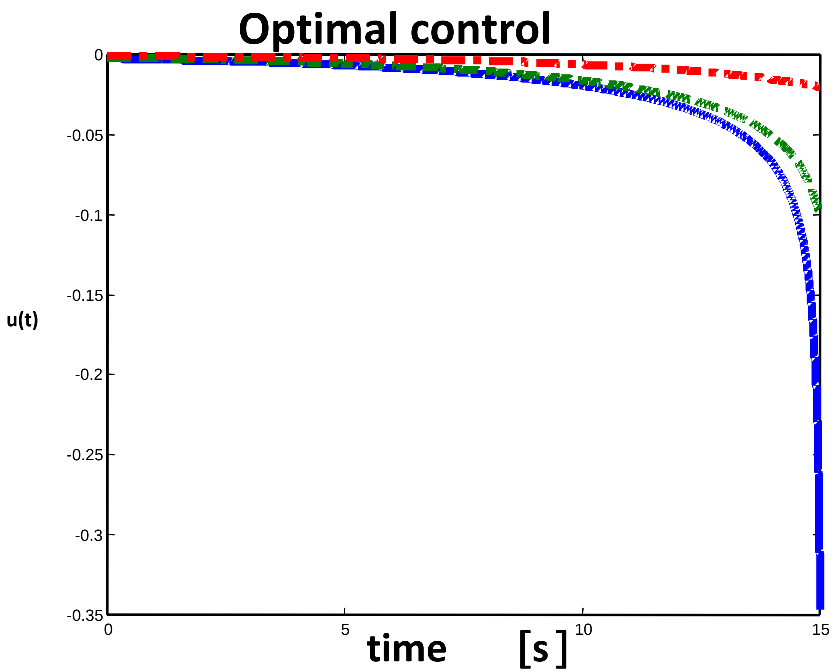


H=5
H=0.5
H=0.05
a=-0.2, T=15s,
x(0)=5



H=5
H=0.5
H=0.05
a=0.2, T=15s,
x(0)=5

Example 1



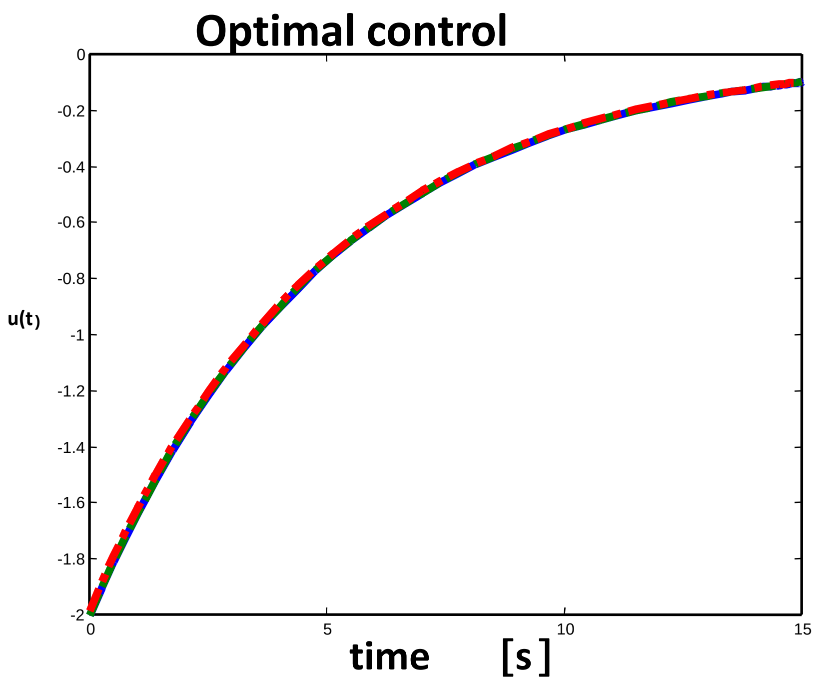
H=5

H=0.5

H=0.05

a=-0.2, T=15s,

x(0)=5



H=5

H=0.5

H=0.05

a=0.2, T=15s,

x(0)=5

Example 2

$$\dot{x}_1(t) = x_2(t)$$

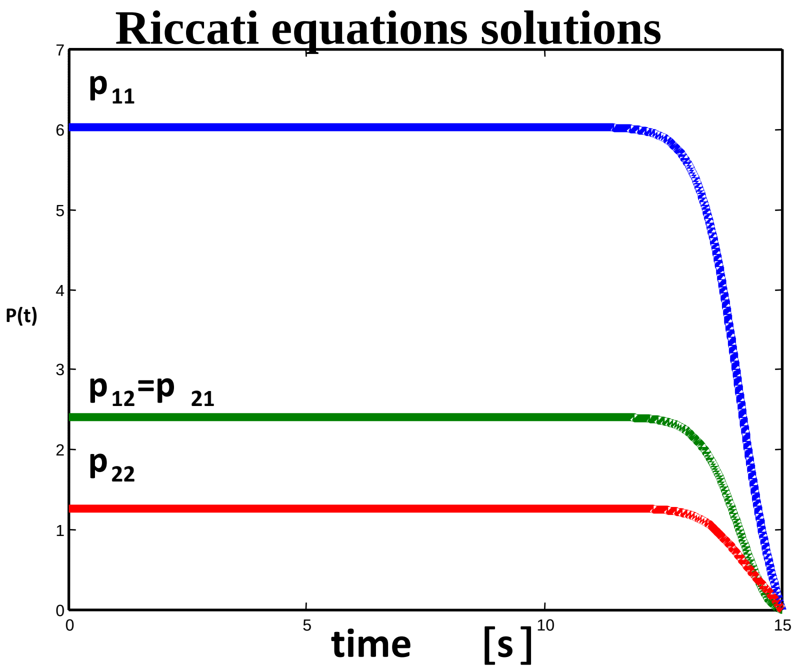
$$\dot{x}_2(t) = 2x_1(t) - x_2(t) + u(t)$$

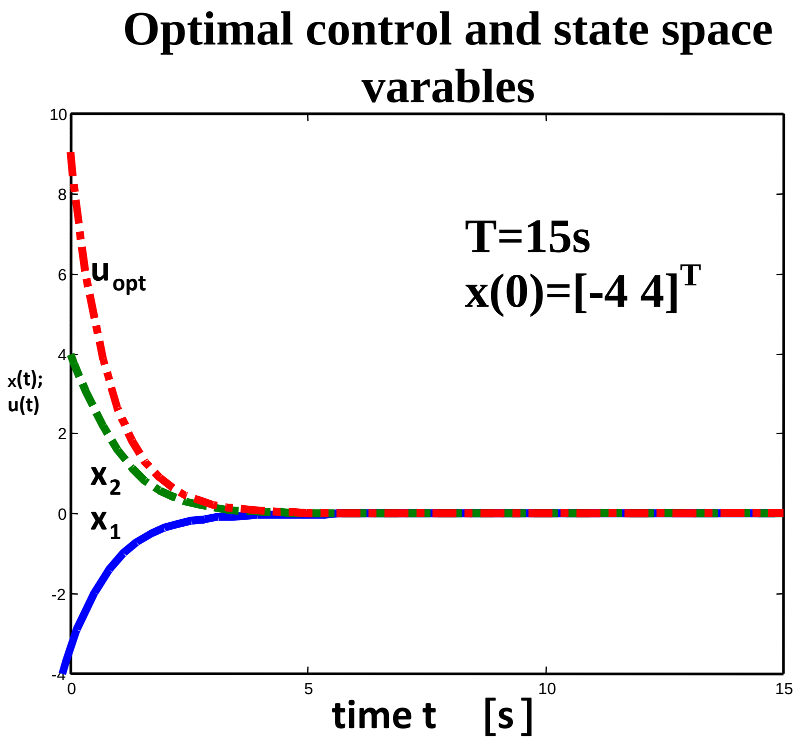
$$I = \int_0^T \left(x_1^2(t) + \tfrac{1}{2}x_2^2(t) + \tfrac{1}{4}u^2(t)\right)dt$$

$$H = x_1^2 + \tfrac{1}{2}x_2^2 + \tfrac{1}{4}u^2 + p_1 x_2 + p_2\left(2x_1 - x_2 + u\right)$$

$$u_{opt} = -2\left[p_{12}(t) \quad p_{22}(t)\right]\mathbf{x}$$

Example 2

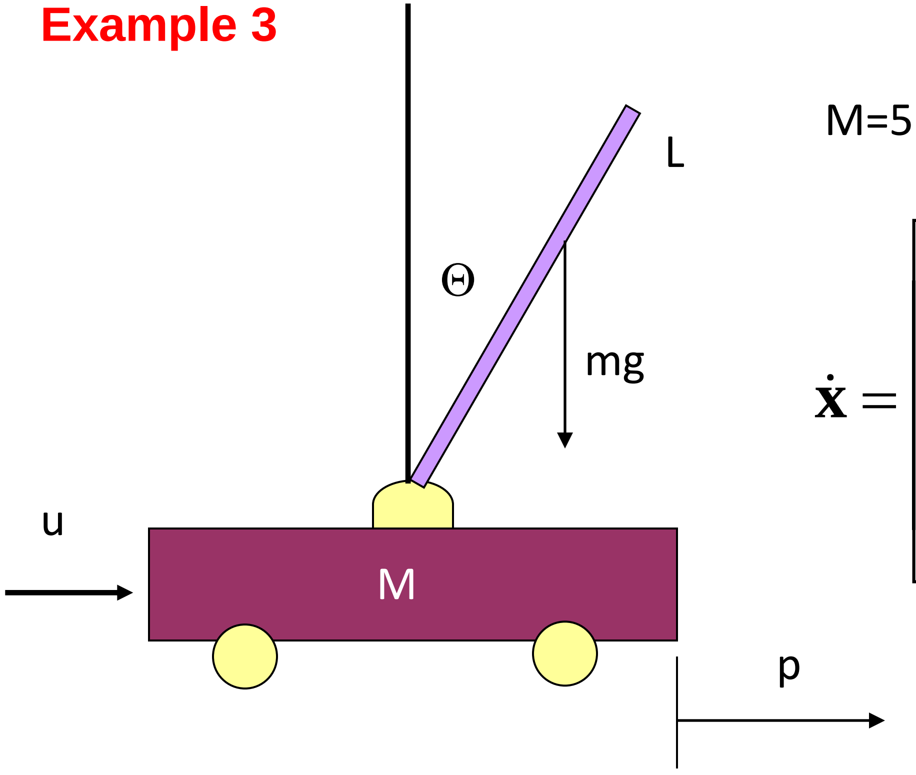




- System is controllable
- Ψ=0 (Lagrange problem)
- Matrices A,B,Q,R are time invariant

$$\Big\} \quad P(t) \rightarrow P, t_f \rightarrow \infty$$

Example 3



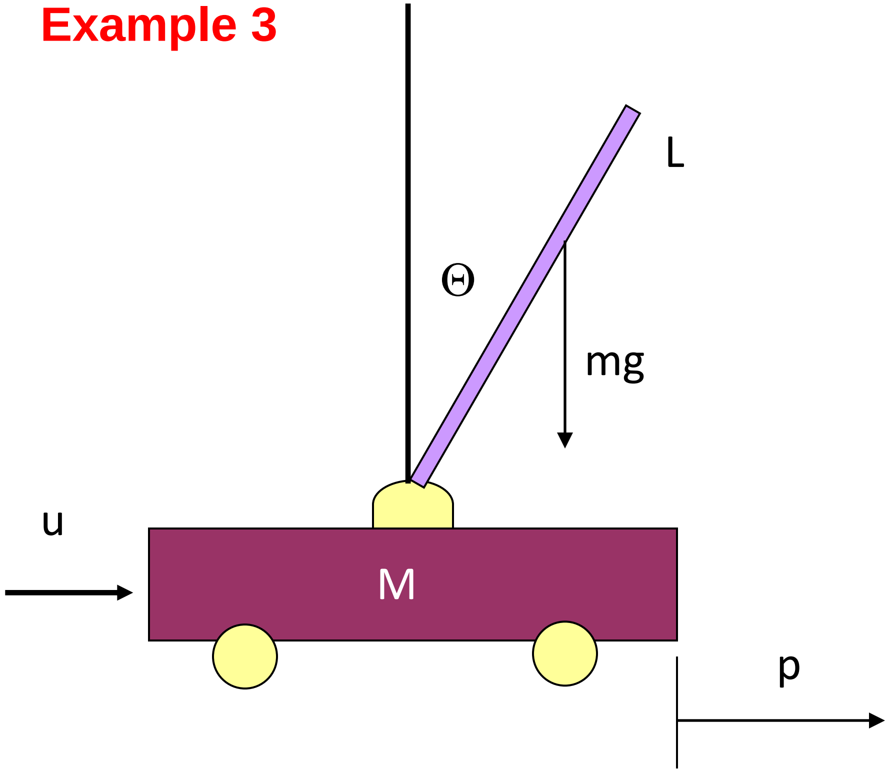
M=5kg, m=0.5kg, L=1m

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 10.78 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -0.98 & 0 & 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ -0.2 \\ 0 \\ 0.2 \end{bmatrix} \mathbf{u}$$

$$\mathbf{x}(0) = \begin{bmatrix} \Theta \\ \dot{\Theta} \\ p \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0.1 \\ 0 \\ 0.1 \\ 0 \end{bmatrix}$$

$$\mathbf{Q} = \begin{bmatrix} 100 & 0 & 0 & 0 \\ 0 & 100 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix} \qquad R = 1$$

Example 3



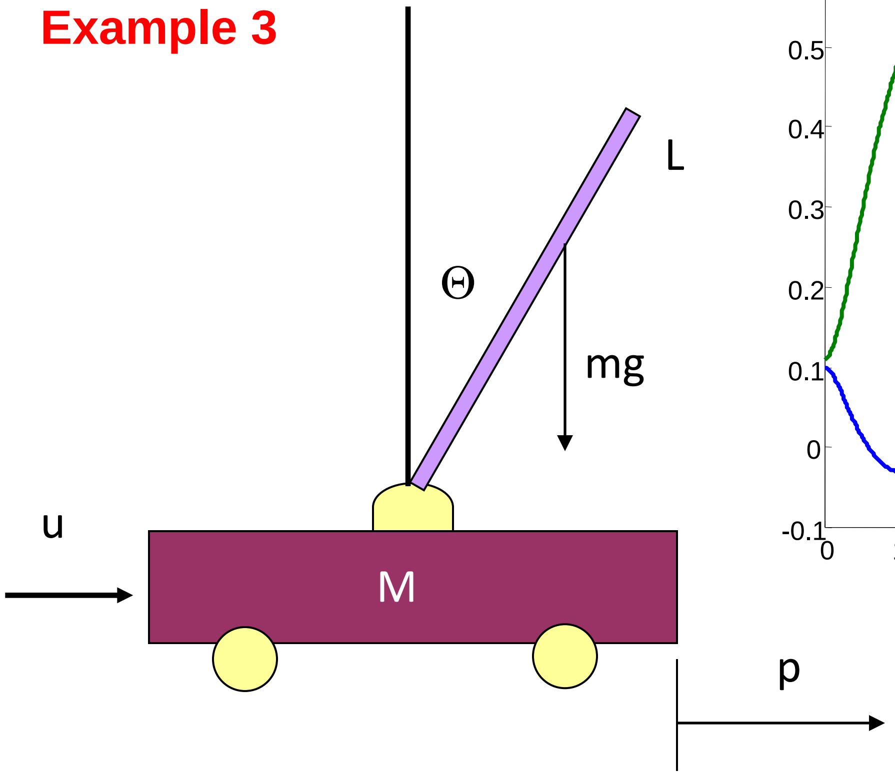
System without feedback control, eigenvalues A

0
0
3.2833
-3.2833

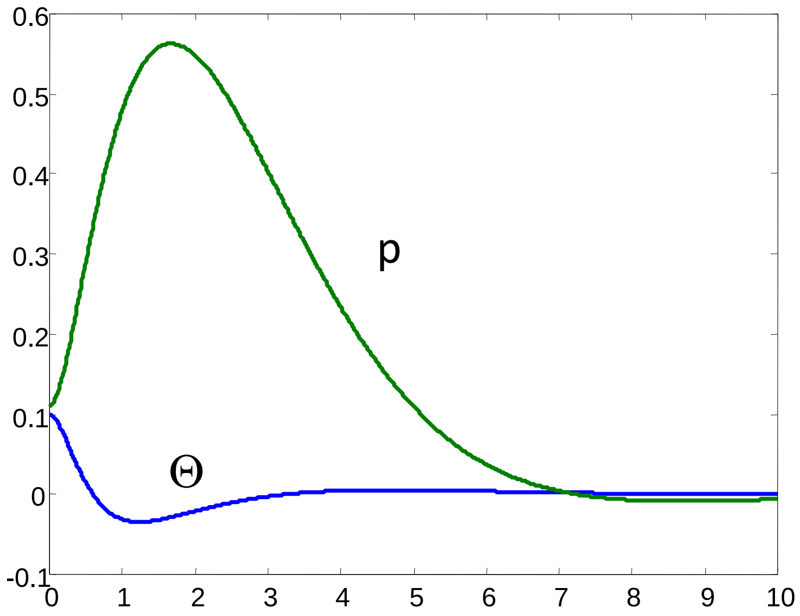
System with feedback cotnrol, eigenvalues Ac

-4.4141
-2.4817
-0.60181 + 0.45126i
-0.60181 - 0.45126i

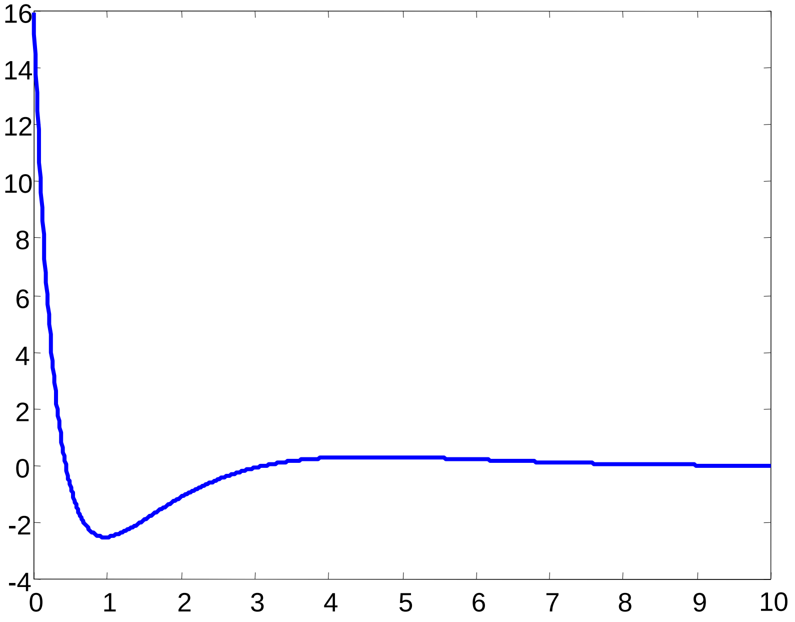
Example 3



L
Θ
mg
u
M
p



p
Θ
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
0 1 2 3 4 5 6 7 8 9 10



16
14
12
10
8
6
4
2
0
-2
-4
0 1 2 3 4 5 6 7 8 9 10

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$$\dot{x} = ax + bu$$

$$x(t_0) = x_0 \qquad t_0 \le t \le t_1 \qquad I = \frac{1}{2}\Psi x^2(t_1) + \frac{1}{2}\int_{t_0}^{t_1}\left[qx^2 + ru^2\right]dt$$

$$H = \frac{1}{2}\left(qx^2 + ru^2\right) + p(ax + bu) \qquad \frac{\partial H}{\partial u} = 0 \qquad u_{opt} = -br^{-1}p(t)$$

$$H^* = \frac{1}{2}\left(qx^2 + r\left(-br^{-1}p(t)\right)^2\right) + p\left(ax + b\left(-br^{-1}p(t)\right)\right)$$

$$H^* = \frac{1}{2}qx^2 + pax - \frac{1}{2}b^2r^{-1}p^2 \qquad p = \frac{\partial S(t,x)}{\partial x}$$

$$\frac{\partial S}{\partial t} + H^*\left(t, \mathbf{x}, \frac{\partial S}{\partial \mathbf{x}}\right) = 0$$

$$\frac{\partial S}{\partial t} + \frac{1}{2}qx^2 - \frac{1}{2}b^2r^{-1}\left(\frac{\partial S}{\partial x}\right)^2 + a\frac{\partial S}{\partial x}x = 0$$

$$\frac{\partial S}{\partial t}+\frac{1}{2}qx^2-\frac{1}{2}b^2r^{-1}\left(\frac{\partial S}{\partial x}\right)^2+a\frac{\partial S}{\partial x}x=0$$

$$S=\frac{1}{2}P(t)x^2; \qquad \frac{\partial S}{\partial x}=P(t)x; \qquad \frac{\partial S}{\partial t}=\frac{1}{2}\dot{P}(t)x$$

$$\left(\dot{P}+2aP-b^2r^{-1}P^2+q\right)x^2=0$$

$$\dot{P}+2aP-b^2r^{-1}P^2+q=0 \qquad P(t_1)=\Psi$$

General form–multivariable system

$$\dot{P}+A^TP+PA-PBR^{-1}B^TP+Q=0$$

$$u_{opt}=K(t)x(t)$$

$$K(t)=-R^{-1}B^TP(t)$$