1. Riemannian Geometry: A Computationally Friendly Approach.

We present a methodical procedure for computing important geometric quantities on a Riemannian manifold. We want to emphasize that the purpose of this document is not meant to provide a systematic introduction to Riemannian geometry, or motivate/give intuition for the definitions (although we will occasionally give quick consistency checks); rather, our aim is to provide a formalism that makes the objects of interest (Christoffel symbols, Riemann tensor, etc.) "manageable". This method is not the most streamlined (see, for example, E. Cartan's theory of moving frames, which is arguably easier to work with; furthermore, if the manifold in question is a Lie group equipped with an invariant metric, the required computations reduce drastically), it is certainly friendlier than what is found in most classical differential geometry texts. Furthermore, most of these techniques carry over to vector and principal bundles, equipped with a connection, and so what follows may also be readily applied there. Before beginning the exposition, we summarize the objectives of this document:

1. Compute several examples of Riemannian metrics.
2. Compute the Levi-Cevita connection in terms of the metric.
3. Compute the Riemann tensor in terms of the connection coefficients.
4. Compute the Ricci and scalar curvatures in terms of the Riemann tensor, as well as the Einstein tensor (in some special cases).

1.1. Riemannian metrics.

We will always start by fixing a local coordinate patch (U, x), and work within this coordinate patch. For simplicity in our examples, we will choose manifolds/metrics which admit global coordinates. These examples will be carried throughout the text and built upon, in order to show the utility of the methods we present.

Example. *The Sphere S^2.* Most commonly (and often most intuitively), a manifold is endowed with the metric arising from the restriction Euclidean metric in its embedding space to the manifold. In plain English, this means that distances are measured as they are in the embedding space, but with the additional restriction that we are required to only measure distances restricted to the surface of the space of interest. As a first example, we consider the sphere $S^2 \subset \mathbb{R}^3$. The sphere admits coordinates

$$x = \sin\theta\cos\varphi, \qquad y = \sin\theta\sin\varphi, \qquad z = \cos\theta, \tag{1.1}$$

where $0 \leq \theta \leq \pi, 0 \leq \varphi \leq 2\pi$[1]. In these coordinates, the differentials dx, dy, and dz may be computed readily:

$$dx = \cos\theta\cos\varphi d\theta - \sin\theta\sin\varphi d\varphi,$$
$$dy = \cos\theta\sin\varphi d\theta + \sin\theta\cos\varphi d\varphi,$$
$$dz = -\sin\theta d\theta.$$

Taking tensor products of each coordinate is a little more tedious, but is still a feasible task (we will write expressions like dx^2 or $dxdy$ instead of $dx \otimes dx$ or $dx \otimes dy$, to simplify notations):

$$dx^2 = \cos^2\theta\cos^2\varphi d\theta^2 + \sin^2\theta\sin^2\varphi d\varphi^2 - \cos\theta\sin\theta\cos\varphi\sin\varphi d\theta d\varphi - \cos\theta\sin\theta\cos\varphi\sin\varphi d\varphi d\theta,$$
$$dy^2 = \cos^2\theta\sin^2\varphi d\theta^2 + \sin^2\theta\cos^2\varphi d\varphi^2 + \cos\theta\sin\theta\cos\varphi\sin\varphi d\theta d\varphi + \cos\theta\sin\theta\cos\varphi\sin\varphi d\varphi d\theta,$$
$$dz^2 = \sin^2\theta d\theta^2.$$

Summing these three terms, we obtain the restriction of the Euclidean metric $g = dx^2 + dy^2 + dz^2$ in $\mathbb{R}^3$ to the sphere. We see that the "cross terms" (the expressions containing $d\theta d\varphi$ and $d\varphi d\theta$) cancel, and repeated use of trigonometric identities on the remaining terms yields the expression

$$g|_{S^2} = d\theta^2 + \sin^2\theta d\varphi^2. \tag{1.2}$$

Instead of numbering the coordinates (which will make the notation cumbersome), we will simply use the coordinates themselves as indices, and subsequent sums over indices will be taken over these symbols. This is less complicated than it sounds: For example, the metric (1.2) has components: $g_{\theta\theta} = 1$, $g_{\theta\varphi} = g_{\varphi\theta} = 0$, and $g_{\varphi\varphi} = \sin^2\theta$. We will use a similar notation for the inverse metric tensor g, but with the indices in the upper position. Since g is diagonal, it is easy to compute the inverse metric tensor: g^{-1} has components $g^{\theta\theta} = 1$, $g^{\theta\varphi} = g^{\varphi\theta} = 0$, and $g^{\varphi\varphi} = 1/\sin^2\theta$.

Exercise. Compute the metric arising from the embedding of the 2-sphere in $\mathbb{R}^3$ by stereographic projection:

$$x = \frac{2X}{1 + X^2 + Y^2}, \qquad y = \frac{2Y}{1 + X^2 + Y^2}, \qquad z = \frac{-1 + X^2 + Y^2}{1 + X^2 + Y^2}, \tag{1.3}$$

[1] We are using the standard physics notation for spherical coordinates; we apologize to anyone used to the other notation, and hope it won't cause too much confusion.

with $(X,Y)\in\mathbb{R}^2$. [2] *Solution:*

$$g|_{S^2} = \frac{4}{(1+X^2+Y^2)^2}(dX^2+dY^2) \tag{1.4}$$

Example. *The Torus* $\mathbb{T}^2$. We apply the same procedure as before to the torus $\mathbb{T}^2$ embedded in $\mathbb{R}^3$. The torus (with $R>r$ being the larger and smaller radii, respectively) has coordinates

$$x=(R+r\cos\theta)\cos\varphi, \qquad y=(R+r\cos\theta)\sin\varphi, \qquad z=r\sin\theta, \tag{1.5}$$

where $0\le\theta,\varphi\le 2\pi$. The differentials of these coordinates are then

$$\begin{aligned}
dx &= -r\sin\theta\cos\varphi d\theta - (R+r\cos\theta)\sin\varphi d\varphi,\\
dy &= -r\sin\theta\sin\varphi d\theta + (R+r\cos\theta)\cos\varphi d\varphi,\\
dz &= r\cos\theta d\theta.
\end{aligned}$$

Taking tensor products of each coordinate, we find that

$$\begin{aligned}
dx^2 &= r^2\sin^2\theta\cos^2\varphi d\theta^2 + (R+r\cos\theta)^2\sin^2\varphi d\varphi^2 + r(R+r\cos\theta)\sin\theta\cos\varphi\sin\varphi d\theta d\varphi + (\cdot)\,d\varphi d\theta,\\
dy^2 &= r^2\sin^2\theta\sin^2\varphi d\theta^2 + (R+r\cos\theta)^2\cos^2\varphi d\varphi^2 - r(R+r\cos\theta)\sin\theta\sin\varphi\cos\varphi d\theta d\varphi - (\cdot)d\varphi d\theta,\\
dz^2 &= r^2\cos^2\theta d\theta^2.
\end{aligned}$$

(We have suppressed the coefficient of the $d\varphi d\theta$ terms, as they are the same as the coefficient of the $d\theta d\varphi$ terms in both cases). Summing, we see that the "cross-terms" cancel, and what remains can be simplified via trigonometric identities. The metric then takes the form:

$$g|_{\mathbb{T}^2} = r^2 d\theta^2 + (R+r\cos\theta)^2 d\varphi^2. \tag{1.6}$$

Thus, the metric is again diagonal, with nonzero components $g_{\theta\theta}=r^2$, $g_{\varphi\varphi}=(R+r\cos\theta)^2$. Since g is diagonal, we can easily compute the inverse metric tensor: g^{-1} has components $g^{\theta\theta}=1/r^2$, $g^{\varphi\varphi}=1/(R+r\cos\theta)^2$.

Example. *The Upper Half Plane* $\mathbb{H}$. Of course, the metric on a Riemannian manifold need not come from

[2] Technically, two stereographic projection charts are required to cover the sphere (say, one from the north pole and one from the south), but it can be checked that the metric arising from these projections has the same form in both charts. Therefore, it is enough to consider only one.

the embedding space: a good example of this is the *hyperbolic metric* on the upper half plane $\mathbb{H} := \{(x, y) \in \mathbb{R}^2 : y > 0\}$. This metric is defined as

$$g|_{\mathbb{H}} = \frac{dx^2 + dy^2}{y^2} = \frac{|dz|^2}{|\mathrm{Im} z|^2}, \tag{1.7}$$

where here $z = x + iy$. The nonzero components of the metric are $g_{xx} = g_{yy} = \frac{1}{y^2}$, and the nonzero components of the inverse metric tensor are $g^{xx} = g^{yy} = y^2$. The relevance of this metric is that it is the unique metric on the upper half plane that is preserved under conformal automorphisms of the upper half plane (i.e., setting $w(z) = \frac{az+b}{cz+d}$ with a, b, c, d real, then $\frac{|dz|^2}{|\mathrm{Im} z|^2} = \frac{|dw|^2}{|\mathrm{Im} w|^2}$). We shall not discuss further properties of this metric here.

Exercise. Use the conformal map $w : \mathbb{D} \to \mathbb{H}$, defined by $w(z) = \frac{z+i}{iz+1}$, to pull back the metric (1.7) to the unit disc $\mathbb{D} := \{z||z| < 1\}$. *Solution:*

$$g|_{\mathbb{D}} = \frac{4|dz|^2}{(1 - |z|^2)^2} = \frac{4}{(1 - x^2 - y^2)^2}(dx^2 + dy^2). \tag{1.8}$$

Example. *The 3-sphere S^3.* Consider the 3-sphere $S^3 := \{(x_0, x_1, x_2, x_3) \mid x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1\}$. We can parameterize the 3-sphere with the coordinates

$$\begin{cases} x_0 &= \cos\psi, \\ x_1 &= \sin\psi\cos\theta, \\ x_2 &= \sin\psi\sin\theta\cos\varphi, \\ x_3 &= \sin\psi\sin\theta\sin\varphi, \end{cases} \tag{1.9}$$

Where $0 \leq \psi, \theta \leq \pi$, and $0 \leq \varphi \leq 2\pi$.

Exercise. Use these coordinates to compute the embedding metric on S^3. *Solution:*

$$g|_{S^3} = d\psi^2 + \sin^2\psi d\theta^2 + \sin^2\psi\sin^2\theta d\varphi^2. \tag{1.10}$$

Example. *The left-invariant metric on $S^3 \cong SU(2)$ & Hopf coordinates.* Let us consider new coordinates on the sphere, called the *Hopf coordinates*:

$$\begin{cases} x_0 &= \cos\xi_1 \sin\eta, \\ x_1 &= \sin\xi_1 \sin\eta, \\ x_2 &= \cos\xi_2 \cos\eta, \\ x_3 &= \sin\xi_2 \cos\eta, \end{cases} \tag{1.11}$$

Where $0 \le \eta \le \frac{\pi}{2}$, $0 \le \xi_1, \xi_2 \le 2\pi$. It is easy to check that the embedding metric in these coordinates is

$$g|_{S^3} = d\eta^2 + \sin^2\eta d\xi_1^2 + \cos^2\eta d\xi_2^2. \tag{1.12}$$

Putting $z_1 = x_0 + ix_1$, $z_2 = x_2 + ix_3$, we can write

$$U(\xi_1, \xi_2, \eta) = \begin{pmatrix} e^{i\xi_1}\sin\eta & -e^{-i\xi_2}\cos\eta \\ e^{i\xi_2}\cos\eta & e^{-i\xi_1}\sin\eta \end{pmatrix} = \begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix}, \tag{1.13}$$

Which is the generic form of a matrix in $SU(2) := \{U \in GL_2(\mathbb{C}) \mid U^\dagger U = \mathbb{I}, \det U = 1\}$. The fact that (x_0, x_1, x_2, x_3) parameterize S^3 guarantees that the matrix $U(\xi_1, \xi_2, \eta)$ is unitary, and has unit determinant. This establishes a diffeomorphism between the sphere S^3 and the Lie group $SU(2)$. Because of the group structure of S^3, we now search for a metric which is invariant under the group action on itself. In fact, we shall see that, up to an overall constant, the unique left-invariant metric on $SU(2)$ is just the usual embedding metric (1.12). We make use of the fact that the metric (1.12) may be expressed as $g = |dz_1|^2 + |dz_2|^2$. Let

$$W := \begin{pmatrix} w_1 & -\bar{w}_2 \\ w_2 & \bar{w}_1 \end{pmatrix} \tag{1.14}$$

Be another matrix in $SU(2)$, i.e. $|w_1|^2 + |w_2|^2 = 1$. Then, under a left translation,

$$W \cdot Z = \begin{pmatrix} w_1 & -\bar{w}_2 \\ w_2 & \bar{w}_1 \end{pmatrix} \begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix} = \begin{pmatrix} w_1 z_1 - \bar{w}_2 z_2 & -\overline{w_2 z_1 + \bar{w}_1 z_2} \\ w_2 z_1 + \bar{w}_1 z_2 & \overline{w_1 z_1 - \bar{w}_2 z_2}, \end{pmatrix} \tag{1.15}$$

i.e., the translated coordinates are

$$\zeta_1 = w_1 z_1 - \bar{w}_2 z_2,$$
$$\zeta_2 = w_2 z_1 + \bar{w}_1 z_2.$$

Invariance of the metric under left translations is equivalent to checking that $|dz_1|^2 + |dz_2|^2 = |d\zeta_1|^2 + |d\zeta_2|^2$. Indeed, we find that

$$d\zeta_1 = w_1 dz_1 - \bar{w}_2 dz_2,$$
$$d\zeta_2 = w_2 dz_1 + \bar{w}_1 dz_2,$$

so that

$$|d\zeta_1|^2 = |w_1|^2|dz_1|^2 - \bar{w}_1\bar{w}_2 dz_2 d\bar{z}_1 - w_1 w_2 dz_1 d\bar{z}_2 + |w_2|^2|dz_2|^2,$$
$$|d\zeta_2|^2 = |w_2|^2|dz_1|^2 + \bar{w}_1\bar{w}_2 dz_2 d\bar{z}_1 + w_1 w_2 dz_1 d\bar{z}_2 + |w_1|^2|dz_2|^2;$$

adding these expressions yields the identity $|dz_1|^2 + |dz_2|^2 = |d\zeta_1|^2 + |d\zeta_2|^2$.

Exercise. Check that the vector fields

$$\begin{aligned}
\sigma_1 &:= \frac{1}{2}\left(-\tan\eta\cos(\xi_1+\xi_2)\frac{\partial}{\partial\xi_1} + \cot\eta\cos(\xi_1+\xi_2)\frac{\partial}{\partial\xi_2} + \sin(\xi_1+\xi_2)\frac{\partial}{\partial\eta}\right),\\
\sigma_2 &:= \frac{1}{2}\left(\tan\eta\sin(\xi_1+\xi_2)\frac{\partial}{\partial\xi_1} - \cot\eta\sin(\xi_1+\xi_2)\frac{\partial}{\partial\xi_2} + \cos(\xi_1+\xi_2)\frac{\partial}{\partial\eta}\right),\\
\sigma_3 &:= \frac{1}{2}\left(\frac{\partial}{\partial\xi_1} + \frac{\partial}{\partial\xi_2}\right),
\end{aligned}$$

satisfy the commutation relations $[\sigma_i, \sigma_j] = \epsilon_{ijk}\sigma_k$.

Example. *The Schwarzschild metric.* Riemannian geometry goes hand-in-hand with Einstein's theory of general relativity. One of the first exact solutions to Einstein's field equations is the *Schwarzschild solution*:

$$g = -\left(1 - \frac{r_s}{r}\right)dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2, \tag{1.16}$$

which (as we shall see) describes a δ-function mass at the origin of an otherwise empty spacetime. g is a pseudo-Riemannian metric on $\mathbb{R}^4$, and has an apparent singularity at $r = r_s$, the *Schwarzschild radius.*

However, this "singularity" can be removed by an appropriate change of coordinates (see, for example, [1]).

1.2. The Levi-Cevita Connection.

There are several ways to derive the Levi-Cevita connection (i.e., the unique connection on the tangent bundle which annihilates the metric and has vanishing torsion) from the metric; however, it is not our purpose to derive the Levi-Cevita connection, and so we will present the result here. The components of the connection, called the *Christoffel symbols*, have three associated indices: Γ^i_{jk}; in terms of the metric, the component Γ^i_{jk} is given by

$$\Gamma^i_{jk} = \frac{1}{2} \sum_{\ell} g^{i\ell} \left(\partial_j \, g_{\ell k} + \partial_k \, g_{j\ell} - \partial_\ell \, g_{jk} \right). \tag{1.17}$$

Here, $\partial_j = \frac{\partial}{\partial \, x_j}$, the derivative with respect to the j^{th} local coordinate. It is worth committing formula (1.17) to memory. Here are a few useful techniques for doing this: Looking closely at the indices, we see that the first two terms are obtained by switching one of the lower indices in Γ^i_{jk} with ℓ, and then differentiating with respect to this variable; note that both terms have a "+" sign in front of them. The last term is obtained by differentiating the component of the metric tensor with indices matching the lower indices of Γ^i_{jk} with respect to the summation variable; note that this is the only term with a "−" coefficient. Finally, note that the only place that the upper index appears is in the inverse metric tensor expression.

Some further remarks about the Christoffel symbols:

1. Note that Γ^i_{jk} is symmetric in the indices j and k, as a consequence of the fact that $g_{ij} = g_{ji}$.
2. If the metric is diagonal, i.e. $g_{ij} \neq 0$ only if $i = j$, then g^{ij} is also diagonal, and thus there is no sum over ℓ in (1.17): the Christoffel symbols are simply

$$\Gamma^i_{jk} = \frac{1}{2} g^{ii} \left(\partial_j \, g_{ik} + \partial_k \, g_{ji} - \partial_i \, g_{jk} \right); \tag{1.18}$$

This often drastically reduces the number of computations one has to make.

Although, formally speaking, we can now compute the connection, the Christoffel symbols are somewhat difficult to manage; an object with three indices (and which is moreover not tensorial) can be difficult to deal with in general. However, we can improve our situation, at least conceptually. We will think of the connection as a *matrix-valued 1-form*:

$$\Gamma = \sum_k \Gamma_k dx^k, \tag{1.19}$$

where each of the Γ_k's is a matrix. The $(ij)^{th}$ component of the matrix Γ_k is given by $(\Gamma_k)^i_j = \Gamma^i_{jk}$. Note that the placement of indices on the left hand side is consistent with that of the right. This definition also allows us to clearly write the Levi-Cevita connection as an operator on vector fields: given a vector field X, the connection applied to X can be written as (at least in the local coordinate patch on which we are working)

$$\nabla X = dX + \Gamma X, \tag{1.20}$$

Where Γ acts on X as a matrix applied to a vector. As a consistency check, we see that both terms in the above have one upper and one lower index, and are thus of the same type, so that ∇X is indeed a well-defined and (as we will soon check) tensorial quantity. Let us begin to demonstrate the utility of this notation. Suppose (U, x), (V, y) are overlapping coordinate charts; consider the change of coordinates $y \to x(y)$. Using formula (1.17), we can (rather tediously) compute $\tilde{\Gamma}^i_{jk}$ in the new coordinates:

$$\tilde{\Gamma}^i_{jk} = \sum_{m,n,p} \left(\frac{\partial x^i}{\partial y^m} \frac{\partial y^n}{\partial x^j} \frac{\partial y^p}{\partial x^k} \Gamma^m_{np} + \frac{\partial^2 y^m}{\partial x^j \partial x^k} \frac{\partial x^i}{\partial y^m} \right). \tag{1.21}$$

Now, adding in the contribution of $dx^k \to \frac{\partial x^k}{\partial y^r} dy^r$, we see that one of the $\frac{\partial y}{\partial x}$ factors cancels in the expression for $\tilde{\Gamma}^i_{jk} dy^k$. We can simplify this expression (at least notationally speaking) significantly, Define λ to be the matrix-valued function with entries $(\lambda)^i_j = \frac{\partial y^i}{\partial x^j}$. Then, in matrix notation the new expression for the Christoffel symbols reads

$$\tilde{\Gamma} = \lambda \Gamma \lambda^{-1} - d\lambda \cdot \lambda^{-1}. \tag{1.22}$$

The discerning reader will recognize this as what is known as a *gauge transformation* in physics. This is, of course, no coincidence: a Riemannian manifold can be viewed as a principal- $GL_n(\mathbb{R})$ bundle and (gauge-invariant) connection $\nabla = d + \Gamma$. (If this sentence does not mean anything to you right now, do not worry; it is not relevant to the remainder of this exposition).

Now, we will check that, indeed, for any vector field X, ∇X is tensorial. Let us use the coordinate transformation of the previous section. Under such a transformation, the vector field X transforms as λX. Altogether, we find that, in the new coordinates,

$$\begin{aligned}
\tilde{\nabla} \tilde{X} &= \left(d + \lambda \Gamma \lambda^{-1} - d\lambda \cdot \lambda^{-1} \right) \lambda X \\
&= d\lambda \cdot X + \lambda dX + \lambda \Gamma X - d\lambda \cdot X \\
&= \lambda dX + \lambda \Gamma X = \lambda \nabla X,
\end{aligned}$$

so that ∇X is a tensorial quantity, and transforms like a tensor with one additional lower index than X. We conclude this section with some computations carried over from the previous section.

Example. *The Sphere* S^2*.* Let us compute the connection coefficients of the Levi-Cevita connection arising from the metric we derived in the previous section. Since the metric is diagonal, we can use formula (1.18) to compute the Christoffel symbols. Furthermore, since the metric is diagonal, and the only nonconstant component of the metric is $g_{\varphi\varphi}$, we see that the only non-vanishing Christoffel symbols will be those that involve an expression of the form $\partial_\theta \, g_{\varphi\varphi}$ (since partial derivatives of constants will yield zero). Comparing with formula (1.18), we conclude that the only non-vanishing Christoffel symbols are $\Gamma^{\varphi}_{\varphi\theta} = \Gamma^{\varphi}_{\theta\varphi}$, and $\Gamma^{\theta}_{\varphi\varphi}$. We compute these entries explicitly:

$$\begin{aligned}
\Gamma^{\varphi}_{\varphi\theta} = \Gamma^{\varphi}_{\theta\varphi} &= \frac{1}{2} g^{\varphi\varphi} \left(\partial_\varphi \, g_{\varphi\theta} + \partial_\theta \, g_{\varphi\varphi} - \partial_\varphi \, g_{\varphi\theta} \right) = \frac{1}{2} g^{\varphi\varphi} \left(\partial_\theta \, g_{\varphi\varphi} \right) = \frac{1}{2} \frac{1}{\sin^2\theta} (2 \sin\theta \cos\theta) = \cot\theta, \\
\Gamma^{\theta}_{\varphi\varphi} &= \frac{1}{2} g^{\theta\theta} \left(\partial_\varphi \, g_{\theta\varphi} + \partial_\varphi \, g_{\varphi\theta} - \partial_\theta \, g_{\varphi\varphi} \right) = -\frac{1}{2} g^{\theta\theta} \left(\partial_\theta \, g_{\varphi\varphi} \right) = -\frac{1}{2} (2 \sin\theta \cos\theta) = -\sin\theta\cos\theta.
\end{aligned}$$

We now write the connection coefficients as a matrix-valued 1-form. Formally, this 1-form should look like

$$\Gamma = \Gamma_\theta d\theta + \Gamma_\varphi d\varphi = \begin{pmatrix} \Gamma^{\theta}_{\theta\theta} & \Gamma^{\theta}_{\varphi\theta} \\ \Gamma^{\varphi}_{\theta\theta} & \Gamma^{\varphi}_{\varphi\theta} \end{pmatrix} d\theta + \begin{pmatrix} \Gamma^{\theta}_{\theta\varphi} & \Gamma^{\theta}_{\varphi\varphi} \\ \Gamma^{\varphi}_{\theta\varphi} & \Gamma^{\varphi}_{\varphi\varphi} \end{pmatrix} d\varphi. \tag{1.23}$$

Using the expressions we found earlier for the Christoffel symbols, we find that Γ explicitly takes the form

$$\Gamma = \Gamma_\theta d\theta + \Gamma_\varphi d\varphi = \begin{pmatrix} 0 & 0 \\ 0 & \cot\theta \end{pmatrix} d\theta + \begin{pmatrix} 0 & -\sin\theta\cos\theta \\ \cot\theta & 0 \end{pmatrix} d\varphi. \tag{1.24}$$

Exercise. Compute the connection form of the stereographic metric (1.4) on the sphere. *Solution:* Set $\rho(X, Y) = 1 + X^2 + Y^2$. Then the connection form is

$$\Gamma = \Gamma_X dX + \Gamma_Y dY = \frac{2}{\rho} \begin{pmatrix} -X & -Y \\ Y & -X \end{pmatrix} dX + \frac{2}{\rho} \begin{pmatrix} -Y & X \\ -X & -Y \end{pmatrix} dY. \tag{1.25}$$

Example. *The Torus* $\mathbb{T}^2$*.* Using the explicit expression for the metric on the torus (1.6), and formula (1.18) (because the metric is diagonal), we can compute the coefficients of the Levi-Cevita connection. Let us first determine all nonzero Christoffel symbols. Since the only non-constant entry in the metric is $g_{\varphi\varphi} = (R + r\cos\theta)^2$, and this coefficient depends only on the variable θ, we see that the only nonzero Christoffel symbols will be those which contain the term $\partial_\theta\, g_{\varphi\varphi}$. There are precisely three of these: $\Gamma^\varphi_{\varphi\theta} = \Gamma^\varphi_{\theta\varphi}$, and $\Gamma^\theta_{\varphi\varphi}$. We compute these terms explicitly:

$$\begin{aligned}
\Gamma^\varphi_{\varphi\theta} = \Gamma^\varphi_{\theta\varphi} &= \frac{1}{2} g^{\varphi\varphi}\left(\partial_\varphi\, g_{\varphi\theta} + \partial_\theta\, g_{\varphi\varphi} - \partial_\varphi\, g_{\varphi\theta}\right) = \frac{1}{2} g^{\varphi\varphi}\left(\partial_\theta\, g_{\varphi\varphi}\right) = \frac{-r\sin\theta}{R + r\cos\theta}, \\
\Gamma^\theta_{\varphi\varphi} &= \frac{1}{2} g^{\theta\theta}\left(\partial_\varphi\, g_{\theta\varphi} + \partial_\varphi\, g_{\varphi\theta} - \partial_\theta\, g_{\varphi\varphi}\right) = \frac{(R + r\cos\theta)}{r}\sin\theta.
\end{aligned}$$

Now, writing the connection as a matrix-valued 1-form, we find that

$$\Gamma = \Gamma_\theta d\theta + \Gamma_\varphi d\varphi = \begin{pmatrix} 0 & 0 \\ 0 & \frac{-r\sin\theta}{R+r\cos\theta} \end{pmatrix} d\theta + \begin{pmatrix} 0 & \frac{(R+r\cos\theta)}{r}\sin\theta \\ \frac{-r\sin\theta}{R+r\cos\theta} & 0 \end{pmatrix} d\varphi. \tag{1.26}$$

Example. *The Upper Half Plane* $\mathbb{H}$. We now compute the Christoffel symbols of the metric (1.7) on the upper half plane. Note that only Christoffel symbols containing the terms $\partial_y\, g_{xx}$ or $\partial_y\, g_{yy}$ have a chance of being nonzero. Thus, the nonzero Christoffel symbols are $\Gamma^x_{yx} = \Gamma^x_{xy}$, Γ^y_{xx}, and Γ^y_{yy}. Since the metric is diagonal, we can again use formula (1.18) to compute the Christoffel symbols: we find that

$$\begin{aligned}
\Gamma^x_{yx} = \Gamma^x_{xy} &= \frac{1}{2} g^{xx}\left(\partial_y\, g_{xx} + \partial_x\, g_{yx} - \partial_x\, g_{yx}\right) = \frac{1}{2} g^{xx}\left(\partial_y\, g_{xx}\right) = -1/y, \\
\Gamma^y_{xx} &= \frac{1}{2} g^{yy}\left(\partial_x\, g_{yx} + \partial_x\, g_{xy} - \partial_y\, g_{xx}\right) = -\frac{1}{2} g^{yy}\left(\partial_y\, g_{xx}\right) = 1/y, \\
\Gamma^y_{yy} &= \frac{1}{2} g^{yy}\left(\partial_y\, g_{yy} + \partial_y\, g_{yy} - \partial_y\, g_{yy}\right) = \frac{1}{2} g^{yy}\left(\partial_y\, g_{yy}\right) = -1/y.
\end{aligned}$$

Written as a matrix-valued 1-form, the connection takes the form

$$\Gamma = \Gamma_x dx + \Gamma_y dy = \begin{pmatrix} 0 & -1/y \\ 1/y & 0 \end{pmatrix} dx + \begin{pmatrix} -1/y & 0 \\ 0 & -1/y \end{pmatrix} dy. \tag{1.27}$$

Exercise. Compute the connection form of the hyperbolic metric on the disc (1.8). *Solution:* Set $\sigma(x, y) = 1 - x^2 - y^2$; then the connection form is given as

$$\Gamma = \Gamma_x dx + \Gamma_y dy = \frac{2}{\sigma}\begin{pmatrix} x & y \\ -y & x \end{pmatrix} dx + \frac{2}{\sigma}\begin{pmatrix} y & -x \\ x & y \end{pmatrix} dy. \tag{1.28}$$

Example. *The 3-sphere* S^3. We compute the connection form from the metric (1.10). Since the only nonzero components of the metric are $g_{\psi\psi} = 1$, $g_{\theta\theta} = \sin^2\psi$, and $g_{\varphi\varphi} = \sin^2\psi\sin^2\theta$, the only nonvanishing derivatives of the metric will be $\partial_\psi\, g_{\theta\theta}$, $\partial_\psi\, g_{\varphi\varphi}$, and $\partial_\theta\, g_{\varphi\varphi}$. Thus, we must compute the 9 nontrivial Christoffel symbols: $\Gamma^\psi_{\theta\theta}$, $\Gamma^\theta_{\theta\psi} = \Gamma^\theta_{\psi\theta}$, $\Gamma^\psi_{\varphi\varphi}$, $\Gamma^\varphi_{\psi\varphi} = \Gamma^\varphi_{\varphi\psi}$, $\Gamma^\theta_{\varphi\varphi}$, and $\Gamma^\varphi_{\theta\varphi} = \Gamma^\varphi_{\varphi\theta}$. By direct computation, we find that

$$\begin{aligned}
\Gamma^\psi_{\theta\theta} &= -\sin\psi\cos\psi, \\
\Gamma^\theta_{\theta\psi} &= \Gamma^\theta_{\psi\theta} = \cot\psi, \\
\Gamma^\psi_{\varphi\varphi} &= -\sin\psi\cos\psi\sin^2\theta, \\
\Gamma^\varphi_{\psi\varphi} &= \Gamma^\varphi_{\varphi\psi} = \cot\theta, \\
\Gamma^\theta_{\varphi\varphi} &= -\sin\theta\cos\theta, \\
\Gamma^\varphi_{\theta\varphi} &= \Gamma^\varphi_{\varphi\theta} = \cot\theta.
\end{aligned}$$

The connection form is thus

$$\Gamma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \cot\psi & 0 \\ 0 & 0 & \cot\psi \end{pmatrix} d\psi + \begin{pmatrix} 0 & -\sin\psi\cos\psi & 0 \\ \cot\psi & 0 & 0 \\ 0 & 0 & \cot\theta \end{pmatrix} d\theta + \begin{pmatrix} 0 & 0 & -\sin\psi\cos\psi\sin^2\theta \\ 0 & 0 & -\sin\theta\cos\theta \\ \cot\psi & \cot\theta & 0 \end{pmatrix} d\varphi \tag{1.29}$$

Example. *The left-invariant metric on $S^3 \cong SU(2)$ & Hopf coordinates.* Recall the metric (1.12) had entries $g_{\eta\eta} = 1$, $g_{\xi_1\xi_1} = \sin^2\eta$, and $g_{\xi_2\xi_2} = \cos^2\eta$. Thus, the only non-vanishing derivatives of the metric tensor will be $\partial_\eta g_{\xi_1\xi_1}$ and $\partial_\eta g_{\xi_2\xi_2}$; consequentially, there are only 6 non-vanishing connection coefficients: $\Gamma^\eta_{\xi_1\xi_1}$, $\Gamma^{\xi_1}_{\eta\xi_1} = \Gamma^{\xi_1}_{\xi_1\eta}$, $\Gamma^\eta_{\xi_2\xi_2}$, and $\Gamma^{\xi_2}_{\eta\xi_2} = \Gamma^{\xi_2}_{\xi_2\eta}$. We compute that:

$$\begin{aligned}
\Gamma^\eta_{\xi_1\xi_1} &= -\cos\eta\sin\eta, \\
\Gamma^{\xi_1}_{\eta\xi_1} &= \Gamma^{\xi_1}_{\xi_1\eta} = \cot\eta, \\
\Gamma^\eta_{\xi_2\xi_2} &= \cos\eta\sin\eta, \\
\Gamma^{\xi_2}_{\eta\xi_2} &= \Gamma^{\xi_2}_{\xi_2\eta} = -\tan\eta.
\end{aligned}$$

The connection form is thus

$$\Gamma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \cot\eta & 0 \\ 0 & 0 & -\tan\eta \end{pmatrix} d\eta + \begin{pmatrix} 0 & -\cos\eta\sin\eta & 0 \\ \cot\eta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} d\xi_1 + \begin{pmatrix} 0 & 0 & \cos\eta\sin\eta \\ 0 & 0 & 0 \\ -\tan\eta & 0 & 0 \end{pmatrix} d\xi_2. \tag{1.30}$$

Example. *The Schwarzschild metric.* We now compute the connection form from the metric (1.16). By inspection, we see that the only nonvanishing derivatives of the metric are: $\partial_r\, g_{tt}$, $\partial_r\, g_{rr}$, $\partial_r\, g_{\theta\theta}$, $\partial_r\, g_{\varphi\varphi}$, and $\partial_\theta\, g_{\varphi\varphi}$. Since the metric is diagonal, formula (1.18) tells us that the only non-vanishing Christoffel symbols are: Γ^r_{tt}, $\Gamma^t_{rt} = \Gamma^t_{tr}$, Γ^r_{rr}, $\Gamma^r_{\theta\theta}$, $\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r}$, $\Gamma^r_{\varphi\varphi}$, $\Gamma^\varphi_{r\varphi} = \Gamma^\varphi_{\varphi r}$, $\Gamma^\theta_{\varphi\varphi}$, and finally $\Gamma^\varphi_{\theta\varphi} = \Gamma^\varphi_{\varphi\theta}$. These quantities are computed below:

$$\begin{aligned}
\Gamma^r_{tt} &= -\frac{1}{2} g^{rr}\left(\partial_r\, g_{tt}\right) = \frac{1}{2}\frac{r_s}{r^2}\left(1 - \frac{r_s}{r}\right), \\
\Gamma^t_{rt} = \Gamma^t_{tr} &= \frac{1}{2} g^{tt}\left(\partial_r\, g_{tt}\right) = \frac{1}{2}\frac{r_s}{r^2}\left(1 - \frac{r_s}{r}\right)^{-1}, \\
\Gamma^r_{rr} &= \frac{1}{2} g^{rr}\left(\partial_r\, g_{rr}\right) = -\frac{1}{2}\frac{r_s}{r^2}\left(1 - \frac{r_s}{r}\right)^{-1}, \\
\Gamma^r_{\theta\theta} &= -\frac{1}{2} g^{rr}\left(\partial_r\, g_{\theta\theta}\right) = -r\left(1 - \frac{r_s}{r}\right), \\
\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} &= \frac{1}{2} g^{\theta\theta}\left(\partial_r\, g_{\theta\theta}\right) = \frac{1}{r}, \\
\Gamma^r_{\varphi\varphi} &= -\frac{1}{2} g^{rr}\left(\partial_r\, g_{\varphi\varphi}\right) = -r\sin^2\theta\left(1 - \frac{r_s}{r}\right), \\
\Gamma^\varphi_{r\varphi} = \Gamma^\varphi_{\varphi r} &= \frac{1}{2} g^{\varphi\varphi}\left(\partial_r\, g_{\varphi\varphi}\right) = \frac{1}{r}, \\
\Gamma^\theta_{\varphi\varphi} &= -\frac{1}{2} g^{\theta\theta}\left(\partial_\theta\, g_{\varphi\varphi}\right) = -\sin\theta\cos\theta, \\
\Gamma^\varphi_{\theta\varphi} = \Gamma^\varphi_{\varphi\theta} &= \frac{1}{2} g^{\varphi\varphi}\left(\partial_\theta\, g_{\varphi\varphi}\right) = \cot\theta.
\end{aligned}$$

Therefore, in matrix form, the Schwarzschild connection looks like

$$\Gamma = \begin{pmatrix} 0 & \frac{1}{2}\frac{r_s}{r^2}\left(1-\frac{r_s}{r}\right)^{-1} & 0 & 0 \\ \frac{1}{2}\frac{r_s}{r^2}\left(1-\frac{r_s}{r}\right) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} dt + \begin{pmatrix} \frac{1}{2}\frac{r_s}{r^2}\left(1-\frac{r_s}{r}\right)^{-1} & 0 & 0 & 0 \\ 0 & -\frac{1}{2}\frac{r_s}{r^2}\left(1-\frac{r_s}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & 0 & 0 & \frac{1}{r} \end{pmatrix} dr$$
$$+ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -r\left(1-\frac{r_s}{r}\right) & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & \cot\theta \end{pmatrix} d\theta + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -r\sin^2\theta\left(1-\frac{r_s}{r}\right) \\ 0 & 0 & 0 & -\sin\theta\cos\theta \\ 0 & \frac{1}{r} & \cot\theta & 0 \end{pmatrix} d\varphi. \tag{1.31}$$

1.3. The Riemann Tensor.

The next object we would like to be able to compute is the Riemann curvature tensor. Seemingly, we are in a much worse situation than before, as we now have to describe an object with 4 indices. In fact, this will turn out to be manageable: just as we expressed the connection coefficients as a matrix-valued 1-form, we will express the Riemann tensor as a matrix-valued 2-form. Contrary to the notation popular in Riemannian geometry, we will denote the Riemann tensor by Ω, and establish a correspondence to the usual notation R^i_{jkl}. As usual, the Riemann tensor is defined as $\Omega := \nabla \circ \nabla$, the connection applied to itself. In our formalism, computing the Riemann tensor amounts to the following computation. Let X be any vector field; then

$$\begin{aligned} \Omega X = (\nabla\circ\nabla)X = \nabla(dX + \Gamma X) &= d(dX+\Gamma X) + \Gamma(dX + \Gamma X) \\ &= d(dX) + d\Gamma\cdot X - \Gamma dX + \Gamma dX + (\Gamma\wedge\Gamma)X \\ &= (d\Gamma + \Gamma\wedge\Gamma)X, \end{aligned}$$

Since $d^2 = 0$. Therefore, locally, we can write the curvature tensor as the matrix-valued 2-form $\Omega = d\Gamma + \Gamma\wedge\Gamma$[3]; this is consistent with the Riemann tensor having 1 upper and 3 lower indices (1 upper and 1 lower contributing to the matrix coefficient, and the remaining 2 lower contributing to the 2-form). Write Ω as $\Omega = \Omega_{kl}dx^k\wedge dx^l$, where each Ω_{kl} is a matrix. In the usual notation of the Riemann tensor, the $(ij)^{th}$ component of the matrix Ω_{kl} is given by $(\Omega_{kl})^i_j = R^i_{jkl}$. As an immediate consequence, we see that the

[3]Note: $\Gamma\wedge\Gamma$ is nonzero in general, since Γ has *matrix* coefficients. This will be made apparent in the examples.

Riemann tensor is antisymmetric in two of its indices.

Furthermore, this notation allows for an almost trivial proof of the second Bianchi identity: Note that $\Omega\nabla = \{(d+\Gamma)(d+\Gamma)\}(d+\Gamma) = (d+\Gamma)\{(d+\Gamma)(d+\Gamma)\} = \nabla\Omega$. Applying this identity to a vector field X, we see that $\Omega\nabla X = \nabla(\Omega X) = \Omega\nabla X + (\nabla\Omega)X$, implying that $\nabla\Omega = 0$.

The Riemann tensor is now a more manageable quantity, and we now show by direct computation (through the examples of the previous sections) that working with it is not too difficult.

Example. *The Sphere.* We now compute the Riemann tensor of the sphere using the connection form we found in the last section. We must compute two quantities: $d\Gamma$ and $\Gamma \wedge \Gamma$. Let us begin by computing $d\Gamma$: since $\Gamma = \Gamma_\theta d\theta + \Gamma_\varphi d\varphi$, we find that $d\Gamma = (\partial_\theta \Gamma_\varphi - \partial_\varphi \Gamma_\theta)\, d\theta \wedge d\varphi$. Now, since Γ_θ is independent of φ, we see that this expression simplifies to $d\Gamma = \partial_\theta \Gamma_\varphi d\theta \wedge d\varphi$. Explicitly (i.e., applying this prescription to equation (1.24)), we find that

$$d\Gamma = \partial_\theta \begin{pmatrix} 0 & -\sin\theta\cos\theta \\ \cot\theta & 0 \end{pmatrix} d\theta \wedge d\varphi = \begin{pmatrix} 0 & \sin^2\theta - \cos^2\theta \\ -\csc^2\theta & 0 \end{pmatrix} d\theta \wedge d\varphi. \tag{1.32}$$

Now, we are left to compute $\Gamma \wedge \Gamma$. We find that $\Gamma \wedge \Gamma = (\Gamma_\theta d\theta + \Gamma_\varphi d\varphi) \wedge (\Gamma_\theta d\theta + \Gamma_\varphi d\varphi) = [\Gamma_\theta, \Gamma_\varphi] d\theta \wedge d\varphi$, where $[\cdot,\cdot]$ denotes the matrix commutator. After some simple matrix algebra, one finds that

$$\Gamma \wedge \Gamma = [\Gamma_\theta, \Gamma_\varphi] d\theta \wedge d\varphi = \begin{pmatrix} 0 & \cos^2\theta \\ \cot^2\theta & 0 \end{pmatrix} d\theta \wedge d\varphi. \tag{1.33}$$

Combining (1.32), (1.33), we find that

$$\Omega = d\Gamma + \Gamma \wedge \Gamma = \begin{pmatrix} 0 & \sin^2\theta \\ -1 & 0 \end{pmatrix} d\theta \wedge d\varphi. \tag{1.34}$$

Exercise. Compute the curvature form of the sphere with stereographic connection (1.25). *Solution:* Again, setting $\rho(X,Y) = 1 + X^2 + Y^2$, the curvature tensor takes the form

$$\Omega = \begin{pmatrix} 0 & 4/\rho \\ -4/\rho & 0 \end{pmatrix} dX \wedge dY. \tag{1.35}$$

Example. *The Torus $\mathbb{T}^2$.* Continuing the example of the torus, we now compute the Riemann tensor on the torus from the connection (1.26). We separately compute $d\Gamma$ and $\Gamma\wedge\Gamma$. Now, $d\Gamma = (\partial_\theta\,\Gamma_\varphi - \partial_\varphi\,\Gamma_\theta)\,d\theta\wedge d\varphi = \partial_\theta\,\Gamma_\varphi d\theta\wedge d\varphi$, since Γ_θ is independent of φ. Thus, we find that

$$d\Gamma = \partial_\theta\,\Gamma_\varphi d\theta\wedge d\varphi = \partial_\theta \begin{pmatrix} 0 & \frac{(R+r\cos\theta)}{r}\sin\theta \\ \frac{-r\sin\theta}{R+r\cos\theta} & 0 \end{pmatrix} d\theta\wedge d\varphi = \begin{pmatrix} 0 & \frac{\cos\theta(R+r\cos\theta)}{r} - \sin^2\theta \\ \frac{-r(r+R\cos\theta)}{(R+r\cos\theta)^2} & 0 \end{pmatrix} d\theta\wedge d\varphi. \tag{1.36}$$

Furthermore, we find that $\Gamma\wedge\Gamma = (\Gamma_\theta d\theta + \Gamma_\varphi d\varphi)\wedge(\Gamma_\theta d\theta + \Gamma_\varphi d\varphi) = [\Gamma_\theta, \Gamma_\varphi] d\theta\wedge d\varphi$; after some computations, we see that this simplifies to

$$\Gamma\wedge\Gamma = [\Gamma_\theta, \Gamma_\varphi] d\theta\wedge d\varphi = \begin{pmatrix} 0 & \sin^2\theta \\ \frac{r^2\sin^2\theta}{(R+r\cos\theta)^2} & 0 \end{pmatrix} d\theta\wedge d\varphi. \tag{1.37}$$

Combining results (1.36), (1.37), we obtain the curvature tensor:

$$\Omega = d\Gamma + \Gamma\wedge\Gamma = \begin{pmatrix} 0 & \frac{\cos\theta(R+r\cos\theta)}{r} \\ -\frac{r\cos\theta}{R+r\cos\theta} & 0 \end{pmatrix} d\theta\wedge d\varphi. \tag{1.38}$$

Example. *The Upper Half Plane* $\mathbb{H}$. We now find the Riemann tensor on the upper half plane endowed with the hyperbolic metric; as before, we must compute $d\Gamma$ and $\Gamma\wedge\Gamma$. Now, $d\Gamma = (\partial_x\,\Gamma_y - \partial_y\,\Gamma_x)\,dx\wedge dy = -\,\partial_y\,\Gamma_x dx\wedge dy$, since Γ_y is independent of x. Therefore:

$$d\Gamma = -\,\partial_y\,\Gamma_x dx\wedge dy = -\,\partial_y \begin{pmatrix} 0 & -1/y \\ 1/y & 0 \end{pmatrix} dx\wedge dy = \begin{pmatrix} 0 & -1/y^2 \\ 1/y^2 & 0 \end{pmatrix} dx\wedge dy. \tag{1.39}$$

Furthermore, we must compute $\Gamma\wedge\Gamma = [\Gamma_x, \Gamma_y] dx\wedge dy$. We find that:

$$\Gamma\wedge\Gamma = [\Gamma_x, \Gamma_y] dx\wedge dy = 0. \tag{1.40}$$

(This is due to the fact that Γ_y is a multiple of the identity matrix). Therefore, the curvature tensor is

$$\Omega = d\Gamma + \Gamma\wedge\Gamma = \begin{pmatrix} 0 & -1/y^2 \\ 1/y^2 & 0 \end{pmatrix} dx\wedge dy. \tag{1.41}$$

Exercise. Compute the curvature form of the disc with the hyperbolic connection (See equation (1.28)).

Solution: Setting $\sigma(x, y) = 1 - x^2 - y^2$, the curvature form is

$$\Omega = \begin{pmatrix} 0 & -4/\sigma^2 \\ 4/\sigma^2 & 0 \end{pmatrix} dx \wedge dy. \tag{1.42}$$

Exercise. *The 3-sphere S^3.* Using the expression (1.29) for the connection on S^3, compute the Riemann tensor. *Solution:*

$$\Omega = \begin{pmatrix} 0 & \sin^2\psi & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} d\psi \wedge d\theta + \begin{pmatrix} 0 & 0 & \sin^2\psi \sin^2\theta \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} d\psi \wedge d\varphi + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sin^2\psi \sin^2\theta \\ 0 & \sin^2\psi & 0 \end{pmatrix} d\theta \wedge d\varphi. \tag{1.43}$$

Exercise. *The left-invariant metric on $S^3 \cong SU(2)$ & Hopf coordinates.* Using the expression (1.30) for the invariant connection on $SU(2)$, compute the Riemann tensor. *Solution:*

$$\Omega = \begin{pmatrix} 0 & \sin^2\eta & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} d\eta \wedge d\xi_1 + \begin{pmatrix} 0 & 0 & \cos^2\eta \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} d\eta \wedge d\xi_2 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \cos^2\eta \\ 0 & -\sin^2\eta & 0 \end{pmatrix} d\xi_1 \wedge d\xi_2 \tag{1.44}$$

Example. *The Schwarzschild metric.* Using the connection (1.31), we can compute the curvature form. We begin by computing $d\Gamma$; we have that $d\Gamma = -\partial_r \Gamma_t dt \wedge dr + \partial_r \Gamma_\theta dr \wedge d\theta + \partial_r \Gamma_\varphi dr \wedge d\varphi + \partial_\theta \Gamma_\theta d\theta \wedge d\varphi$.

Using (1.31), we find that:

$$-\partial_r\,\Gamma_t dt\wedge dr = -\partial_r \begin{pmatrix} 0 & \frac{1}{2}\frac{r_s}{r^2}\left(1-\frac{r_s}{r}\right)^{-1} & 0 & 0 \\ \frac{1}{2}\frac{r_s}{r^2}\left(1-\frac{r_s}{r}\right) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} dt\wedge dr = \begin{pmatrix} 0 & \frac{r_s(2r-r_s)}{2r^2(r-r_s)^2} & 0 & 0 \\ \frac{r_s(2r-3r_s)}{2r^4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} dt\wedge dr,$$

$$\partial_r\,\Gamma_\theta dr\wedge d\theta = \partial_r \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -r\left(1-\frac{r_s}{r}\right) & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & \cot\theta \end{pmatrix} dr\wedge d\theta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -\frac{1}{r^2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} dr\wedge d\theta,$$

$$\partial_r\,\Gamma_\varphi dr\wedge d\varphi = \partial_r \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -r\sin^2\theta\left(1-\frac{r_s}{r}\right) \\ 0 & 0 & 0 & -\sin\theta\cos\theta \\ 0 & \frac{1}{r} & \cot\theta & 0 \end{pmatrix} dr\wedge d\varphi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sin^2\theta \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{r^2} & 0 & 0 \end{pmatrix} dr\wedge d\varphi,$$

$$\partial_\theta\,\Gamma_\varphi d\theta\wedge d\varphi = \partial_\theta \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -r\sin^2\theta\left(1-\frac{r_s}{r}\right) \\ 0 & 0 & 0 & -\sin\theta\cos\theta \\ 0 & \frac{1}{r} & \cot\theta & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2r\sin\theta\cos\theta\left(1-\frac{r_s}{r}\right) \\ 0 & 0 & 0 & \sin^2\theta-\cos^2\theta \\ 0 & 0 & -\csc^2\theta & 0 \end{pmatrix} d\theta\wedge d\varphi.$$

We now must compute $\Gamma\wedge\Gamma$. It is easily shown that $\Gamma\wedge\Gamma = [\Gamma_t,\Gamma_r]dt\wedge dr[\Gamma_t,\Gamma_\theta]dt\wedge d\theta + [\Gamma_t,\Gamma_\varphi]dt\wedge d\varphi + [\Gamma_r,\Gamma_\theta]dr\wedge d\theta + [\Gamma_r,\Gamma_\varphi]dr\wedge d\varphi + [\Gamma_\theta,\Gamma_\varphi]d\theta\wedge d\varphi$. Again, using (1.31), and some (rather tedious, but still elementary) matrix algebra, we find that:

Altogether, the Riemann tensor is given by:

$$
\begin{aligned}
\Omega = &\begin{pmatrix} 0 & \frac{r_s}{r^2(r-r_s)} & 0 & 0 \\ \frac{r_s(r-r_s)}{r^4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} dt \wedge dr + \begin{pmatrix} 0 & 0 & -\frac{r_s}{2r} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{r_s(r-r_s)}{2r^4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} dt \wedge d\theta \\
&+ \begin{pmatrix} 0 & 0 & 0 & -\frac{r_s \sin^2\theta}{2r} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{r_s(r-r_s)}{2r^4} & 0 & 0 & 0 \end{pmatrix} dt \wedge d\varphi + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{r_s}{2r} & 0 \\ 0 & \frac{r_s}{r^2(r-r_s)} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} dr \wedge d\theta \\
&+ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{r_s \sin^2\theta}{2r} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{r_s}{2r^2(r-r_s)} & 0 & 0 \end{pmatrix} dr \wedge d\varphi + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{r_s \sin^2\theta}{r} \\ 0 & 0 & -\frac{r_s}{r} & 0 \end{pmatrix} d\theta \wedge d\varphi
\end{aligned}
\tag{1.45}
$$

1.4. The Ricci and scalar curvatures.

Commonly, problems in geometry require one to compute derived quantities of the Riemann tensor. Two of the most important are the Ricci and scalar curvatures: the Ricci curvature is defined to be the trace of the Riemann tensor:

$$R_{ij} = \sum_k R^k_{ikj}. \tag{1.46}$$

This quantity is symmetric in its remaining indices. We will carry on our examples from the previous section to make the computations here transparent. Finally, the scalar curvature is defined to be the contraction of both indices the Ricci tensor:

$$R = \sum_{i,j} g^{ij} R_{ij}. \tag{1.47}$$

Let us compute these quantities for the examples from the previous sections.

Example. *The Sphere S^2.* Using equation (1.34), we can compute the Ricci and scalar curvatures. We compute each component of the Ricci tensor individually: let us begin with $R_{\theta\theta}$. By equation (1.46), we have that $R_{\theta\theta} = R^\theta_{\theta\theta\theta} + R^\varphi_{\theta\varphi\theta} = R^\varphi_{\theta\varphi\theta}$, since the Riemann tensor is antisymmetric in the last two indices (it is a matrix-valued 2-form). Write $\Omega = \Omega_{kl} dx^k \wedge dx^l$. Since $(\Omega_{kl})^i_j = R^i_{jkl}$, and using (1.34), we see that

$R^{\varphi}_{\theta\varphi\theta} = 1$. Similarly, we compute $R_{\varphi\varphi} = R^{\theta}_{\varphi\theta\varphi} + R^{\varphi}_{\varphi\varphi\varphi} = R^{\theta}_{\varphi\theta\varphi} = \sin^2\theta$. By symmetry of the Riemann tensor, we see that $R_{\theta\varphi} = R_{\varphi\theta} = 0$. Thus, the Ricci tensor (written in matrix form) is

$$\text{Ricci} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}. \tag{1.48}$$

The scalar curvature is now easy to compute:

$$R = g^{\theta\theta}R_{\theta\theta} + g^{\varphi\varphi}R_{\varphi\varphi} = 1 + \frac{1}{\sin^2\theta}\sin^2\theta = 2, \tag{1.49}$$

so that the sphere is of constant positive curvature, as expected. In fact, the scalar curvature computed in this fashion for surfaces is always twice the Gaussian curvature.

Exercise. Compute the Ricci and scalar curvatures of S^2 using stereographic coordinates (see equation (1.35)). *Solution:* The Ricci curvature is

$$\text{Ricci} = \begin{pmatrix} 4/\rho & 0 \\ 0 & 4/\rho \end{pmatrix}, \tag{1.50}$$

and the scalar curvature is $R = 2$, which is consistent with the result from the previous example.

Example. *The Torus* $\mathbb{T}^2$. The Ricci and scalar curvatures of the torus can be found by directly applying the formalism to (1.38). By symmetry of the Riemann tensor, we again see that the components $R_{\theta\varphi} = R_{\varphi\theta} = 0$. It remains to compute the diagonal entries. By equation (1.46), we have that $R_{\theta\theta} = R^{\theta}_{\theta\theta\theta} + R^{\varphi}_{\theta\varphi\theta} = R^{\varphi}_{\theta\varphi\theta} = \frac{r\cos\theta}{R+r\cos\theta}$. Similarly, we compute $R_{\varphi\varphi}$ to be $R_{\varphi\varphi} = R^{\theta}_{\varphi\theta\varphi} = \frac{\cos\theta(R+r\cos\theta)}{r}$. The Ricci tensor is therefore

$$\text{Ricci} = \begin{pmatrix} \frac{r\cos\theta}{R+r\cos\theta} & 0 \\ 0 & \frac{\cos\theta(R+r\cos\theta)}{r} \end{pmatrix}. \tag{1.51}$$

We can then compute the scalar curvature as

$$R = g^{\theta\theta}R_{\theta\theta} + g^{\varphi\varphi}R_{\varphi\varphi} = \frac{1}{r^2}\left(\frac{r\cos\theta}{R+r\cos\theta}\right) + \frac{1}{(R+r\cos\theta)^2}\left(\frac{\cos\theta\,(R+r\cos\theta)}{r}\right) = \frac{2\cos\theta}{r(R+r\cos\theta)}. \tag{1.52}$$

As a consistency check, let us compute the integrated total curvature:

$$\int RdA = \iint R\sqrt{g}d\theta d\varphi = \iint \left(\frac{2\cos\theta}{r(R+r\cos\theta)}\right) r(R+r\cos\theta)d\theta d\varphi = \iint 2\cos\theta d\theta d\varphi = 0,$$

which is consistent with the Gauss-Bonnet theorem ($\int_{\mathbb{T}^2} RdA = 4\pi\chi(\mathbb{T}^2) = 0$).

Example. *The Upper Half Plane* $\mathbb{H}$. We now compute the derived curvature quantities for the upper half plane. By symmetry, the off-diagonal components of the Ricci tensor are zero. The xx component of the Ricci tensor is: $R_{xx} = R^x_{xxx} + R^y_{xyx} = -1/y^2$. Similarly, $R_{yy} = R^x_{yxy} + R^y_{yyy} = -1/y^2$. Thus, the Ricci tensor is

$$\text{Ricci} = \begin{pmatrix} -1/y^2 & 0 \\ 0 & -1/y^2 \end{pmatrix}. \tag{1.53}$$

We can then compute the scalar curvature to be

$$R = g^{xx}R_{xx} + g^{yy}R_{yy} = -1 - 1 = -2, \tag{1.54}$$

So that the upper half plane is of constant negative curvature, as expected.

Exercise. Compute the Ricci and scalar curvatures of the disc with the hyperbolic metric (see equation (1.42)). *Solution:* Recall that $\sigma(x, y) = 1 - x^2 - y^2$. The Ricci tensor is:

$$\text{Ricci} = \begin{pmatrix} -2/\sigma^2 & 0 \\ 0 & -2/\sigma^2 \end{pmatrix}. \tag{1.55}$$

The scalar curvature is $R = -2$, which agrees with the constant negative-curvature solution of the previous example.

Example. *The 3-sphere* S^3. Using the expression (1.43) for the Riemann tensor of the 3-sphere, and the same procedure as before, we compute the Ricci tensor to be

$$\text{Ricci} = 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin^2\psi & 0 \\ 0 & 0 & \sin^2\psi\sin^2\theta \end{pmatrix}. \tag{1.56}$$

Taking the trace of this against the metric tensor (1.10), we compute the scalar curvature to be

$$R = g^{\psi\psi} R_{\psi\psi} + g^{\theta\theta} R_{\theta\theta} + g^{\varphi\varphi} R_{\varphi\varphi} = 2 + 2 + 2 = 6, \tag{1.57}$$

so that the 3-sphere is a 3-manifold of constant curvature.

Example. *The left-invariant metric on $S^3 \cong SU(2)$ & Hopf coordinates.* Using the expression (1.44), we compute the nontrivial components of the Ricci tensor to be $R_{\eta\eta} = 2$, $R_{\xi_1\xi_1} = 2\sin^2\eta$, and $R_{\xi_2\xi_2} = 2\cos^2\eta$, so that

$$\text{Ricci} = 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin^2\psi & 0 \\ 0 & 0 & \cos^2\eta \end{pmatrix}. \tag{1.58}$$

Contracting with the metric tensor, we obtain an expression for the scalar curvature:

$$R = g^{\eta\eta} R_{\eta\eta} + g^{\xi_1\xi_1} R_{\xi_1\xi_1} + g^{\xi_2\xi_2} R_{\xi_2\xi_2} = 2 + 2 + 2 = 6, \tag{1.59}$$

which coincides with our previous result for the scalar curvature of S^3.

References

[1] R. Wald, *General Relativity*, University of Chicago Press, (1984).