Scientific Computing

Understanding and solving stochastic PDE

USHAKOVA Oxana

August 5, 2016

Notations

- S - Stock price, also called Spot price (or any underlying asset)
- $V(S,t)$ - value of an option, depending on time and spot price
- K - Strike price
- r - risk-free rate
- d - dividend yield
- μ - drift rate of S - the rate at which the average of S changes
- σ - volatility of the stock, standard deviation of $log(S)$ - return on stock
- T_0, T - initial and final time
- θ - long variance : as t tends to infinity, the expected value of ν tends to θ
- κ - the rate at which ν reverts to θ
- ξ - the volatility of volatility

Financial mathematics

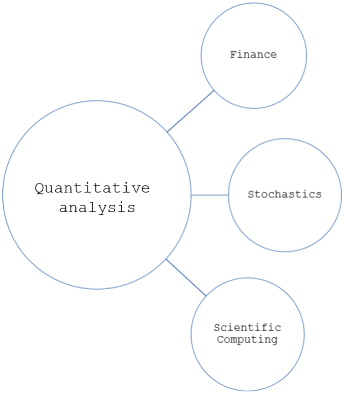


Figure : Financial mathematics

What is an option ?

An option - a contract that gives the buyer the right, **but not the obligation**, to buy or sell an underlying asset (a stock, a bond, gold, other option) at a specific price, called Strike price, on a certain date, called maturity.

What right is proposed ?

- The right to buy – **Call** option
- The right to sell – **Put** option

What are option parameters?

Parameters to fix at t_0 :

- Who buys (long), who sells (short)?
- What is the underlying asset ?
- What is the maturity T of the contract ?
- Does the contract give the right to buy (**call option**) or to sell(**put option**) ?
- What is the Strike price K ?
- What is the price of the option itself, i.e. premium?

Long Call Payoff

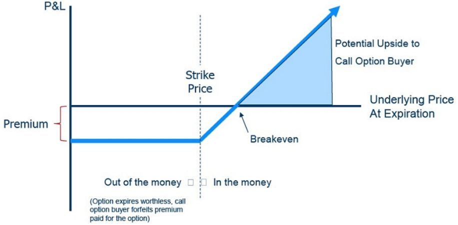


Figure : Options Payoffs

Buying Call Option brings profit (*in the money*), if S at maturity T is higher than the Breakeven point. So the payoff of this option is: (premium ignored):

$$Payoff = \begin{cases} S - K & \text{if } S(T) > K \\ 0 & \text{if not} \end{cases}$$

Other Payoffs

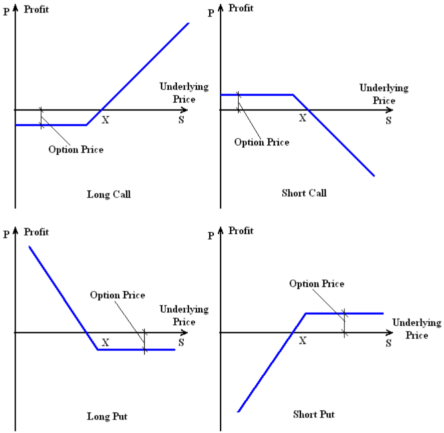


Figure : Other Payoffs

Vanilla VS Exotics

Vanilla Option

- European option

Exotic Option

- American option (Bermudian)
- Barrier option (Paris)
- Asian option
- Lookback option (Russian)
- Binary option
- Cliquer option
- etc, etc..

Greeks

Delta - the rate of change of the option price w.r.t. the price of the underlying asset: $\delta = \frac{\partial C}{\partial S}$

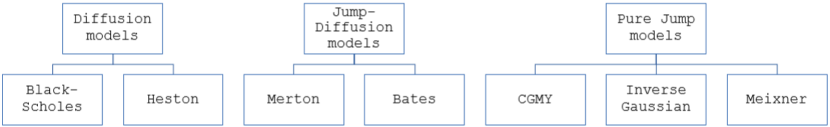
Gamma - the rate of change of the δ w.r.t. the price of the underlying asset: $\Gamma = \frac{\partial^2 \Pi}{\partial S^2}$

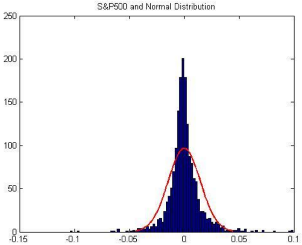
Theta - the rate of change of the portfolio price w.r.t. the time: $\theta = \frac{\partial \Pi}{\partial t}$

Vega - the rate of change of the portfolio price w.r.t. the volatility of the underlying asset: $\upsilon = \frac{\partial \Pi}{\partial \sigma}$

Rho - the rate of change of the portfolio price w.r.t. the interest rate:$\upsilon = \frac{\partial \Pi}{\partial r}$

Pricing models



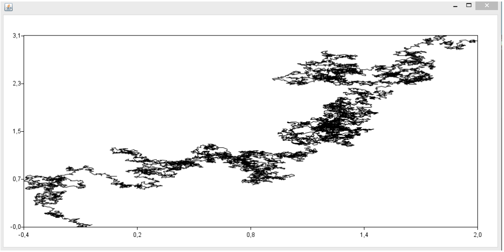


What is a Wiener process

- Lévy processes: independent, stationary increments
- Markov processes: "memoryless"

BUT: no jumps ? → add Compound Poisson process!

Figure : Wiener and Wiener-Poisson process



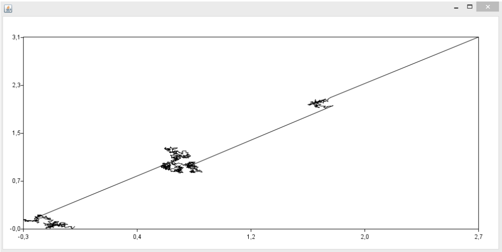
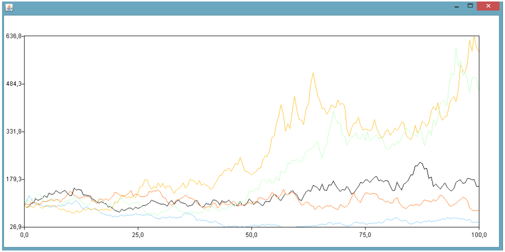
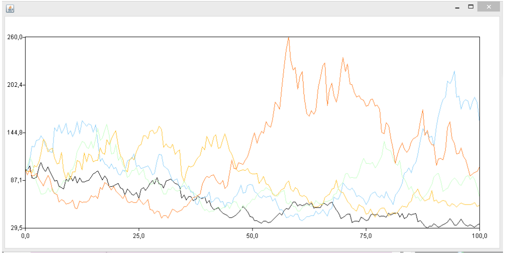


Figure : Black Scholes and Merton





Ito's lemma and Black-Scholes

Ito's lemma :
X_t given by: $dX_t = udt + vdB_t$, $f(t,x) \in C^2$, $Y_t = f(t, X_t)$.

$$dY_t = \frac{\partial f}{\partial t}(t, X_t)dt + \frac{\partial f}{\partial x}(t, X_t)dB_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, X_t)(dB_t)^2 \tag{1}$$

Applying to Black-Scholes:

$$dS_t = \mu S_t dt + \sigma S_t dB_t, X_0 > 0 \tag{2}$$

with $f(t,x) = lnx$,$f \in C^2$ and $Y_t = lnS_t$, gives:

$$\int_0^T dY_t = (\mu - \frac{1}{2}\sigma^2)T + \sigma B_t \tag{3}$$

Or

$$S_T = e^{Y_T} = S_0 e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma dB_t} \tag{4}$$

Technical aspects

- Derivation
  - Economic approach
  - From Heat PDE
- Numerical methods:
  - Monte-Carlo (Box-Muller Algorithm)
  - Tree methods (Cox-Ross-Rubenstein model)
  - **Solving PDE**
- S_t follows Geometric Brownian motion: $dS_t = S_t \mu dt + S_t \sigma dB_t$
- Put-Call Parity:

$$C + Ke^{-rT} = P + S_0 \tag{5}$$

Why use the Geometric Brownian motion?

- The expected returns of GBM are independent of the value of the process (stock price), which agrees with what we would expect in reality.
- A GBM process only assumes positive values, just like real stock prices.
- A GBM process shows the same kind of 'roughness' in its paths as we see in real stock prices.
- Calculations with GBM processes are relatively easy.

GBM path in C++ with gnuplot

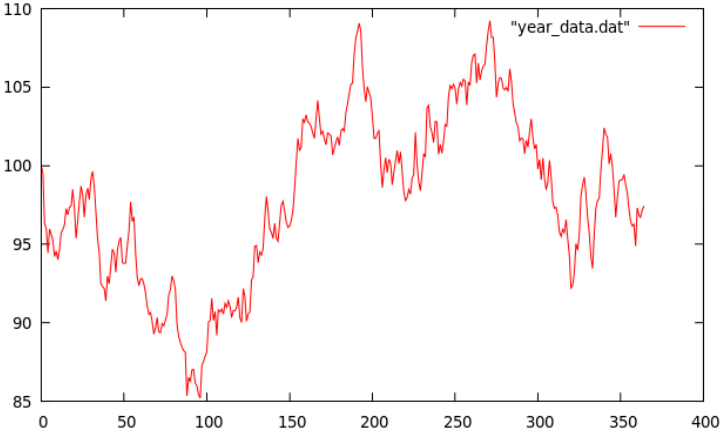


Figure : GBM path

Black-Scholes analytical solution

With Ito's lemma we find :

$$S_T = e^{Y_T} = e^{Y_0 + (\mu - \frac{1}{2}\sigma^2)T + \sigma dB_t} = S_0 e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma dB_t} \tag{6}$$

and

$$C = e^{-dT} S_0 \mathcal{N}(d_1) - e^{-rT} K \mathcal{N}(d_2)) \tag{7}$$

$$P = e^{-rT} K \mathcal{N}(-d_2)) - e^{-dT} S_0 \mathcal{N}(-d_1) \tag{8}$$

with

$$d_1 = \frac{\ln(\frac{S_0}{K}) + (r - d + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(\frac{S_0}{K}) + (r - d - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

Weak Formulation

Linear parabolic PDE with non-constant coefficients and non-homogenous boundary conditions and, possibly, non-differentiable or discontinuous final conditions:

- Choosing space : Weighted Sobolev
  $\forall u \in V, V = \{v \in L^2(\mathbb{R}_+) : S\frac{\partial v}{\partial S} \in L^2(\mathbb{R}_+)\}$
- Bilinear form a:

$$a(u,w) = \int_0^\infty \frac{\partial u}{\partial S}\frac{\sigma^2}{2}\frac{S^2 w}{\partial S} - rSw\frac{\partial u}{\partial S} + ruw \tag{9}$$

- Weak formulation (for put):

$$(\frac{\partial u}{\partial t}, w) + a(u,w) = 0, \forall w \in V \tag{10}$$

Black-Scholes PDE - Call and Put

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S(t)^2} + rS(t)\frac{\partial C}{\partial S(t)} - rC = 0 \tag{11}$$

$C(0, t) = 0$
$C(S, t) = e^{-dT}S_0 - e^{-rT}K$ when $S \longrightarrow \infty$
$C(S, T) = max(S - K, 0)$

$$\frac{\partial P}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S(t)^2} - rS(t)\frac{\partial P}{\partial S(t)} + rP = 0 \tag{12}$$

$P(0, t) = e^{-rT}K$
$P(S, t) = 0$ when $S \longrightarrow \infty$
$P(S, T) = max(K - S, 0)$

Black-Scholes PDE terms

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S(t)^2} + rS(t)\frac{\partial C}{\partial S(t)} - rC = 0 \tag{13}$$

- Diffusion term: $\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S(t)^2}$
- Convection term: $rS(t)\frac{\partial C}{\partial S(t)}$

Peclet number : $\frac{diff}{conv} = \frac{rS}{\frac{1}{2}\sigma^2 S^2} = \frac{2rS}{\sigma^2 S^2} = \frac{2r}{\sigma^2 S}$

Remarque : $Pe < \frac{r}{\sigma^2}$, if not, then small σ is not compensated by a small r.

Existence and Uniqueness

Martingales + Filtration + Ito Calculus

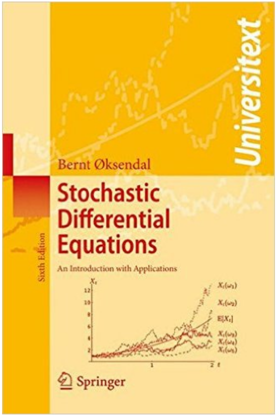


Figure : Oksendal - SDE (6ed)

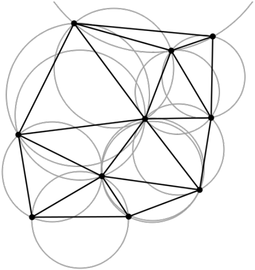
Mesh adaptation and Delaunay Triangulation

Delanay algorithm: interpolation error is bounded by:

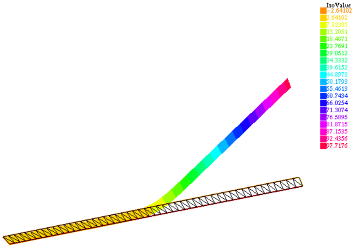
$$||u - u_h|| < C||\nabla(\nabla u)h^2 \tag{14}$$

- no obtuse triangles
- neighbour triangles with the same size

For each edge the circle circumscribing one triangle does not contain the fourth vertex.



Vanilla put



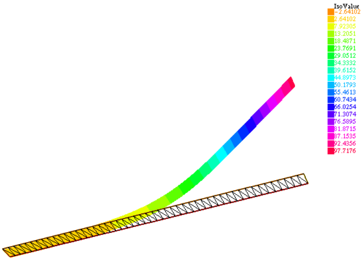


Figure : $\sigma = 0.1$ and $\sigma = 0.3$

Barrier put : 30, 90, 100 with K=100

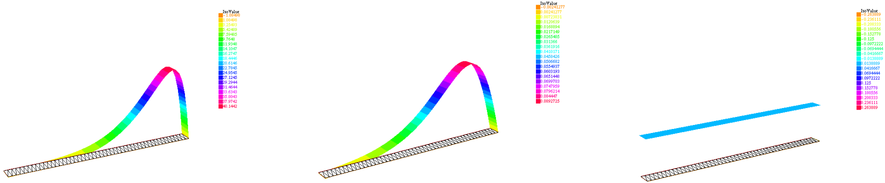


Figure : barriers $= 30, 90, 100$

Asian put

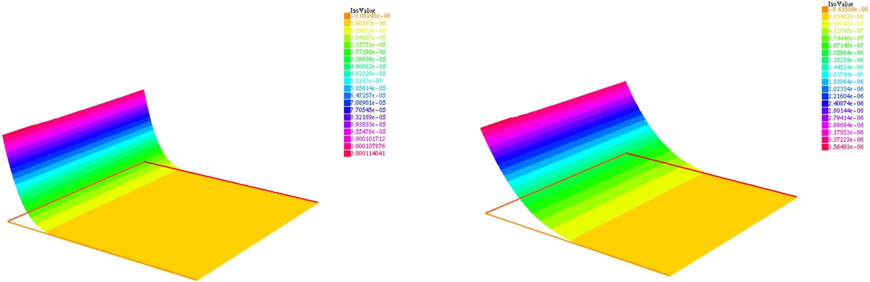
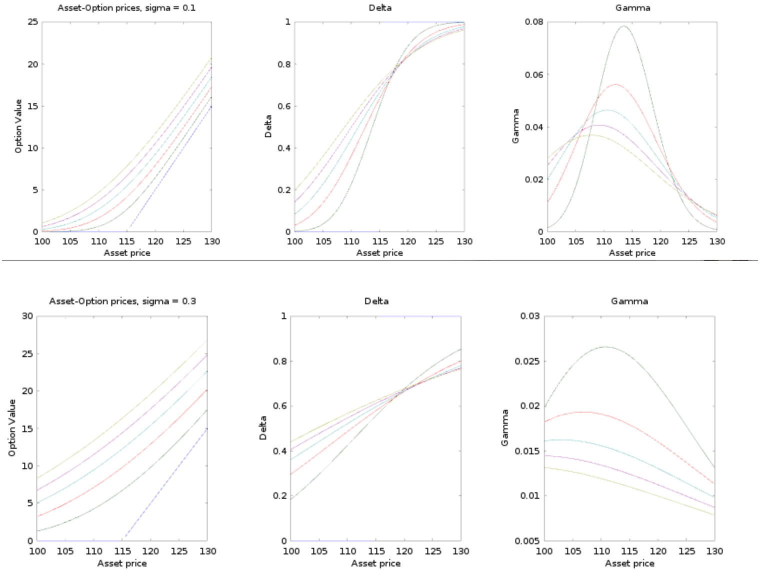


Figure : $\mu = 0.1$ and $\mu = 0.3$

Delta and Gamma for $\sigma = 0.1, 0.3$



Delta for $\sigma = 0.1$

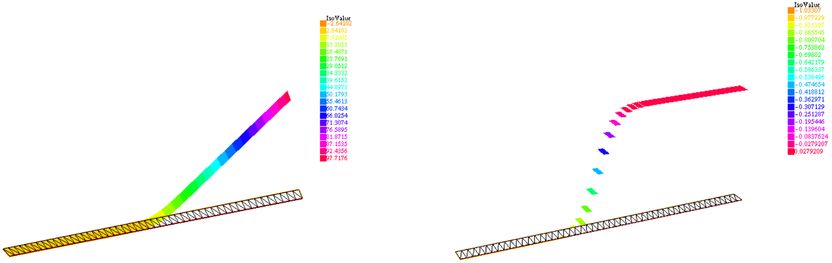
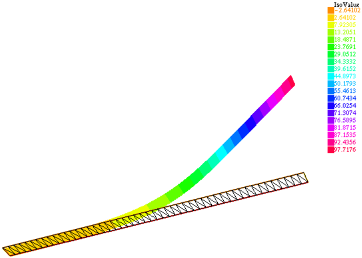


Figure : $\sigma = 0.1$ and $\sigma = 0.3$

Delta for $\sigma = 0.3$



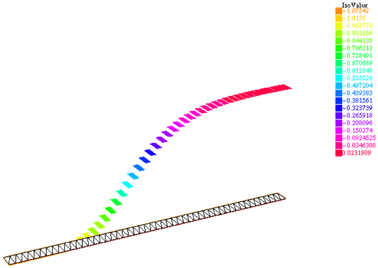
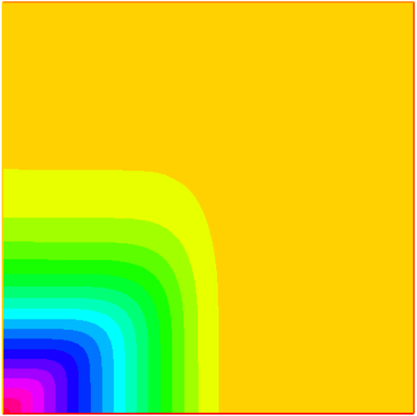
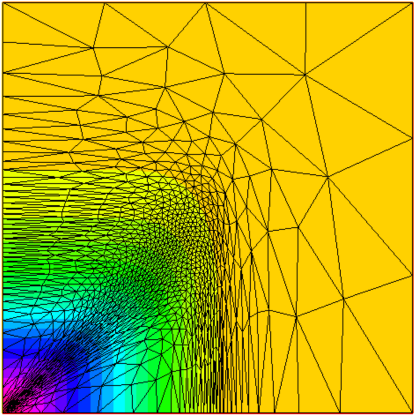


Figure : Delta for $\sigma = 0.1$ and $\sigma = 0.3$

Vanilla 2D

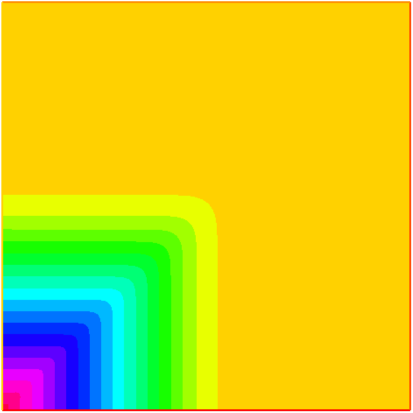
Classic asymmetric data

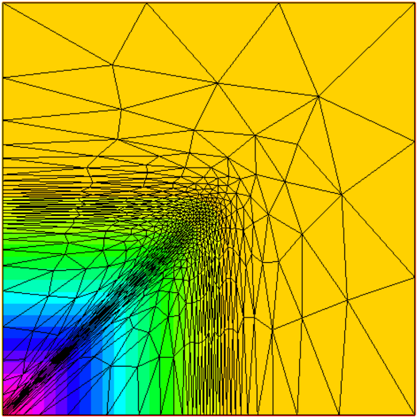




Vanilla 2D

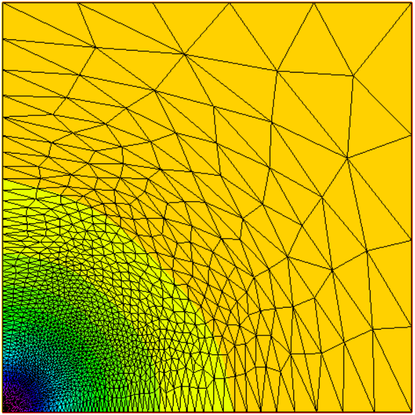
Low volatility with high correlation





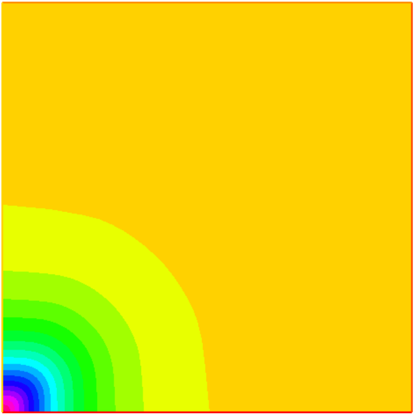
Vanilla 2D

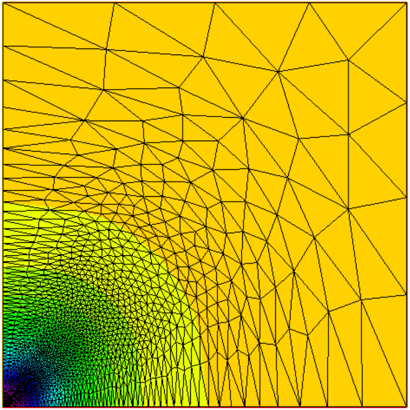
High volatility but low correlation



Vanilla 2D

High volatility with high correlation





Basics

- Forward-Time Central-Space (FTCS) or **Explicit**
- Backward-Time Central-Space (BTCS)or Implicit
- Central-Time Central-Space (CTCS) or Crank-Nicolson

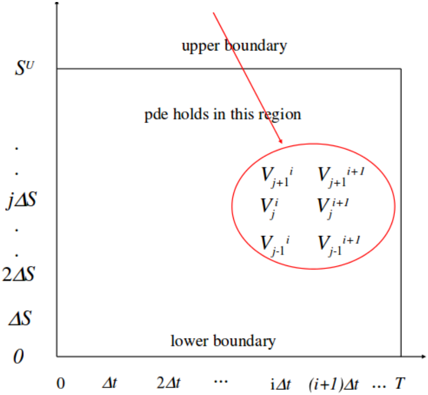
$$V_j^i = \frac{1}{1 + r\Delta t}(AV_{j+1}^{i+1} + BV_j^{i+1} + CV_{j-1}^{i+1}) \tag{15}$$

with

- $A = (\frac{1}{2}\sigma^2 j^2 + \frac{1}{2}rj)\Delta t$
- $B = 1 - \sigma^2 j^2 \Delta t$
- $C = (\frac{1}{2}\sigma^2 j^2 - \frac{1}{2}rj)\Delta t$

"Explicit" grid

- Space interval $[0, S^U]$ in *jmax* intervals of length $\Delta S = S^U/jmax$
- Time interval $[0, T]$ in *imax* intervals of length $\Delta t = T/imax$
- The value at each node $V(j\Delta S, i\Delta t) \longrightarrow V_j^i$



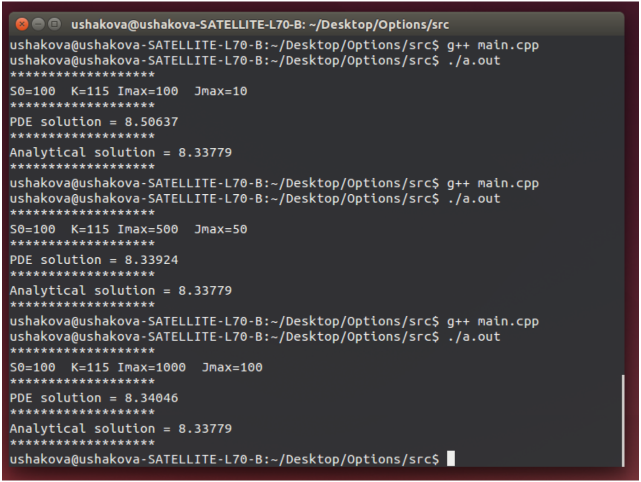
Stability

Explicit method $\Rightarrow$ unstable !
Conditions of stability derived by using probabilistic approach.
If A, B, C as probabilities $\Rightarrow$ positive.

- A and $C \rightarrow j > |\frac{r}{\sigma^2}|$
- $B \rightarrow \Delta t < \frac{1}{\sigma^2 j^2}$

Increase *jmax* by 10 $\rightarrow$ Increase *imax* by 100.

FDM and Analytical solutions in C++



```
ushakova@ushakova-SATELLITE-L70-B: ~/Desktop/Options/src
ushakova@ushakova-SATELLITE-L70-B:~/Desktop/Options/src$ g++ main.cpp
ushakova@ushakova-SATELLITE-L70-B:~/Desktop/Options/src$ ./a.out
*******************
S0=100  K=115 Imax=100  Jmax=10
*******************
PDE solution = 8.50637
*******************
Analytical solution = 8.33779
*******************
ushakova@ushakova-SATELLITE-L70-B:~/Desktop/Options/src$ g++ main.cpp
ushakova@ushakova-SATELLITE-L70-B:~/Desktop/Options/src$ ./a.out
*******************
S0=100  K=115 Imax=500  Jmax=50
*******************
PDE solution = 8.33924
*******************
Analytical solution = 8.33779
*******************
ushakova@ushakova-SATELLITE-L70-B:~/Desktop/Options/src$ g++ main.cpp
ushakova@ushakova-SATELLITE-L70-B:~/Desktop/Options/src$ ./a.out
*******************
S0=100  K=115 Imax=1000  Jmax=100
*******************
PDE solution = 8.34046
*******************
Analytical solution = 8.33779
*******************
ushakova@ushakova-SATELLITE-L70-B:~/Desktop/Options/src$
```

Figure : FDM and Analytical solutions

Comparing FDM and FEM for SDE

- Why use FEM?
  - Much more accurate
  - Lots of free libraries
- Why use FDM?
  - No usual boundary problems
  - Easier to implement