Scientific Computing

Understanding and solving stochastic PDE

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What is an option ?

An option - a contract that gives the buyer the right, **but not the obligation**, to buy or sell an underlying asset (a stock, a bond, gold, other option) at a specific price, called Strike price, on a certain date, called maturity.

What right is proposed ?

- The right to buy – **Call** option
- The right to sell – **Put** option

What are option parameters?

Parameters to fix at t_0 :

- Who buys (long), who sells (short)?
- What is the underlying asset ?
- What is the maturity T of the contract ?
- Does the contract give the right to buy (**call option**) or to sell(**put option**) ?
- What is the Strike price K ?
- What is the price of the option itself, i.e. premium?

Long Call Payoff

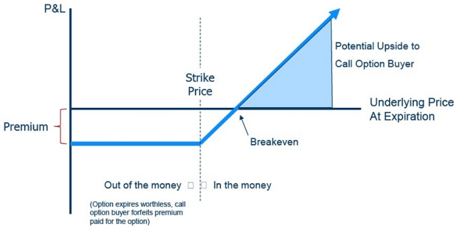


Figure : Options Payoffs

Buying Call Option brings profit (*in the money*), if S at maturity T is higher than the Breakeven point. So the payoff of this option is: (premium ignored):

$$Payoff = \begin{cases} S - K & \text{if } S(T) > K \\ 0 & \text{if not} \end{cases}$$

Other Payoffs

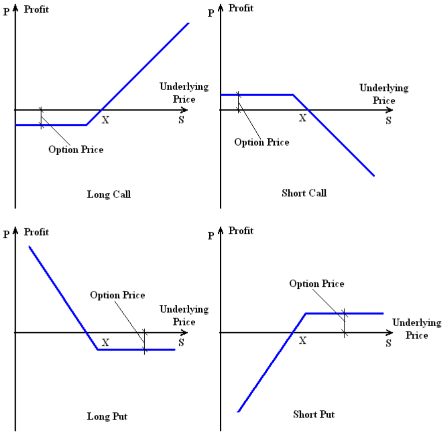


Figure : Other Payoffs

Vanilla VS Exotics

Vanilla Option

- European option

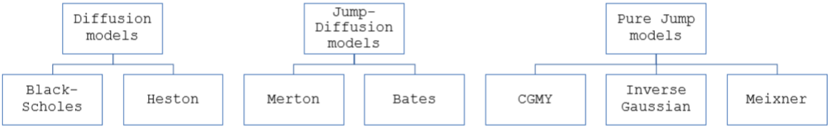
Exotic Option

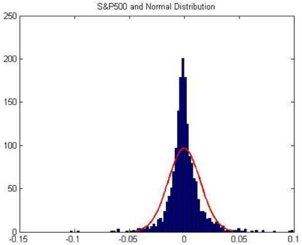
- American option (Bermudian)
- Barrier option (Paris)
- Asian option
- Lookback option (Russian)
- Binary option
- Cliquer option
- etc, etc..

Notations

- S - Stock price, also called Spot price (or any underlying asset)
- $V(S, t)$ - value of an option, depending on time and spot price
- K - Strike price
- r - risk-free rate
- d - dividend yield
- μ - drift rate of S - the rate at which the average of S changes
- σ - volatility of the stock, standard deviation of $log(S)$ - return on stock
- T_0, T - initial and final time
- θ - long variance : as t tends to infinity, the expected value of ν tends to θ
- κ - the rate at which ν reverts to θ
- ξ - the volatility of volatility

Pricing models





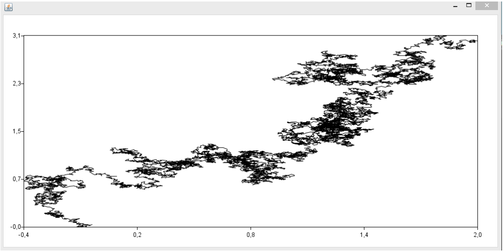
What is a Wiener process

Wiener Process (Browinan Motion) is a stochastic process that lives in family of:

- Lévy processes: independent, stationary increments
- Markov processes: "memoryless"

BUT: no jumps $\rightarrow$ add Compound Poisson process

Figure : Wiener and Poisson process



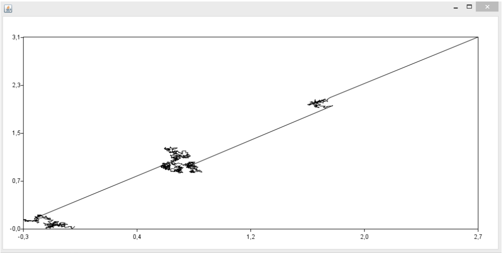
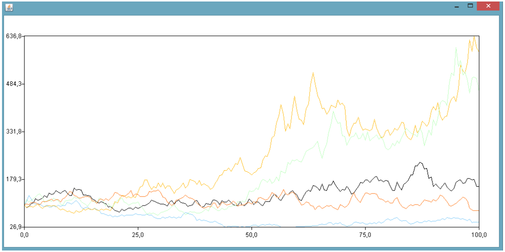
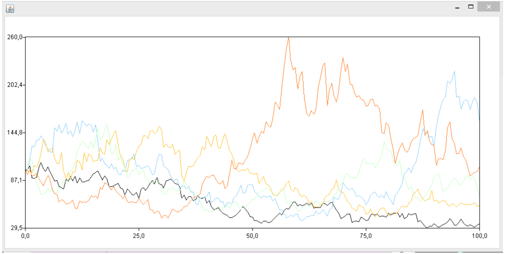


Figure : Black Scholes and Merton





Ito's lemma - derivation

1. X_t - stochastic process, that satisfies $dX_t = \mu dt + \sigma dB_t$.
2. Taylor series expansion for $f(t,x) \in C^2$:

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dx + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} dx^2 ... \tag{1}$$

3. Change $x \longrightarrow X_t$, $dx \longrightarrow \mu dt + \sigma dB_t$:

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} (\mu dt + \sigma dB_t) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} (\mu^2 dt^2 + 2\mu\sigma dt dB_t + \sigma^2 dB_t^2) + ... \tag{2}$$

4. Using $dt * dt = dt * dB_t = dB_t * dt = 0, dB_t * dB_t = dt$, we get:

$$df = (\frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 f}{\partial x^2}) dt + \sigma \frac{\partial f}{\partial x} dB_t \tag{3}$$

Technical aspects

- Numerical methods:
  - Monte-Carlo (Box-Muller Algorithm)
  - Tree methods (Cox-Ross-Rubenstein model)
  - **Solving PDE**
- S_t follows Geometric Brownian motion: $dS_t = S_t \mu dt + S_t \sigma dB_t$
- Put-Call Parity:

$$C + Ke^{-rT} = P + S_0 \tag{4}$$

Why use the Geometric Brownian motion?

- The expected returns of GBM are independent of the value of the process (stock price), which agrees with what we would expect in reality.
- A GBM process only assumes positive values, just like real stock prices.
- A GBM process shows the same kind of 'roughness' in its paths as we see in real stock prices.
- Calculations with GBM processes are relatively easy.

Analytic Solution

With Ito's lemma we find :

$$S_T = e^{Y_T} = e^{Y_0+(\mu-\frac{1}{2}\sigma^2)T+\sigma dB_t} = S_0 e^{(\mu-\frac{1}{2}\sigma^2)T+\sigma dB_t} \tag{5}$$

and

- $C = e^{-rT}\mathbb{E}[max(S_0 e^{(r-d-\frac{\sigma^2}{2})T+\sigma B_T} - K, 0)]$

$\Rightarrow \mathrm{C} = \mathrm{e}^{-dT} S_0 \mathcal{N}(d_1) - e^{-rT} K \mathcal{N}(d_2))$

- $P = e^{-rT}\mathbb{E}[K - max(S_0 e^{(r-d-\frac{\sigma^2}{2})T+\sigma B_T}, 0)]$

$\Rightarrow \mathrm{P} = \mathrm{e}^{-rT} K \mathcal{N}(-d_2)) - e^{-dT} S_0 \mathcal{N}(-d_1)$

with

$d_1 = \frac{\ln(\frac{S_0}{K})+(r-d+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$

$d_2 = \frac{\ln(\frac{S_0}{K})+(r-d-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$

Black-Scholes PDE - Call and Put

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S(t)^2} + rS(t)\frac{\partial C}{\partial S(t)} - rC = 0 \tag{6}$$

$C(0, t) = 0$
$C(S, t) = e^{-dT} S_0 - e^{-rT} K$ when $S \longrightarrow \infty$
$C(S, T) = max(S - E, 0)$

$$\frac{\partial P}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S(t)^2} - rS(t)\frac{\partial P}{\partial S(t)} + rP = 0 \tag{7}$$

$P(0, t) = e^{-rT} K$
$P(S, t) = 0$ when $S \longrightarrow \infty$
$P(S, T) = max(E - S, 0)$

Weak Formulation

Linear parabolic PDE with non-constant coefficients and non-homogenous boundary conditions and, possibly, non-differentiable or discontinuous final conditions:

- Choosing space : Weighted Sobolev
  $\forall u \in V, V = \{v \in L^2(\mathbb{R}_+) : S\frac{\partial v}{\partial S} \in L^2(\mathbb{R}_+)\}$
- Coercive form a:

$$a(u, w) = \int_0^\infty \frac{\partial u}{\partial S}\frac{\sigma^2}{2}\frac{S^2 w}{\partial S} - rSw\frac{\partial u}{\partial S} + ruw \tag{8}$$

- Weak formulation (for put):

$$(\frac{\partial u}{\partial t}, w) + a(u, w) = 0, \forall w \in V \tag{9}$$

Existence and Uniqueness

Martingales + Filtration + Ito Calculus

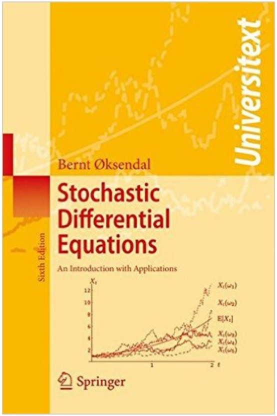


Figure : Oksendal - SDE (6ed)

Discretization

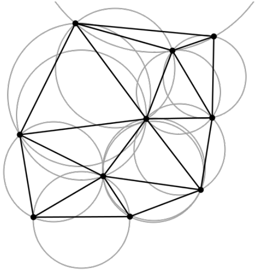
ADD!!!!!!!!!!!!!!

Mesh adaptation and Delaunay Triangulation

Delanay algorithm keeps the error of interpolation bounded by:

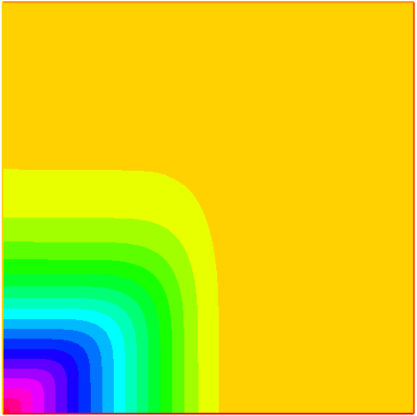
$$||u - u_h|| < C||\nabla(\nabla u)h^2 \qquad (10)$$

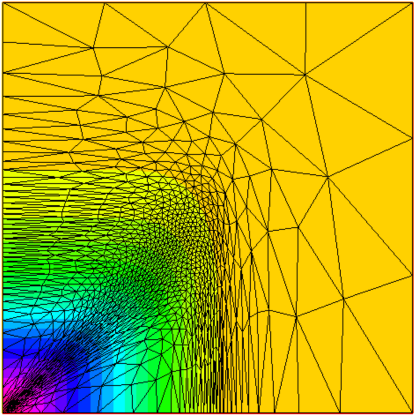
Delaunay triangulation helps to create a "good" mesh : no obtuse triangles, neighbor triangles have more or less the same size.
In other words, the Delaunay triangulation create a mesh where for each edge the circle circumscribing one triangle does not contain the fourth vertex.



Freefem++ output

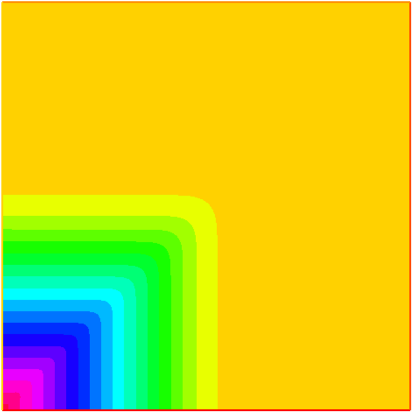
Classic asymmetric data

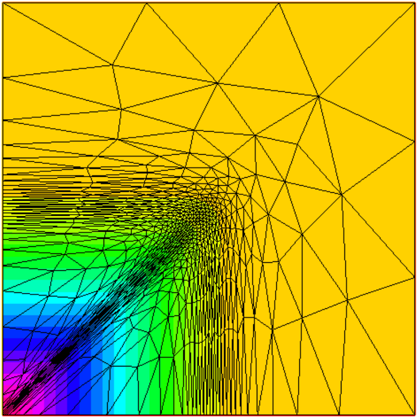




Freefem++ output

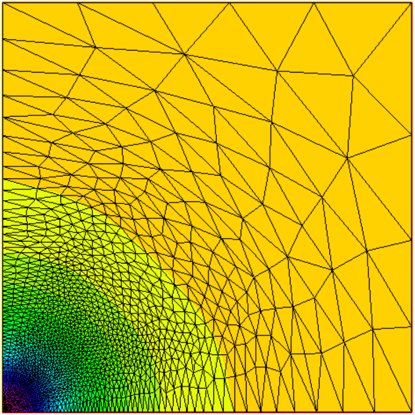
Low volatility with high correlation





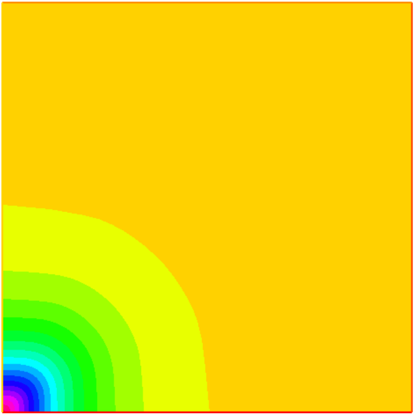
Freefem++ output

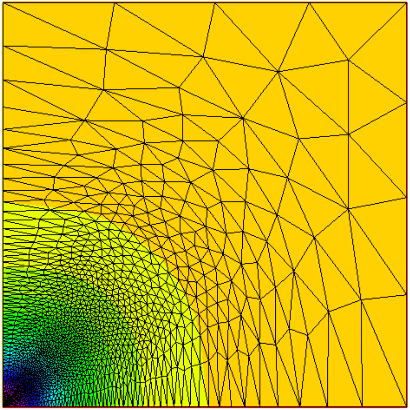
High volatility but low correlation



Freefem++ output

High volatility with high correlation





Black-Scholes Heat equations

- $x = \ln \frac{S}{K} \Rightarrow$S=Ke^{x}
- $\tau = \frac{\sigma^2}{2}(T - t) \Rightarrow$t=T-2$\tau_{\overline{\sigma^2}}$
- $U(x, \tau) = \frac{1}{K} V(S, t) = \frac{1}{K} V(Ke^x, T - 2\frac{\tau}{\sigma^2}$

- $\frac{\partial V}{\partial t} = K \frac{\partial U}{\partial \tau} \frac{\partial \tau}{\partial t} = \frac{-K\sigma^2}{2} \frac{\partial U}{\partial \tau}$
- $\frac{\partial V}{\partial S} = K \frac{\partial U}{\partial x} \frac{\partial x}{\partial S} = \frac{K}{S} \frac{\partial U}{\partial x} = e^{-x} \frac{\partial U}{\partial x}$
- $\frac{\partial^2 V}{\partial S^2} = \frac{e^{-2x}}{K}(\frac{\partial^2 U}{\partial x^2} - \frac{\partial U}{\partial x})$

$$-\frac{\partial U}{\partial \tau} + (k - 1)\frac{\partial U}{\partial x} + \frac{\partial^2 U}{\partial x^2} - kU = 0 \tag{11}$$

$k = \frac{2r}{\sigma^2}$

$U_0(x_T) = U(x_T, 0) = \frac{1}{K} V(S_T - K)^+ = \frac{1}{K}(Ke^{x_T} - K)^+ = (e^{x_T} - 1)^+$

Cont

$W(x,\tau) = e^{\alpha x + \beta^2 \tau} U(x,\tau)$

- $\alpha = \frac{1}{2}(k-1)$
- $\beta = \frac{1}{2}(k+1) = \alpha + 1$

- $\frac{\partial U}{\partial \tau} = e^{-\alpha x - \beta^2 \tau}(\frac{\partial W}{\partial \tau} - W(x,\tau)\beta^2)$
- $\frac{\partial U}{\partial x} = e^{-\alpha x - \beta^2 \tau}(\frac{\partial W}{\partial x} - \alpha W(x,\tau))$
- $\frac{\partial^2 U}{\partial x^2} = e^{-\alpha x - \beta^2 \tau}(\alpha^2 W(x,\tau) - 2\alpha \frac{\partial W}{\partial x} + \frac{\partial^2 W}{\partial x^2}$

$\frac{\partial W}{\partial \tau} = \frac{\partial^2 W}{\partial x^2}$
$W_0(x_T) = W(x_T, 0) = e^{\alpha x_T} U(x_T, 0) = (e^{(\alpha+1)x_T} - e^{\alpha x_T})^+ = (e^{\beta x_T} - e^{\alpha x_T})^+$

$$V(S,t) = \frac{1}{K} e^{-\alpha x - \beta^2 \tau} W(x,\tau) \tag{12}$$

$$W(x,\tau) = \frac{1}{\sqrt{4\pi\tau}} \int_{-\infty}^{\infty} e^{-(x-\xi)^2/4\tau} W_0(\xi) d\xi = \frac{1}{\sqrt{4\pi\tau}} \int_{-\infty}^{\infty} e^{-(x-\xi)^2/4\tau} (e^{\beta\xi} - e \tag{13}$$

$z = \frac{\xi - x}{\sqrt{2\tau}} \Rightarrow \xi = \sqrt{2\tau} z$ and $d\xi = \sqrt{2\tau} dz$

$$W(x,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} exp(-\frac{1}{2} z^2) \times exp(\beta[\sqrt{2\tau} z + x] - \alpha[\sqrt{2\tau} z + x])^+ dz \tag{14}$$

$\beta[\sqrt{2\tau} z + x] > \alpha[\sqrt{2\tau} z + x]$
$z > -\frac{x}{\sqrt{2\tau}}$

Cont

$$W(x,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} exp(-\frac{1}{2}z^2) exp(\beta[\sqrt{2\tau}z + x])dz$$
$$-\frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} exp(-\frac{1}{2}z^2) exp(\alpha[\sqrt{2\tau}z + x])dz = I_1 - I_2$$
$$-\frac{1}{2}z^2 + \beta\sqrt{2\tau}z + \beta x = -\frac{1}{2(z-\beta}\sqrt{2\tau})^2 + +\beta^2\tau$$
$$y = z - \beta\sqrt{2\tau}$$
$$I_1 = e^{\beta x + \beta^2 \tau}\Phi(\frac{x}{\sqrt{2\tau}} + \beta\sqrt{2\tau})$$
$$I_2 = e^{\alpha x + \alpha^2 \tau}\Phi(\frac{x}{\sqrt{2\tau}} + \alpha\sqrt{2\tau})$$

- $\frac{x}{\sqrt{2\tau}} + \beta\sqrt{2\tau} = \frac{\ln\frac{S}{K} + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} = d_1$
- $\frac{x}{\sqrt{2\tau}} + \alpha\sqrt{2\tau} = d_1 - \sigma\sqrt{T-t} = d_2$

$$I_1 = exp(\beta x + \beta^2\tau)\Phi(d_1)$$
$$I_2 = exp(\alpha x + \alpha^2\tau)\Phi(d_2)$$

Cont

$$V(S,t) = Ke^{-\alpha x - \beta^2 \tau} W(x,\tau) = Ke^{-\alpha x - \beta^2 \tau}(I_1 - I_2) = Ke^{-\alpha x - \beta^2 \tau} exp(\beta x + \beta^2 \tag{15}$$

$\alpha^2 - \beta^2 = -\frac{2r}{\sigma^2}$

Cont

Theorem

Let $T > 0$. *Let* $b(.,.) : [0, T] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *and* $\sigma(.,.) : [0, T] \times \mathbb{R}^n \longrightarrow \mathbb{R}^{n\times m}$ *be measurable functions, satisfying two following properties:*

$$|b(t,x)| + |\sigma(t,x)| \leq C(1 + |x|), \tag{16}$$

with $x \in \mathbb{R}, t \in [0, T]$, *for some constant* C, *(where* $|\sigma|^2 = \sum |\sigma_{ij}|^2$*) and*

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq D|x - y|, \tag{17}$$

with $x, y \in \mathbb{R}, t \in [0, T]$, *for some constant* D.
Let Z *be a random variable which is independent of the*
$\sigma - algebra\mathbb{F}_{\infty}^{(m)} generatedbyB - s(.),$*s¿0suchthat*$\mathcal{E}[|Z|^2] < \infty.$
Then the stochastic SDE $dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t$ *with* $X_0 = Z$ *has a unique solution*
$X_t(w)withthepropertythatX_t(w)isadaptedtothrfiltration\mathbb{F}^Z andB_s(.), s \leq$