PDE - Final Project

Oxana USHAKOVA

July 11, 2016

Introduction

The main objective of this project is to learn what are stochastic PDE's and how to solve them with FEM. The numerical results are presented in Feel++ and FreeFem++ environments.

The paper can be divided in 4 parts. As the most common example of stochastic PDEs is the financial model Black-Scholes for option pricing, the first and the second chapters give the basics both in financial theory and stochastic calculus. The third and the forth chapters show the implementation of FEM in theory and in practice respectfully.

Chapter 1

Financial Market : basics

Computational finance is an interdisciplinary field which joins financial mathematics, stochastics, numerics and scientific computing. Its task is to estimate as accurately and efficiently as possible the risks that financial instruments generate. In this project we will focus on options.

1.1 Option as a financial instrument

An **Option** is a contract that gives the buyer the right, *but not the obligation*, to buy or sell an underlying asset (a stock, a bond, gold, other option, etc) at a specific price, called **Strike** price, on or before a certain date. An option is a security, just as stockes or bonds, it has its own price called **Premium**.

Every option contract has several parameters to be pre-set at $t = T_0$:

- What is the underlying asset ?
- What is the maturity T of the contract ?
- Does the contract give the right to buy (**call option**) or to sell(**put option**) ?
- What is the Strike price K ?
- What is the price of the option itself, i.e. premium?
  Rq: note that premium is ignored in following computations

Obviously, the asymmetry of the option contract leads us to the question if we buy this contract it or we sell it. A buyer of an option contract is said to be in **long** positions, the seller - in **short** position.

Let's take a look on potential gain diagram for Long Call Option:

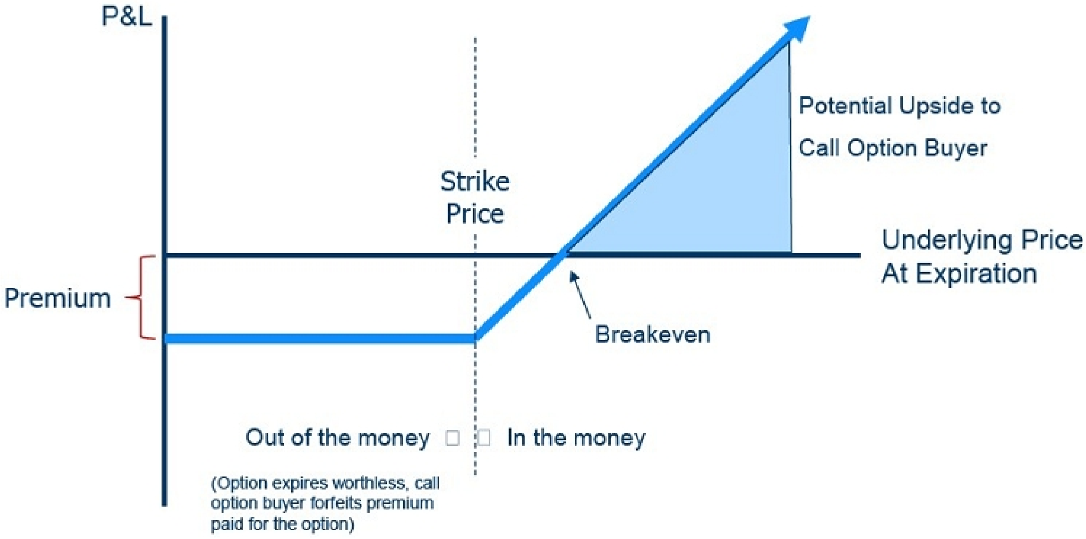

Figure 1.1: Long Call Option

As it comes from the figure 1, Long Call Option brings profit (« in the money ») , if S at maturity T is higher than the Breakeven point. So the payoff of this option is premium ignored):

- $S - K$ if $S(T) > K$

- 0 if $S(T) < K$

1.1.1 Put-Call Parity

The put–call parity defines a relationship between the price of a European call option and European put option, both with the identical strike price and maturity.
We consider two portfolios:one European call (C) and cash (Ke^{-rT}) one European put (P) and one share (S_0)

At time T they both worth $max(S_T, K)$, hence their values should be equal today, i.e.:

$$C + Ke^{-rT} = P + S_0 \tag{1.1}$$

1.1.2 Vanilla vs Exotics

This type of of options is said to be **Vanilla** : the simplest version of all, without any optional extras, by analogy with the default ice cream flavour, vanilla. There exist a rich family of, so called **Exotic** options. Here we list only several of them:

- American option (Bermudian)
  can be executed not only at T,but on any time of life of the option (T_0, T);
  Bermudian option can be executed on a specific period during the life of the option, i.e. every second Monday, June, etc.
- Barrier option (Paris)
  can be activated for be executed only if the asset price touches (or not) a specific barrier;
  Paris barrier option can be activated for be executed only if the asset price satisfy the barrier condition for a certain period of time (i.e. 15min, 1 day, 30
- Asian option
  its payoff is determined by the average underlying price over some pre-set period of time.

- Lookback option (Russian)
  its payoff depends not at S at final time T, $S(T)$, but on $max(S)$ over the life of the option.
  Russian lookback option is a special case of lookback : it has no pre-set expiration time, it's up to buyer of the option when to execute it. It's also called 'no regret' option.

In this project we will focus on Barrier options, so let's look at its payoffs more closely:

- Up-and-in barrier call option
  Standard European call option with strike K when its maximum lies above the barrier H, while it is worthless otherwise.
- Up-and-out barrier call option
  Standard European call option with strike K when its maximum lies below the barrier H, while it is worthless otherwise.
- Down-and-in barrier
  Standard European call option with strike K when its minimum lies below the barrier H,while it is worthless otherwise.
- Down-and-out barrier
  Standard European call with strike K when its lies above some barrier H, while it is worthless otherwise.

1.2 Option Pricing models

1.2.1 General notations and definitions

- S - Stock price, also called Spot price (or any underlying asset)
- $V(S,t)$ - value of an option, depending on time and spot price
- K - Strike price
- r - risk-free rate
- d - dividend yield
- μ - drift rate of S - the rate at which the average of S changes
- σ - volatility of the stock, standard deviation of $log(S)$ - return on stock
- T_0,T - initial and final time
- θ - long variance : as t tends to infinity, the expected value of ν tends to θ
- κ - the rate at which ν reverts to θ
- ξ - the volatility of volatility

1.2.2 Pricing Models

- **Diffusion models** have only a diffusion component, given by Wiener process : *Black-Scholes, Heston*;

  - *Black-Scholes*
    W, and consequently its increment dW,represents the only source of 'diffusion' uncertainty in the price history of the stock :
    $dS = \mu S dt + \sigma S dW$
  - *Heston*
    W, and consequently its increment dW, and ν, and consequently root from it, represent two sources of 'diffusion' uncertainty in the price history of the stock :
    $dS = \mu S dt + \sqrt{\nu} S dW$
    $d\nu = \kappa(\theta - \nu)dt + \xi\sqrt{\nu} dW$

- **Jump diffusion models** have both diffusion component, given by Wiener process, and jump component, given by compounded Poisson process :

  Rq: For financial applications, it is of little interest to have a process with a single possible jump size. The compound Poisson process is a generalization where the waiting times between jumps are exponential but the jump sizes can have an arbitrary distribution.

  - *Merton*
    dW represents the source of 'diffusion' uncertainty and the last term represents is the source of 'jump' uncertainty (compound Poisson process with Gaussian jumps) in the price history of the stock. dW and ν represent the source of 'diffusion' uncertainty and the last term represents is the source of 'jump' uncertainty (compound Poisson process with Gaussian jumps) in the price history of the stock.
    It mixes Black-Scholes model Compounded Poisson process :
    $dS = \mu S dt + \sigma S dW + \sum_{n=1}^{N_t} Y_i$
  - *Bates*
    dW and ν represent the source of 'diffusion' uncertainty and the last term represents is the source of 'jump' uncertainty (compound Poisson process with Gaussian jumps) in the price history of the stock.
    It mixes Black-Scholes model Compounded Poisson process : It mixes Merton and Heston models:

    $dS = \mu S dt + \sqrt{\nu} S dW + \sum_{n=1}^{N_t} Y_i$

    $d\nu = \kappa(\theta - \nu)dt + \xi\sqrt{\nu}dW$

- **Pure jump models** have no diffusion, just a random process :

  - *CGMY*
  - *NIG*

1.3 Greeks

Greeks are the quantities representing the sensitivity of the price of options to a change in underlying parameters. We list some of them, futher in the project we will study them:

- **Delta** measures the rate of change of the theoretical option value with respect to changes in the underlying asset's price. **Delta** is the first derivative of the value V of the option with respect to the underlying instrument's price S.
- **Vega** measures sensitivity to volatility. **Vega** is the derivative of the option value with respect to the volatility of the underlying asset.
- **Theta** measures the sensitivity of the value of the derivative to the passage of time : the "time decay."
- **Rho** measures sensitivity to the interest rate: it is the derivative of the option value with respect to the risk free interest rate (for the relevant outstanding term).

Chapter 2

Stochastics : basics

2.1 Stochastic process

A **stochastic process** in discrete time, denoted by $(X_t)_{t\in\mathcal{T}}$, is a sequence of random variables $X_0, X-1, ..X_T$ defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with values in $\mathbb{R}$. For a given $\omega \in \Omega$, the sequence $X_t(w), t \in \mathcal{T}$ is called a **path** of the process (X_t), where $\mathcal{T}$ stands for time series (dates) : $\mathcal{T} = \{0, 1, ..T\}$

2.2 Filtrations

A **filtration** $\mathcal{F}$ on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ is a sequence of σ-algebras $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq ... \subseteq \mathcal{F}_T \subseteq \mathcal{A}$. A quadruple $(\Omega, \mathcal{A}, \mathbb{P}, \mathcal{F})$ is called a **filtered probability space**.
A stochastic process $(X_t)_{t\in\mathcal{T}}$ is sais to be **adapted to a filtration** $\mathcal{F}$ if for all $t \in \mathcal{T}$, the random variable X_t is $\mathcal{F}_t$ -measurable.

2.3 Martingales

Let $(\Omega, \mathcal{A}, \mathbb{P}, \mathcal{F}))$ be a filtered probability space. A stochastic process $X = (X_t)_{t\in\mathcal{T}}$ is called a **martingale** if

1. X is adapted to $\mathcal{F}$
2. $\mathbb{E}[|X_t|] < \inf \forall \in \mathcal{T}$
3. $\mathbb{E}[X_t|\mathcal{F}_s] = X_s \forall t \geq s$

Note, the following brownian martingale is used in the Black-Scholes theory:

$$(e^{\alpha B_t - \frac{\alpha^2}{2}t}) \tag{2.1}$$

and

$$\mathbb{E}[e^{\alpha B_t - \frac{\alpha^2}{2}t}|\mathcal{F}_f = e^{\alpha B_s - \frac{\alpha^2}{2}s}, 0 \leq s < t \tag{2.2}$$

2.4 Itô Calculus

The **Itô integral** is the central concept of Itô calculus. The integrand and the integrator are stochastic processes:

$$Y_t = \int_0^t X_s dB_s, \tag{2.3}$$

where X is a locally square integrable process adapted to the filtration generated by B, which is a Wiener process or Brownian motion. The result of the integration is another process.

If we assume that $\{\pi_n\}$ is a sequence of partitions of $[0,t]$ with min mesh size, then the integration by parts for Itô integral is given by :

$\int_0^t XdB = \lim_{n\to\infty} \sum_{t_{i-1},t_i\in\pi_n} X_{t_{i-1}}(B_{t_i} - B_{t_{i-1}})$.

X is square integrable, then its integral w.r.t. B can be defined and X is said to be B-integrable. As $\int_0^t (X-X_n)^2 ds \to 0$ in probability, the Itô integral becomes:

$\int_0^t XdB = \lim_{n\to\infty} \int_0^t X_n dB$

where we faal again on convergence in probability. This leads us to a very important property: **Itô isometry**:

$$\mathbb{E}[(\int_0^t X_s dB_s)^2] = \mathbb{E}[\int_0^t X_s^2 ds] \tag{2.4}$$

The most crucial notion for solving stochastic PDE's is the **Itô's formula (lemma)**.

Given a process X_t described by an SDE, the Itô formula tells us how another process $Y_t = f(t, X_t)$ that is given in terms of t and X_t is itself described by an SDE. The formula is so useful because it can be used to transform an SDE that is hard to solve (integrate) into another SDE that can be solved, and then transforming the solution back into a solution for the original equation.
Let X_t be a process given by the SDE:

$dX_t = udt + vdB_t$

Let $f(t,x) \in C^2$ and $Y_t = f(t, X_t)$. Then Y_t follows the SDE:

$$dY_t = \frac{\partial f}{\partial t}(t, X_t)dX_t + \frac{\partial f}{\partial x}(t, X_t)dX_t + \frac{1\partial^2 f}{2\partial x^2}(t, X_t)(dX_t)^2 \tag{2.5}$$

To calculate $(dX_t)^2$ we must keep in mind following rules:

$$dt * dt = dt * dB_t = dB_t * dt = 0, dB_t * dB_t = dt \tag{2.6}$$

A good way to remember Itô's formula is look at it as a Taylor's formula for a function with two variables together with the rules given above.

Let's solve the Black-Scholes SDE:

$$dX_t = \mu X_t dt + \sigma X_t dB_t, X_0 > 0 \tag{2.7}$$

We take $f(t,x) = \ln x$ and $Y_t = \ln X_t$ and with **Itô's lemma** we get:

$$dY_t = \frac{1}{X_t}dX_t + \frac{1}{2}(-\frac{1}{X_t^2})(dX_t)^2 = \mu dt + \sigma dB_t - \frac{1}{2}\frac{1}{X_t^2}\sigma^2 X_t^2 dt = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dB_t. \tag{2.8}$$

Now lets integrate it:

$Y_t - Y_0 = \int_0^T dY_t = (\mu - \frac{1}{2}\sigma^2)\int_0^T dt + \sigma \int_0^t dB_t = (\mu - \frac{1}{2}\sigma^2)T + \sigma B_t$

And we finally get the solution:

$$X_T = e^{Y_T} = e^{Y_0 + (\mu - \frac{1}{2}\sigma^2)T + \sigma dB_t} = X_0 e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma dB_t} \tag{2.9}$$

Chapter 3

Black-Scholes PDE

3.1 Theoretical assumptions

The Black-Scholes model is one of the most important concepts in modern option pricing theory. Before introducing the underlying idea, we list the model's assumptions.

- The stock price follows the Geometric Brownian motion with μ and σ constant.
- The short selling of securities with full use of proceeds is permitted.(*No time limits or conditions on transaction*).
- No transaction costs or taxes. All securities are perfectly divisible.
- There are no dividends during the life of derivative. No arbitrage.
- Security trading is continuous.
- Risk-free rate is constant and the same of all maturities.

3.2 Geometric Brownian Motion

The Black-Scholes SDE describing the process for a stock price S_t is given by:

$$dS_t = \mu S_t dt + \sigma S_t dB_t \tag{3.1}$$

This process is called **Geometric Brownian Motion** - a continuous-time stochastic process in which the logarithm of the randomly varying quantity follows a Brownian motion with drift.
We have already seen that its solution is given by:

$$S_T = S_0 e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma dB_t} \tag{3.2}$$

where the stock price S_t is log-normally distributed.
Recall : μ is a drift rate of the diffusion and σ is the volatility.
So why to use GBM and not the Brownian motion itself?

- The expected returns of GBM are independent of the value of the process (stock price), which agrees with what we would expect in reality.
- A GBM process only assumes positive values, just like real stock prices.
- A GBM process shows the same kind of 'roughness' in its paths as we see in real stock prices.
- Calculations with GBM processes are relatively easy.

However, GBM has no jumps in it and the volatility is constant, which is not the case for a stock price. That's why we will take a look on Heston and Merton models.
For further computations we need to find mean and variance.Lets first find $\mathbb{E}[S_t]$ using martingales theory:

$$\mathbb{E}[S_t] = S_0 e^{\mu T} \mathbb{E}[e^{\sigma B_t - \frac{1}{2}\sigma^2 T}] = S_0 e^{T}. \tag{3.3}$$

since $\mathbb{E}[e^{\sigma B_T - \frac{1}{2}\sigma^2 T}] = e^{\sigma B_0} = 1$.
Now, lets find $\mathbb{E}[S_t]$ with moment generating function.We know that for a normally distributed variable $X \sim \mathcal{N}(\mu, \sigma^2)$ the moment generating function is given by $M_X(t) := \mathbb{E}[e^{tX}] = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$.
When we apply this to:

$$X = \sigma B_T - \tfrac{1}{2}\sigma^2 T \sim \mathcal{N}(-\tfrac{1}{2}\sigma^2 T, \sigma^2 T)$$

we get:

- $\mathbb{E}[e^X] = M_X(1) = e^{-\frac{1}{2}\sigma^2 T + \frac{1}{2}\sigma T} = 1$
- $\mathbb{E}[(e^X)^2] = \mathbb{E}[e^{2X}] = M_X(2) = e^{-\sigma^2 T + 2\sigma T}$

For the second moment we get:

$$\mathbb{E}[S_t^2] = S_0^2 e^{2\mu T}\mathbb{E}[e^{2X}] = S_0^2 e^{2\mu T} e^{\sigma^2 T}$$

And finally the variance:

$$\mathbb{V}[S_T] = \mathbb{E}[S_T^2] - (\mathbb{E}[S_T])^2 = S_0^2 e^{2\mu T}(e^{\sigma^2 T} - 1)$$

3.3 Black-Scholes formulas

Lets find the discounted expectation $e^{-rT}\mathbb{E}[max(S_T - K, 0)]$ for a Vanilla Call option C. Please note, that we need to take in account that $\mu = r - d$, where d corresponds to a dividend yield.

$$\begin{aligned}
C &= e^{-rT}\mathbb{E}[max(S_0 e^{(r-d-\frac{\sigma^2}{2})T + \sigma B_T} - K, 0)] \\
&= \frac{1}{\sqrt{2\pi}} \int_{\frac{\ln(\frac{K}{S_0}) - (r-d-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}}^{\infty} (S_0 e^{\sigma\sqrt{T}x - (d + \frac{\sigma^2}{2}T)} - e^{-rT}K) e^{-\frac{x^2}{2}} dx \\
&= e^{-dT} S_0 \frac{1}{\sqrt{2\pi}} \int_{-d_2}^{\infty} e^{-\frac{(x-\sigma\sqrt{T})^2}{2}} dx - e^{-rT} K \frac{1}{\sqrt{2\pi}} \int_{-d_2}^{\infty} e^{-\frac{x^2}{2}} dx \\
&= e^{-dT} S_0 (1 - \mathcal{N}(-d_2 - \sigma\sqrt{T})) - e^{-rT} K (1 - \mathcal{N}(-d_2)) \\
&= e^{-dT} S_0 \mathcal{N}(d_1) - e^{-rT} K \mathcal{N}(d_2))
\end{aligned}$$

with

- $d_1 = \ln(\frac{S_0}{K}) + (r - d + \frac{\sigma^2}{2})T\sigma\sqrt{T}$
- $d_2 = \ln(\frac{S_0}{K}) + (r - d - \frac{\sigma^2}{2})T\sigma\sqrt{T} = d_1 - \sigma\sqrt{T}$

With similar computation we get a formula for Put option:

$$P = e^{-rT} K \mathcal{N}(-d_2)) - e^{-dT} S_0 \mathcal{N}(-d_1)$$

Chapter 4

Applying FEM

4.1 Stochastic PDE and its Weak Formulation

Consider a vanilla put option with maturity T and payoff function u_0. Let u be the pricing function, i.e., the price of the option at time Tt and when the spot price is S is $u(S,t)$. The function u solves the initial SDE:

$$\frac{\partial u}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 u)}{\partial S^2} - rS\frac{\partial u}{\partial S} + ru = 0 \tag{4.1}$$

Now lets multiply it by a test function and integrate the whole expression. As always , we apply the integration by parts and obtain:

$$\frac{d}{dt}(\int_{\mathbb{R}+} u(S,t)w(S)dS) + a_t(v,w) = 0, \tag{4.2}$$

where a_t is a bilinear form defined as:

$$a_t(v,w) = \int_{\mathbb{R}+} (\frac{1}{2}S^2\sigma^2(S,t)\frac{\partial v}{\partial S}\frac{\partial w}{\partial S} + r(t)vw)ds + \int_{\mathbb{R}+} (-r(t) + \sigma^2(S,t) + S\sigma(S,t)\frac{\partial \sigma}{\partial S}(S,t))S\frac{\partial v}{\partial S}wdS. \tag{4.3}$$

Let's check if u_0 is a unique solution.

- As the volatility is always positive and bounded, we can conclude that a_t is continuous on V, càd there exists a positive constant M, such that for all $v, w \in V$,

$$|a_t(v,w)| \leq M|v|_V|w|_V. \tag{4.4}$$

- To prove the coercivity we use **Gârding's inequity**:

$$a_t(v,v) \geq C_1||v||_V^2 - C_2||v||_{L^2}^2. \tag{4.5}$$

So, under these conditions we may say that by Lax-Milgram if $u_0 \in L^2$ then it is the unique solution and we can write the weak formulation:
Weak Formulation

$$\begin{gathered}\text{Find } u \in \mathcal{C}^0([0;T]), u \in L^2 \cap V \\ \text{and } u|_{t=0} = u_0 \text{ and for all } t \in (0,T), \\ \forall v \in V, (\tfrac{\partial u}{\partial t}(t), v) + a_t(u(t), v) = 0.\end{gathered}$$

4.1.1 Artificial Boundary Conditions

Most option pricing problems have a semi-infinite domain. The usual approach is to truncate some parts of the domain to make it finite and to construct artificial boundary conditions. So we have to determine where to put a condition and of what type it should be. Obviously, this choice is crucial for financial modelling.
A differential equation located at the boundary is closely connected to a Dirichlet BC, since the solution of the differential equation is the corresponding Dirichlet BC. The location of the boundary depends on the contract except in the case of a barrier, the behavior of the option at infinity is emulated at the boundary. The task is to find a point where the infinite or semi-infinite domain can be cut off without severely affecting the solution.
So, at time T the price of the option is the profit made by realizing the option, more precisely:

$$C(x,T) = \phi(x) = (x-K)^+ \tag{4.6}$$

$$P = (x,T) = (K-x)^+ \tag{4.7}$$

We also know that $0 < C < x$ because the price of an option can not be higher than the price of its underlying asset, that's how we get a boundary condition $C = 0$ at $x = 0$. But at $x = 0$ we have the singularity, so the PDE contains a hidden boundary condition:

$$\frac{\partial u}{\partial t} - ru = 0, x = 0 \tag{4.8}$$

and it gives us:

$$u(0,t) = u(0,T)e^{r(t-T)} \tag{4.9}$$

There is one more boundary to impose at the infinity part of the domain, so here instead we just replace infinity by an arbitrary big L: $x = L$:

$$u(L,t) = \phi(L)e^{r(t-T)} \tag{4.10}$$

This approach is called *'localization'*.

The Maximum Principle

Plain Vanilla options

At first we look at Vanilla call options.

$$\frac{\partial(V,t)}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2(V,t)}{\partial S^2} + rS\frac{\partial V(S,t)}{\partial S} - rV = 0 \tag{4.11}$$

$$V(S,T) = max(S-K,0) \tag{4.12}$$

$$V(0,t) = 0 \tag{4.13}$$

To make this problem well-posed, we also need a condition that prohibits unlimited growth of S. There exist several techniques to deal with the boundary problem, but we use the most common approach : we will to cut off the domain at some point.

4.2 Mesh Adaptation and Delaunay triangulation

Mesh adaptation is an important tool in problems with free boundary. The procedure is done w.r.t. Delanay algorithm and keeps the error of interpolation bounded by:

$$||u - u_h|| < C||\nabla(\nabla u)h^2 \tag{4.14}$$

where $\nabla(\nabla u)$ is a Hessian matrix of u.

The, so called, Delaunay triangulation helps to create a "good" mesh : no obtuse triangles, neighbor triangles have more or less the same size.

In other words, the Delaunay triangulation create a mesh where for each edge the circle circumscribing one triangle does not contain the fourth vertex.

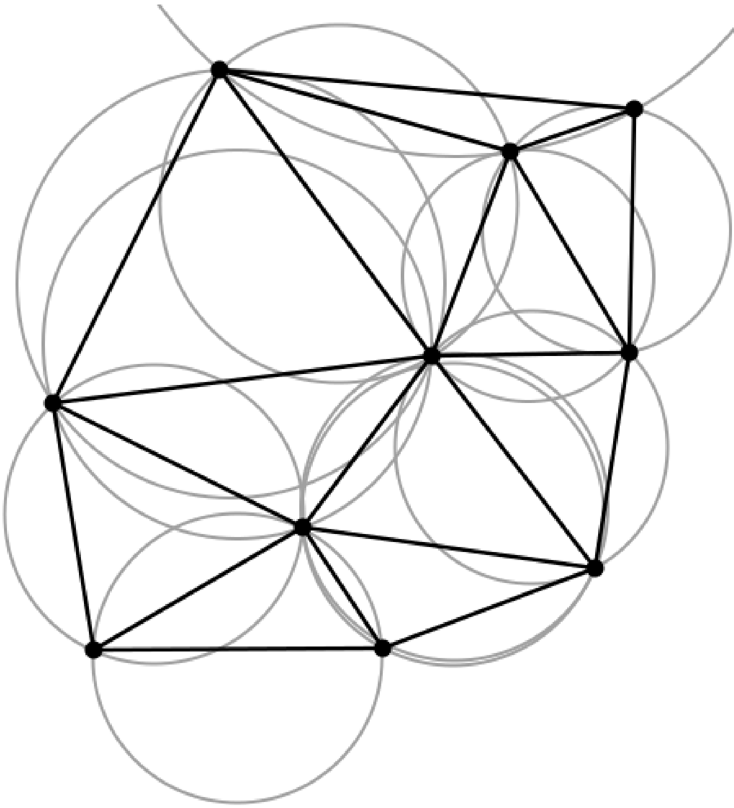

Figure 4.1: Delaunay triangulation

In freefem++ the mesh adaptation is easily done with **adaptmesh** command.

4.3 Numerical results

We solve a 2D PDE for an american put with $= 40, r = 5$

4.3.1 FreeFem++

The source code:

```
 mesh th=square(m,m,[L*x,LL*y]);
fespace Vh(th,P1);

Vh u=max(K-max(x,y),0.);
Vh xveloc, yveloc, v,uold;

for (int n=0; n*dt <= 1.0; n++)
{
if(j>20)  { th = adaptmesh(th,u,verbosity=1,abserror=1,nbjacoby=2,
err=0.001, nbvx=5000, omega=1.8, ratio=1.8, nbsmooth=3,
splitpbedge=1, maxsubdiv=5,rescaling=1) ;
j=0;
xveloc = -x*r+x*sigx^2+x*rho*sigx*sigy/2; //discount factor
yveloc = -y*r+y*sigy^2+y*rho*sigx*sigy/2;

u=u;

};
uold=u;
solve eq1(u,v,init=j,solver=LU) = int2d(th)( u*v*(r+1/dt)
```

```
+ dx(u)*dx(v)*(x*sigx)^2/2 + dy(u)*dy(v)*(y*sigy)^2/2
+ (dy(u)*dx(v) + dx(u)*dy(v))*rho*sigx*sigy*x*y/2)
- int2d(th)(v*convect([xveloc,yveloc],dt,uold)/dt) + on(2,3,u=0);
j=j+1;
};
plot(u,wait=1,value=1);
```

Here are the results:

- Classic asymmetric data

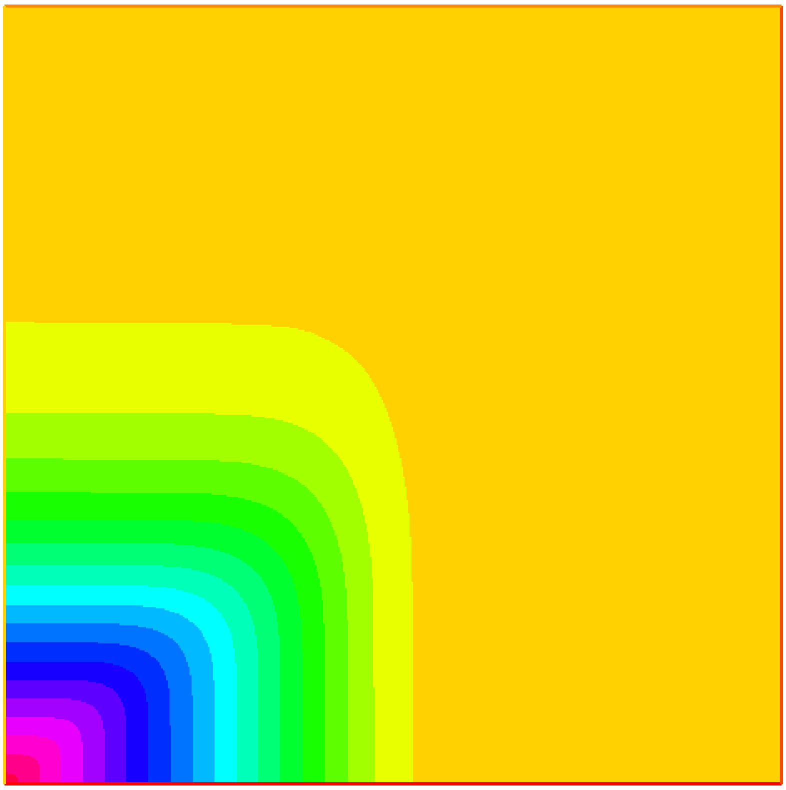

Figure 4.2: Mesh : $\sigma_x = 0.1, \sigma_y = 0.3, \rho = 0.3$

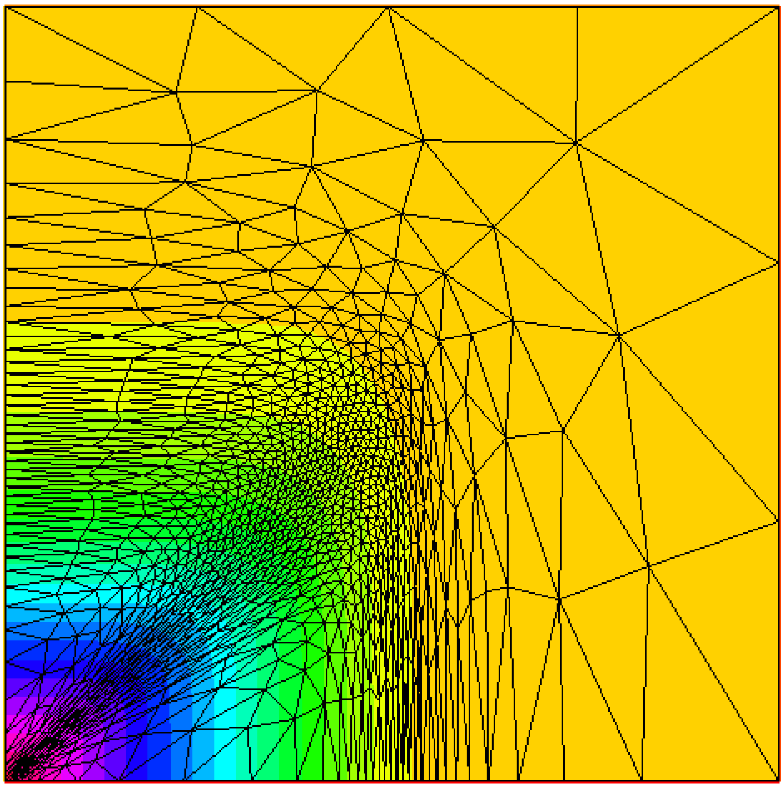

Figure 4.3: PDE : $\sigma_x = 0.1, \sigma_y = 0.3, \rho = 0.3$

- Low volatility with high correlation

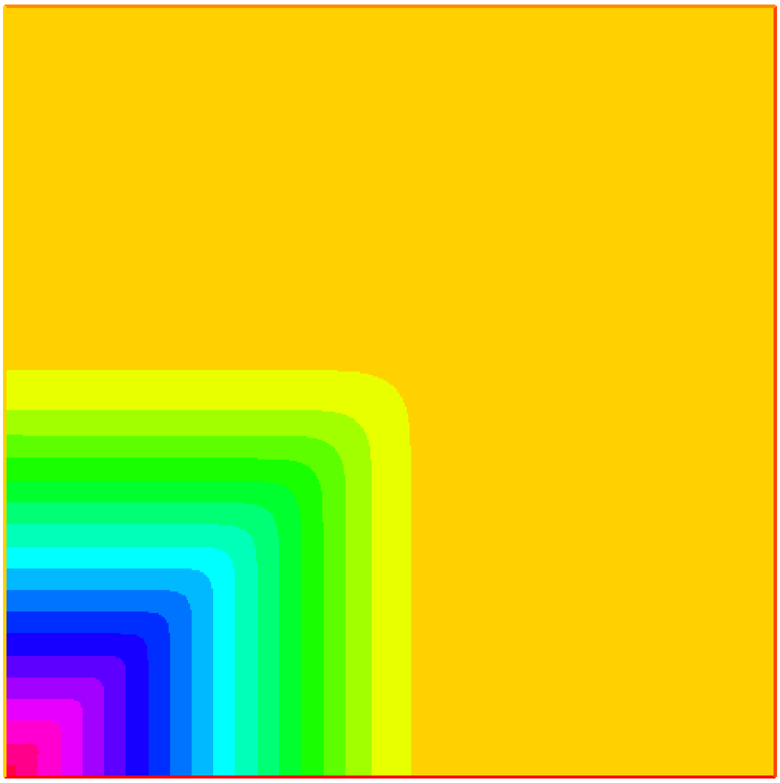

Figure 4.4: Mesh : $\sigma_x = 0.1, \sigma_y = 0.1, \rho = 0.6$

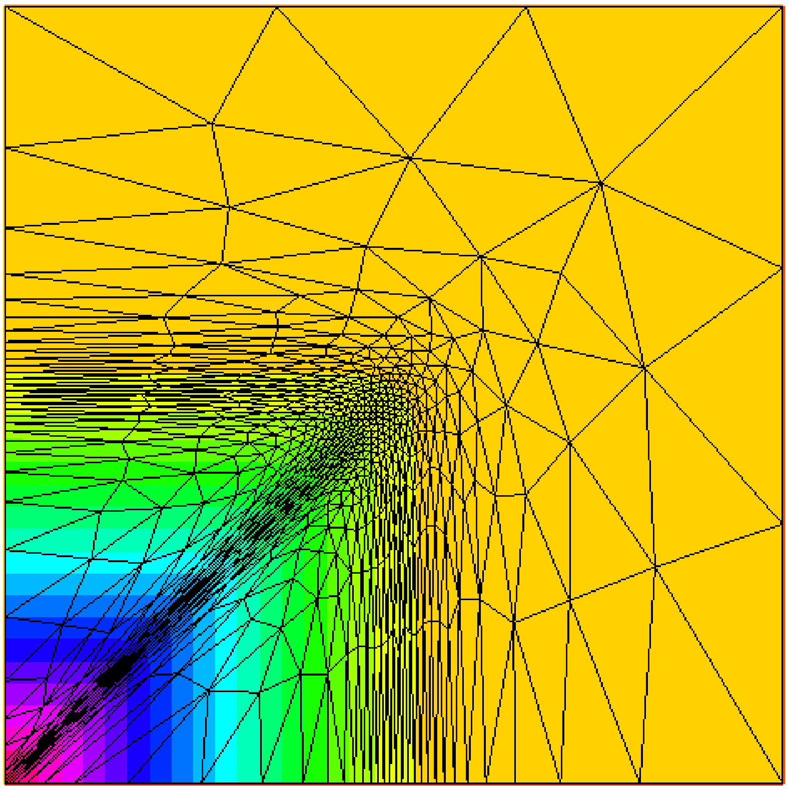

Figure 4.5: PDE : $\sigma_x = 0.1, \sigma_y = 0.1, \rho = 0.6$

- High volatility but low correlation

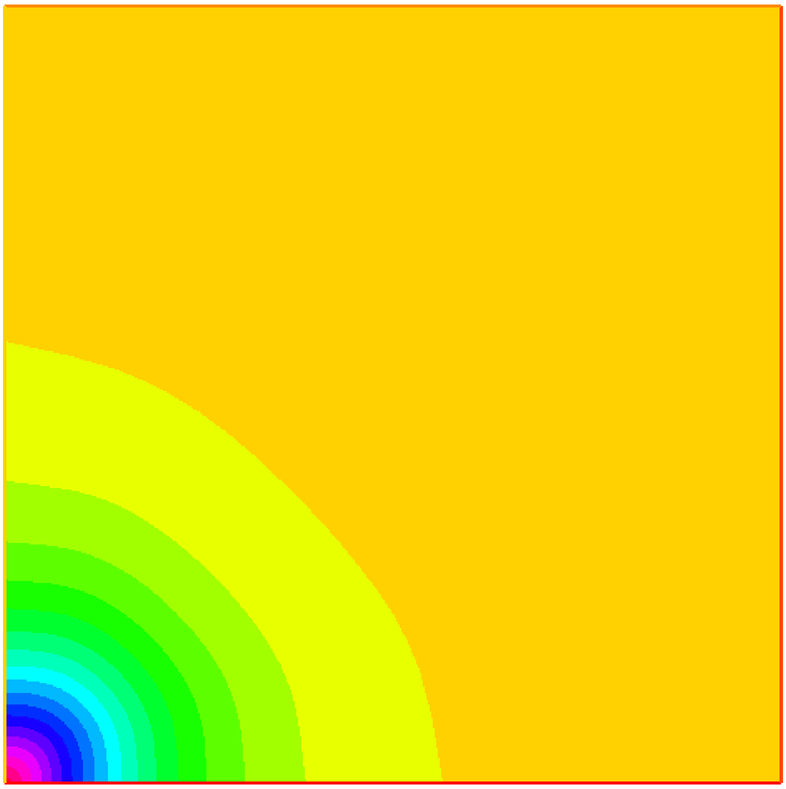

Figure 4.6: Mesh : $\sigma_x = 0.6, \sigma_y = 0.6, \rho = 0.3$

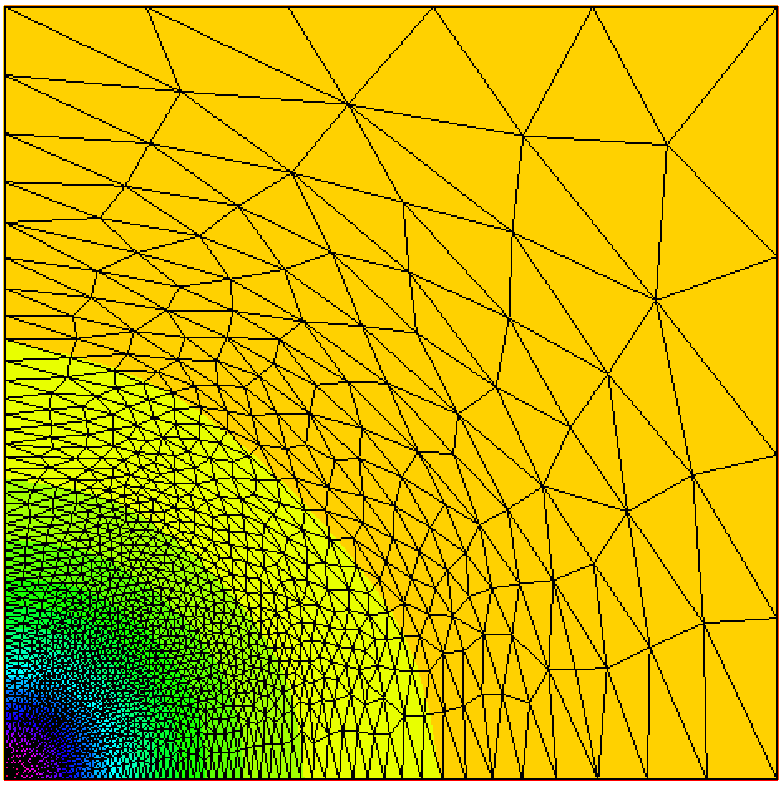

Figure 4.7: PDE : $\sigma_x = 0.6, \sigma_y = 0.6, \rho = 0.3$

- High volatility with high correlation

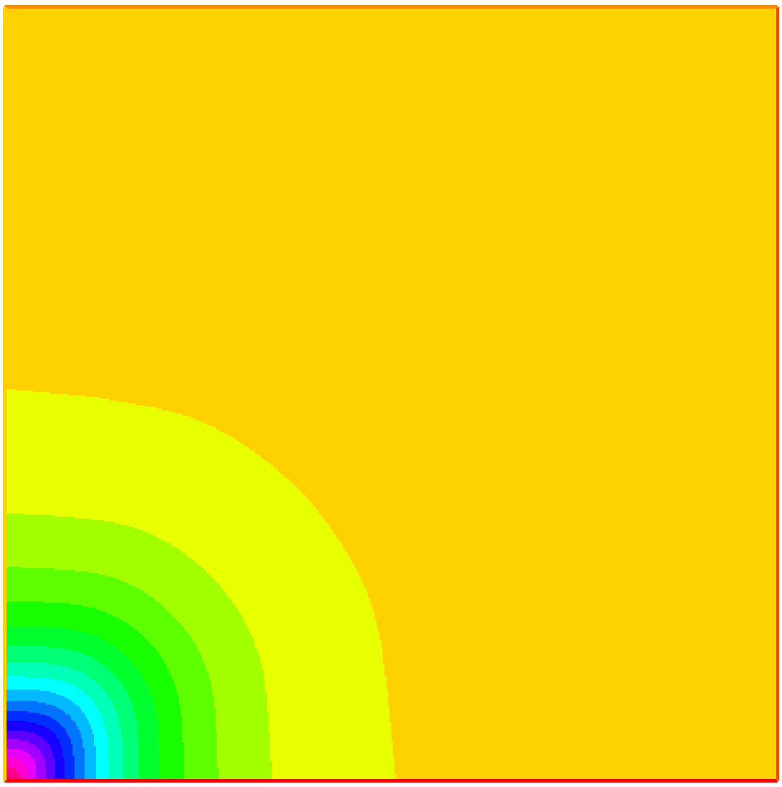

Figure 4.8: Mesh : $\sigma_x = 0.6, \sigma_y = 0.6, \rho = 0.6$

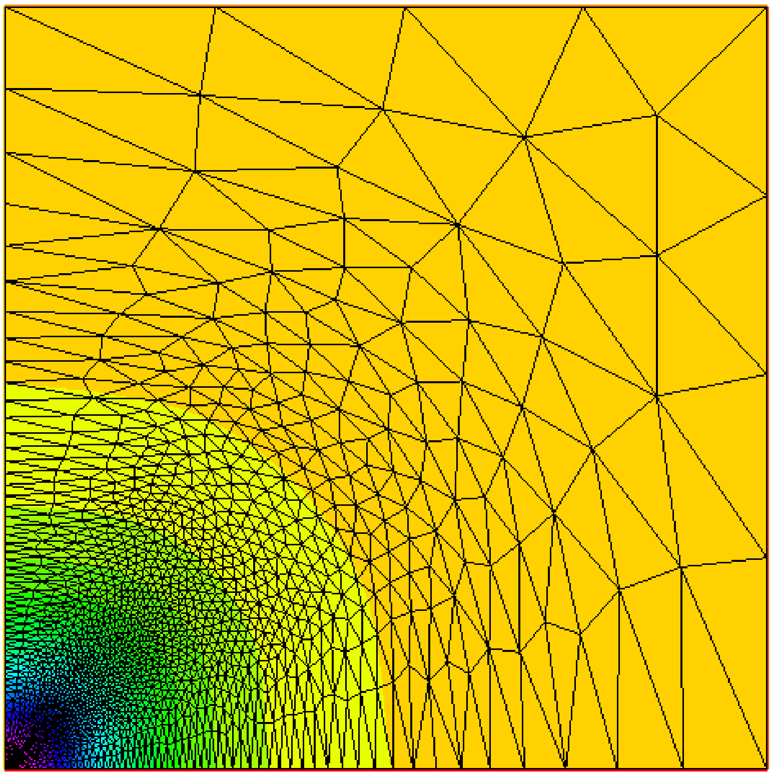

Figure 4.9: PDE : $\sigma_x = 0.6, \sigma_y = 0.6, \rho = 0.6$

4.3.2 Fell++

//////////////////////////////

Before we considered only European options, where we have only one possibility for execution , which is T. Now, we will take a look on American options, where we can execute it at any $t \leq T$. The Black-Scholes formula for a call becomes:

$$C = \mathbb{E}[e^{-r(T-t)}max(S_t - K, 0)] = \mathbb{E}[e^{-r(T-t)}max(S_0 e^{(r-d-\frac{\sigma^2}{2})t+\sigma B_t} - K, 0)] \tag{4.15}$$

The SDE for Black-Scholes is given by:

$$\frac{\partial V(S(t),t)}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V(S(t),t)}{\partial S(t)^2} + rS(t)\frac{\partial V(S(t),t)}{\partial S(t)} - rV(S(t),t) = 0 \tag{4.16}$$

$$V(S(t),0) = (S(t) - K, 0) \tag{4.17}$$