Scientific Computing

Understanding and solving stochastic PDE

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What is an option ?

An option is a contract that gives the buyer the right, but not the obligation, to buy or sell an underlying asset (a stock, a bond, gold, other option) at a specific price, called Strike price, on or before a certain date, called maturity. An option is a security, just as stocks or bonds, it has its own price called premium.

What are option parameters?

Every option contract has several parameters to be pre-set at $t = T_0$:

- What is the underlying asset ?
- What is the maturity T of the contract ?
- Does the contract give the right to buy (**call option**) or to sell(**put option**) ?
- What is the Strike price K ?
- What is the price of the option itself, i.e. premium?

Options types

What right is proposed ?

- The right to buy – **Call** option
- The right to sell – **Put** option

Who are you in this contract?

- You buy the option – **long** position
- You sell the option – **short** position

Long Call Payoff

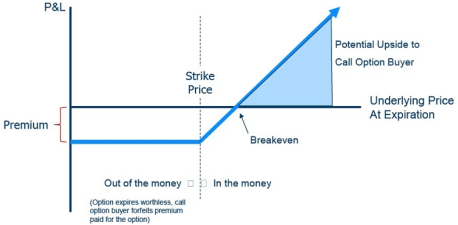

Figure : Long Call Option

As it comes from the figure, Long Call Option brings profit (*in the money*), if S at maturity T is higher than the Breakeven point. So the payoff of this option is: (premium ignored):

$$Payoff = \begin{cases} S - K & \text{if } S(T) > K \\ 0 & \text{if not} \end{cases}$$

Vanilla VS Exotics

Vanilla Option

At time $T0$ you fix the maturity date T and the Strike price K. At time T you decide whenever you want or not execute your option.

Exotic Option

- American option (Bermudian)
- Barrier option (Paris)
- Asian option
- Lookback option (Russian), etc

More closely on Exotics

- **American option** can be executed not only at T,but on any time of life of the option (T_0, T); **Bermudian option** can be executed on a specific period during the life of the option, i.e. every second Monday, June, etc.
- **Barrier option** can be activated for be executed only if the asset price touches (or not) a specific barrier; **Paris barrier option** can be activated for be executed only if the asset price satisfy the barrier condition for a certain period of time (i.e. 15min, 1 day, 30
- **Asian option**, its payoff is determined by the average underlying price over some pre-set period of time.
- **Lookback option**, its payoff depends not at S at final time T, $S(T)$, but on $max(S)$ over the life of the option. **Russian lookback** option is a special case of lookback : it has no pre-set expiration time, it's up to buyer of the option when to execute it. It's also called 'no regret' option.

Put- Call Parity

The put–call parity defines a relationship between the price of a European call option and European put option, both with the identical strike price and maturity.
We consider two portfolios:

- one European call (C) and cash (Ke^{-rT})
- one European put (P) and one share (S_0)

At time T they both worth $max(S_T, K)$, hence their values should be equal today, i.e.:

$$C + Ke^{-rT} = P + S_0 \tag{1}$$

Notations

- S - Stock price, also called Spot price (or any underlying asset)
- $V(S,t)$ - value of an option, depending on time and spot price
- K - Strike price
- r - risk-free rate
- d - dividend yield
- μ - drift rate of S - the rate at which the average of S changes
- σ - volatility of the stock, standard deviation of $log(S)$ - return on stock
- T_0, T - initial and final time
- θ - long variance : as t tends to infinity, the expected value of ν tends to θ
- κ - the rate at which ν reverts to θ
- ξ - the volatility of volatility

Diffusion models

Diffusion models have only a diffusion component, given by Brownian motion* :

- **Black-Scholes**
  W, and consequently its increment dW,represents the only source of 'diffusion' uncertainty in the price history of the stock :
  $dS = \mu S dt + \sigma S dW$
- **Heston**
  W, and consequently its increment dW, and ν, and consequently root from it, represent two sources of 'diffusion' uncertainty in the price history of the stock :
  $dS = \mu S dt + \sqrt{\nu} S dW$
  $d\nu = \kappa(\theta - \nu)dt + \xi\sqrt{\nu} dW$

Jump - Diffusion models

Jump diffusion models have both diffusion component, given by Brownian motion*, and jump component, given by compounded Poisson process.

- **Merton**
  dW represents the source of 'diffusion' uncertainty and the last term represents is the source of 'jump' uncertainty (compound Poisson process with Gaussian jumps) in the price history of the stock.
  It mixes Black-Scholes model Compounded Poisson process :
  $dS = \mu S dt + \sigma S dW + \sum_{n=1}^{N_t} Y_i$
- **Bates**
  dW and ν represent the source of 'diffusion' uncertainty and the last term represents is the source of 'jump' uncertainty (compound Poisson process with Gaussian jumps) in the price history of the stock.
  It mixes Black-Scholes model Compounded Poisson process : It mixes Merton and Heston models:
  $dS = \mu S dt + \sqrt{\nu} S dW + \sum_{n=1}^{N_t} Y_i$
  $d\nu = \kappa(\theta - \nu)dt + \xi\sqrt{\nu}dW$

Jump - Diffusion models

Remarque: For financial applications, it is of little interest to have a process with a single possible jump size. The compound Poisson process is a generalization where the waiting times between jumps are exponential but the jump sizes can have an arbitrary distribution.

Pure Jump models

- CGMY
- Normal Inverse Gaussian
- Meixner
- etc..

Rq: We will see further that actually we work with Geometric Brownian motion

Greeks

Greeks are the quantities representing the sensitivity of the price of options to a change in underlying parameters:

- **Delta** - the rate of change of the option price w.r.t. the price of the underlying asset:

$$\delta = \frac{\partial C}{\partial S} \tag{2}$$

- **Theta** - the rate of change of the portfolio price w.r.t. the time:

$$\theta = \frac{\partial \Pi}{\partial t} \tag{3}$$

Greeks

- **Gamma** - the rate of change of the portfolio w.r.t. the price of the underlying asset:

$$\Gamma = \frac{\partial^2 \Pi}{\partial S^2} \tag{4}$$

- **Vega** - the rate of change of the portfolio price w.r.t. the volatility of the underlying asset:

$$\upsilon = \frac{\partial \Pi}{\partial \sigma} \tag{5}$$

- **Rho** - the rate of change of the portfolio price w.r.t. the interest rate:

$$\upsilon = \frac{\partial \Pi}{\partial r} \tag{6}$$

What is a Wiener process

Wiener Process (Browinan Motion) is a stochastic process that lives in family of:

- Lévy processes (martingale)
  - X is adapted to $\mathcal{F}$-filtration
  - $\mathbb{E}[|X_t|] < \inf \forall \in \mathcal{T}$
  - $\mathbb{E}[X_t|\mathcal{F}_s] = X_s \forall t \geq s$
- Markov processes ("memoryless")
  - $\mathbb{P}(X_{t+1} \in \mathcal{A}\ \mathcal{F}_t) = \mathbb{P}(X_{t+1} \in \mathcal{A}\ \mathcal{X}_t)$

Filtration $\mathcal{F}$ on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ is a sequence of σ-algebras $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq ... \subseteq \mathcal{F}_T \subseteq \mathcal{A}$.
A stochastic process $(X_t)_{t \in \mathcal{T}}$ is sais to be **adapted to a filtration** $\mathcal{F}$ if for all $t \in \mathcal{T}$, the random variable X_t is $\mathcal{F}_t$ - measurable.

Ito's integral

The Ito integral is the central concept of Itô calculus.

$$Y_t = \int_0^t X_s dB_s, \tag{7}$$

where X is a locally square integrable process adapted to the filtration generated by B, which is a Brownian motion.
The result of the integration is another stochastic process.

Ito's lemma

Ito's formula transforms an SDE that is hard to solve (integrate) into another SDE that can be solved, and then transforming the solution back into a solution for the original equation.
Procedure:

1. Let X_t be a process given by the SDE: $dX_t = udt + vdB_t$
2. Let $f(t,x) \in C^2$ and $Y_t = f(t, X_t)$
3. So we have to solve the following equation:

$$dY_t = \frac{\partial f}{\partial t}(t, X_t)dX_t + \frac{\partial f}{\partial x}(t, X_t)dX_t + \frac{1\partial^2 f}{2\partial x^2}(t, X_t)(dX_t)^2 \tag{8}$$

4. When solving, keep in mind :

$$dt * dt = dt * dB_t = dB_t * dt = 0, dB_t * dB_t = dt \tag{9}$$

Theoretical assumptions

The Black-Scholes model is one of the most important concepts in modern option pricing theory. Before introducing the underlying idea, we list the model's assumptions.

- The stock price follows the Geometric Brownian motion with μ and σ constant.
- The short selling of securities with full use of proceeds is permitted.(*No time limits or conditions on transaction*).
- No transaction costs or taxes. All securities are perfectly divisible.
- There are no dividends during the life of derivative. No arbitrage.
- Security trading is continuous.
- Risk-free rate is constant and the same of all maturities.

Geometric Brownian Motion

Geometric Brownian Motion - a continuous-time stochastic process in which the logarithm of the randomly varying quantity follows a Brownian motion with drift.

So why to use GBM and not the Brownian motion itself?

- The expected returns of GBM are independent of the value of the process (stock price), which agrees with what we would expect in reality.
- A GBM process only assumes positive values, just like real stock prices.
- A GBM process shows the same kind of 'roughness' in its paths as we see in real stock prices.
- Calculations with GBM processes are relatively easy.

Solve the PDE with Ito's lemma

We solve $dX_t = \mu X_t dt + \sigma X_t dB_t$, $X_0 > 0$, with $f(t,x) = \ln x$ and $Y_t = \ln X_t$.

- By Ito's lemma we get:

$$dY_t = \frac{1}{X_t} dX_t + \frac{1}{2}(-\frac{1}{X_t^2})(dX_t)^2 = \mu dt + \sigma dB_t - \frac{1}{2}\frac{1}{X_t^2}\sigma^2 X_t^2 dt = (\mu - \frac{1}{2}\sigma^2) \tag{10}$$

- Applying the integration by parts:

$$Y_t - Y_0 = \int_0^T dY_t = (\mu - \frac{1}{2}\sigma^2)\int_0^T dt + \sigma \int_0^t dB_t = (\mu - \frac{1}{2}\sigma^2)T + \sigma B_t \tag{11}$$

- So, the final solution is:

$$X_T = e^{Y_T} = e^{Y_0 + (\mu - \frac{1}{2}\sigma^2)T + \sigma dB_t} = X_0 e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma dB_t} \tag{12}$$

Pricing formulas for options

Lets find the discounted expectation $e^{-rT}\mathbb{E}[max(S_T - K, 0)]$ for a Vanilla Call option C.

Please note, that we need to take in account that $\mu = r - d$, where d corresponds to a dividend yield.

Using notations

- $d_1 = \frac{\ln(\frac{S_0}{K}) + (r - d + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$
- $d_2 = \frac{\ln(\frac{S_0}{K}) + (r - d - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$

we try to solve $C = e^{-rT}\mathbb{E}[max(S_0 e^{(r-d-\frac{\sigma^2}{2})T + \sigma B_T} - K, 0)]$

Pricing formulas for options

$$
\begin{aligned}
C &= e^{-rT}\mathbb{E}[max(S_0 e^{(r-d-\frac{\sigma^2}{2})T+\sigma B_T} - K, 0)] \\
&= \frac{1}{\sqrt{2\pi}} \int_{\frac{\ln(\frac{K}{S_0})-(r-d-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}}^{\infty} (S_0 e^{\sigma\sqrt{T}x-(d+\frac{\sigma^2}{2}T)} - e^{-rT}K) e^{-\frac{x^2}{2}} dx \\
&= e^{-dT} S_0 \frac{1}{\sqrt{2\pi}} \int_{-d_2}^{\infty} e^{-\frac{(x-\sigma\sqrt{T})^2}{2}} dx - e^{-rT} K \frac{1}{\sqrt{2\pi}} \int_{-d_2}^{\infty} e^{-\frac{x^2}{2}} dx \\
&= e^{-dT} S_0 (1 - \mathcal{N}(-d_2 - \sigma\sqrt{T})) - e^{-rT} K (1 - \mathcal{N}(-d_2)) \\
&= e^{-dT} S_0 \mathcal{N}(d_1) - e^{-rT} K \mathcal{N}(d_2))
\end{aligned}
$$

With similar computation we get a formula for Put option:
$P = e^{-rT} K \mathcal{N}(-d_2)) - e^{-dT} S_0 \mathcal{N}(-d_1)$