Financial Computing: Black-Scholes PDE

Solving with Feel++

Plan

- Finances: Overview
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- Probabilities : Overview
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Finance : Options

What is an option ?

An option is a contract that gives the buyer the right, but not the obligation, to buy or sell an underlying asset (a stock, a bond, gold, other option) at a specific price, called *Strike price*, on or before a certain date, called *maturity*. An option is a security, just as stocks or bonds, it has its own price called premium.

Finance : Options

Every option contract has several parameters to be pre-set :

- What is the underlying asset ?
- What is the maturity T of the contract ?
- Does the contract give you the right to buy (call option) or to sell (put option) ?
- What is the Strike price K ?
- What is the price of the option itself ? (Premium)* we'll see further that premium is always ignored

Finance : Options

What right is proposed ?

- The right to buy – **Call** option
- The right to sell – **Put** option

Who are you in this contract ?

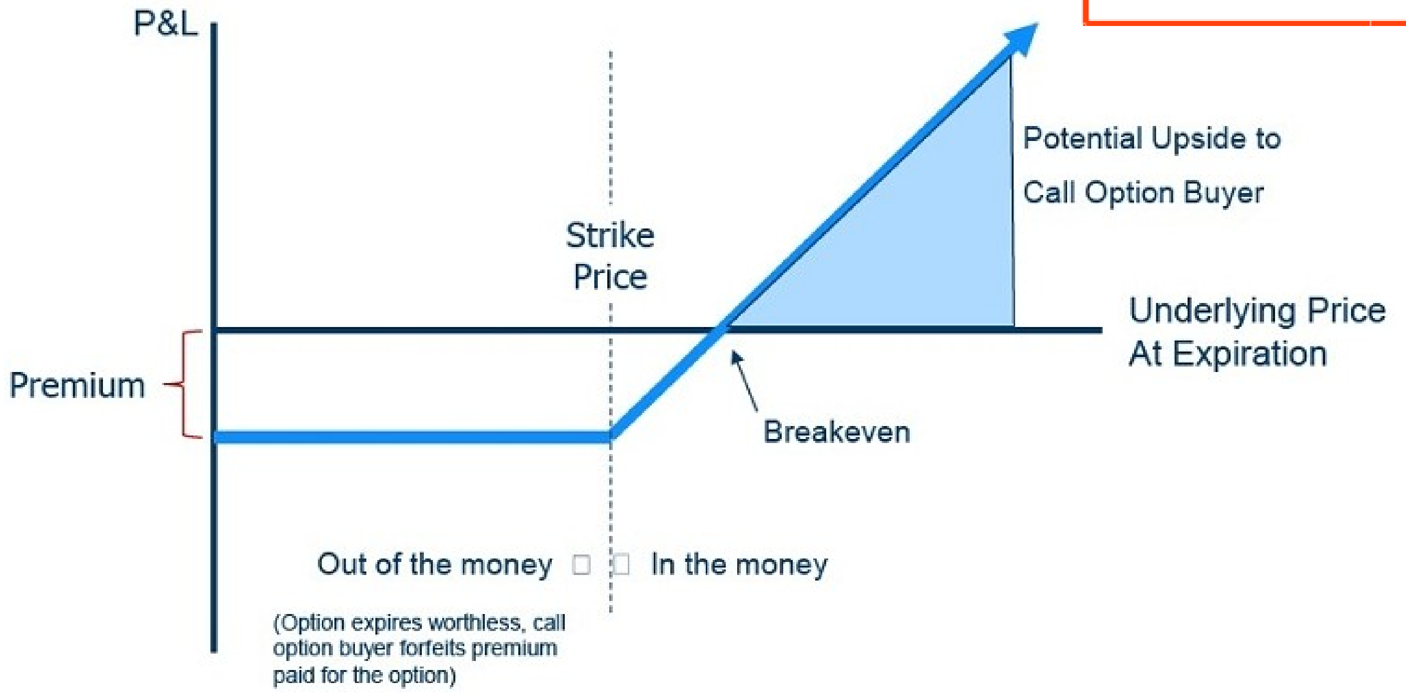
- You buy the option – **long** position
- You sell the option – **short** position

Finance : Pay-offs

Payoff for Long Call Option

With S – price of the underlying asset, K – Strike price, we have (ignore the premium):

- If S(T)>K then your gain is S-K
- If S(T)<K then your gain is 0 (not negative !!)



As it comes from the plot – Long Call Option brings profit (« in the money ») , if S at maturity T is higher than the breakeven point.

Finance : Vanilla VS Exotics

Vanilla Option :

At time T0 you fixe the maturity date T and the Strike price K. At time T you decide whenever you want or not execute your option .

Exotic Options :

- American(Bermudian) Option
- Barrier(Paris) Option
- Asian Option
- Look Back(Russian) Option
  Etc, etc, etc …

Finance : Exotic Options

American(sub : Bermudian) option – can be executed not only at T, but on any time (T0,T).

If **Berdumian** type – then on a specific period(s) of (T0,T), i.e. every second Monday.

Asian option – its payoff is determined by the average underlying price over some pre-set period of time.

Barrier*(sub : Paris) option – can be executed only if the asset price touchs (or not) a specific barrier.

If **Paris** type – then the asset price must satisfy the barrier condition for a certain period of time (i.e. 15min, 1 day, 30%).

Look-Back(sub : Russian) option – its payoff depends not at S at T, but on max(S) over the life of the option.

If **Russian** type – no expiration date, no pre-set T, so you can execute the option when you want .

Finance : Exotic Barrier Options

The **up-and-in barrier** call option is a standard European call option with strike K when its maximum lies above the barrier H, while it is worthless otherwise.

The **up-and-out barrier** call option is a standard European call option with strike K when its maximum lies below the barrier H, while it is worthless otherwise.

The **down-and-in barrier** call option is a standard European call option with strike K when its minimum lies below the barrier H, while it is worthless otherwis

The **down-and-out barrier** call option is a standard European call with strike K when its lies above some barrier H, while it is worthless otherwise

Finances : Pricing Models

General notatons and definitions :

- S – stock price
- V(S,T) – price of the option
- K – strike price
- r – risk-free interest rate
- mu – drift rate of S*
- sigma – volatility of the stock, standard deviation of log(S)
- T – time

Drift rate - the rate at which the average of S changes.

Finance : Pricing models

Diffusion models have only a diffusion component, given by Wiener process :
-Black-Scholes
-Heston

Jump diffusion models have both diffusion component, given by Wiener process, and jump component, given by compounded Poisson process :
-Merton
-Bates

Pure jump models have no diffusion, just a random process :
-CGMY
-NIG

Finance : Diffusion models

Black – Scholes

W, and consequently its increment dW, represents the only source of 'diffusion' uncertainty in the price history of the stock :

$$dS = \mu S dt + \sigma S dW$$

Heston

W, and consequently its increment dW, and nu, and consequently root from it, represent two sources of 'diffusion' uncertainty in the price history of the stock :

$$dS = \mu S dt + \sqrt{\nu} S dW$$

$$d\nu = \kappa(\theta - \nu)dt + \xi\sqrt{\nu} dW$$

- μ is the rate of return of the asset.
- θ is the **long variance**, or long run average price variance; as t tends to infinity, the expected value of ν_t tends to θ.
- κ is the rate at which ν_t reverts to θ.
- ξ is the volatility of the volatility, or **vol of vol**, and determines the variance of ν_t.

Finance: Black – Scholes model

Assumptions:

- The stock price follows the Geometric Brownian motion with mu and sigma constant.
- The short selling of securities with full use of proceeds is permitted.(No time limits or conditions on transaction).
- No transaction costs or taxes. All securities are perfectly divisible.
- There are no dividends during the life of derivative.No arbitrage.
- Security trading is continuous.
- Risk-free rate is constant and the same of all maturities.

Finance: Black – Scholes model

Geometric Brownian Motion

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

h

$$\frac{dS}{S} = \mu\, dt + \sigma\, dW$$

Why?

- The expected returns of GBM are independent of the value of the process (stock price), which agrees with what we would expect in reality.
- A GBM process only assumes positive values, just like real stock prices.
- A GBM process shows the same kind of 'roughness' in its paths as we see in real stock prices.
- Calculations with GBM processes are relatively easy.

Finance: Black – Scholes formulas

$\mu = r - d$, where d corresponds to a dividend yield.

$$C = e^{-rT}\mathbb{E}[max(S_0 e^{(r-d-\frac{\sigma^2}{2})T+\sigma B_T} - K, 0)]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\frac{\ln(\frac{K}{S_0})-(r-d-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}}^{\infty} (S_0 e^{\sigma\sqrt{T}x-(d+\frac{\sigma^2}{2}T)} - e^{-rT}K)e^{-\frac{x^2}{2}}dx$$

$$= e^{-dT}S_0\frac{1}{\sqrt{2\pi}}\int_{-d_2}^{\infty} e^{-\frac{(x-\sigma\sqrt{T})^2}{2}}dx - e^{-rT}K\frac{1}{\sqrt{2\pi}}\int_{-d_2}^{\infty} e^{-\frac{x^2}{2}}dx$$

$$= e^{-dT}S_0(1-\mathcal{N}(-d_2-\sigma\sqrt{T})) - e^{-rT}K(1-\mathcal{N}(-d_2))$$

$$= e^{-dT}S_0\mathcal{N}(d_1) - e^{-rT}K\mathcal{N}(d_2))$$

- $d_1 = \ln(\frac{S_0}{K}) + (r - d + \frac{\sigma^2}{2})T\sigma\sqrt{T}$
- $d_2 = \ln(\frac{S_0}{K}) + (r - d - \frac{\sigma^2}{2})T\sigma\sqrt{T} = d_1 - \sigma\sqrt{T}$

$$P = e^{-rT}K\mathcal{N}(-d_2)) - e^{-dT}S_0\mathcal{N}(-d_1)$$

Finance : Jump –Diffusion Models

Merton

dW represents the source of 'diffusion' uncertainty and the last term represents is the source of 'jump' uncertainty (compound Poisson process with Gaussian jumps) in the price history of the stock .

It mixes Black-Scholes model and Compounded Poisson process :

$$dS = \mu S dt + \sigma S dW + \sum_{n=1}^{N_t} Y_i$$

Bates

dW and nu represent the source of 'diffusion' uncertainty and the last term represents is the source of 'jump' uncertainty (compound Poisson process with Gaussian jumps) in the price history of the stock.

It mixes Merton and Heston models :

$$dS = \mu S dt + \sqrt{\nu} S dW + \sum_{n=1}^{N_t} Y_i$$

$$d\nu = \kappa(\theta - \nu)dt + \xi\sqrt{\nu}dW$$

Rq : For financial applications, it is of little interest to have a process with a single possible jump size. The compound Poisson process is a generalization where the waiting times between jumps are exponential but the jump sizes can have an arbitrary distribution. More precisely, let N be a Poisson process with parameter λ and { Y i } i ≥ 1 be a sequence of independent random variables with law f. The process is called compound Poisson process. Its trajectories piecewise constant but the jump sizes are now random with law f. The compound Poisson process has independent and stationary increments.

Finance : Pure Jump Models

Normal Inverse Gaussian process
CGMY
Generalized Hyperbolic process
Meixner process

(I used to see these processes through characteristic function, I will finish this part after)

Finance : Greeks

The Greeks are the quantities representing the sensitivity of the price of options to a change in underlying parameters :

Delta measures the rate of change of the theoretical option value with respect to changes in the underlying asset's price. Delta is the first derivative of the value V of the option with respect to the underlying instrument's price S.

Vega measures sensitivity to volatility. Vega is the derivative of the option value with respect to the volatility of the underlying asset.

Theta measures the sensitivity of the value of the derivative to the passage of time : the "time decay."

Rho measures sensitivity to the interest rate: it is the derivative of the option value with respect to the risk free interest rate (for the relevant outstanding term).

Probabilities: Overview

- Martingales
- Filtrations
- Stochastic Process
- Itô's calculus:
  - extends the methods of calculus to stochastic processes such as Brownian motion
  - its central concept is the Itô stochastic integral, that gives another stochastic process as the result of integration

FEM : Artificial Boundaries

Semi-infinite domain -> Localization technique (truncate infinity at L)

Singularity at x=0 -> Add a condition

$$\frac{\partial u}{\partial t} - ru = 0, x = 0$$

Bo ditions are:

1) $u(0,t) = u(0,T)e^{r(t-T)}$

2) $u(L,t) = \phi(L)e^{r(t-T)}$

3)

FEM: Weak Formulation

1. Multiply and integrate

$$\frac{d}{dt}\left(\int_{\mathbb{R}+} u(S,t)w(S)dS\right) + a_t(v,w) = 0,$$

$$a_t(v,w) = \int_{\mathbb{R}+} \left(\frac{1}{2}S^2\sigma^2(S,t)\frac{\partial v}{\partial S}\frac{\partial w}{\partial S} + r(t)vw\right)ds + \int_{\mathbb{R}+} \left(-r(t) + \sigma^2(S,t) + S\sigma(S,t)\frac{\partial \sigma}{\partial S}(S,t)\right)S\frac{\partial v}{\partial S}wdS.$$

2. Unicity

- As the volatility is always positive and bounded, we can conclude that a_t is continuous on V, càd there exists a positive constant M, such that for all $v, w \in V$,

$$|a_t(v,w)| \leq M|v|_V |w|_V. \tag{16}$$

- To prove the coercivity we use **Gârding's inequity**:

$$a_t(v,v) \geq C_1||v||_V^2 - C_2||v||_{L^2}^2. \tag{17}$$

FEM: Weak Formulation

Weak Formulation

$$\begin{gathered}\text{Find } u \in \mathcal{C}^0([0;T]), u \in L^2 \cap V \\ \text{and } u|_{t=0} = u_0 \text{ and for all } t \in (0,T), \\ \forall v \in V, (\tfrac{\partial u}{\partial t}(t), v) + a_t(u(t), v) = 0.\end{gathered}$$

$$a_t(v,w) = \int_{\mathbb{R}+} (\frac{1}{2} S^2 \sigma^2(S,t) \frac{\partial v}{\partial S} \frac{\partial w}{\partial S} + r(t)vw) ds + \int_{\mathbb{R}+} (-r(t) + \sigma^2(S,t) + S\sigma(S,t) \frac{\partial \sigma}{\partial S}(S,t)) S \frac{\partial v}{\partial S} w dS.$$