1. Financial Market : basics

Computational finance is an interdisciplinary field which joins financial mathematics, stochastics, numerics and scientific computing. Its task is to estimate as accurately and efficiently as possible the risks that financial instruments generate. In this project we will focus on options.

1.1. Option - financial instrument

An **Option** is a contract that gives the buyer the right, *but not the obligation*, to buy or sell an underlying asset (a stock, a bond, gold, other option, etc) at a specific price, called **Strike** price, on or before a certain date. An option is a security, just as stockes or bonds, it has its own price called **Premium**.

Every option contract has several parameters to be pre-set at $t = T_0$:

- What is the underlying asset ?
- What is the maturity T of the contract ?
- Does the contract give the right to buy (**call option**) or to sell(**put option**) ?
- What is the Strike price K ?
- What is the price of the option itself, i.e. premium?
  Rq: note that premium is ignored in following computations

Obviously, the asymmetry of the option contract leads us to the question if we buy this contract it or we sell it. A buyer of an option contract is said to be in **long** positions, the seller - in **short** position.

Let's take a look on potential gain diagram for Long Call Option:

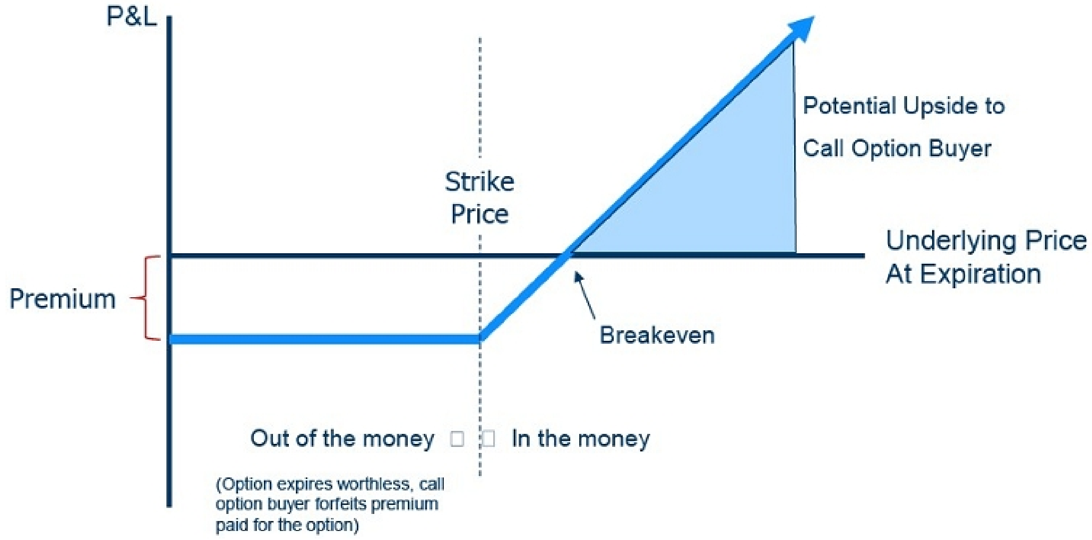

Figure 1: Long Call Option

As it comes from the figure 1, Long Call Option brings profit (« in the money ») , if S at maturity T is higher than the Breakeven point. So the payoff of this option is premium ignored):

- $S - K$ if $S(T) > K$
- 0 if $S(T) < K$

1.1.1. Vanilla vs Exotics

This type of of options is said to be **Vanilla** : the simpliest version of all, without any optional extras, by analogy with the default ice cream flavour, vanilla. There exist a rich family of, so called **Exotic** options. Here we list only several of them:

- American option (Bermudian)
  can be executed not only at T,but on any time of life of the option (T_0, T);
  Bermudian option can be executed on a specific period during the life of the option, i.e. every second Monday, June, etc.
- Barrier option (Paris)
  can be activated for be executed only if the asset price touches (or not) a specific barrier;
  Paris barrier option can be activated for be executed only if the asset price satisfy the barrier condition for a certain period of time (i.e. 15min, 1 day, 30
- Asian option
  its payoff is determined by the average underlying price over some pre-set period of time.
- Lookback option (Russian)
  its payoff depends not at S at final time T, $S(T)$, but on $max(S)$ over the life of the option.
  Russian lookback option is a special case of lookback : it has no pre-set expiration time, it's up to buyer of the option when to execute it. It's also called 'no regret' option.

In this project we will focus on Barrier options, so let's look at its payoffs more closely:

- Up-and-in barrier call option
  Standard European call option with strike K when its maximum lies above the barrier H, while it is worthless otherwise.
- Up-and-out barrier call option
  Standard European call option with strike K when its maximum lies below the barrier H, while it is worthless otherwise.
- Down-and-in barrier
  Standard European call option with strike K when its minimum lies below the barrier H,while it is worthless otherwise.
- Down-and-out barrier
  Standard European call with strike K when its lies above some barrier H, while it is worthless otherwise.

1.2. Option Pricing models

1.2.1. General notations and definitions

- S - Stock price (or any underlying asset)
- $V(S,T)$ - Value of an option, depending on time and stock price
- K - Strike price
- r - risk-free rate
- d - dividend yield
- μ - drift rate of S - the rate at which the average of S changes
- σ - volatility of the stock, standard deviation of $log(S)$ - return on stock
- T_0,T - initial and final time
- θ - long variance : as t tends to infinity, the expected value of ν tends to θ
- κ - the rate at which ν reverts to θ
- ξ - the volatility of volatility

1.2.2. Pricing models

- **Diffusion models** have only a diffusion component, given by Wiener process : *Black-Scholes, Heston*;

  - *Black-Scholes*
    W, and consequently its increment dW,represents the only source of 'diffusion' uncertainty in the price history of the stock :
    $dS = \mu S dt + \sigma S dW$
  - *Heston*
    W, and consequently its increment dW, and ν, and consequently root from it, represent two sources of 'diffusion' uncertainty in the price history of the stock :

$$dS = \mu S dt + \sqrt{\nu} S dW$$
$$d\nu = \kappa(\theta - \nu)dt + \xi\sqrt{\nu} dW$$

- **Jump diffusion models** have both diffusion component, given by Wiener process, and jump component, given by compounded Poisson process :

  Rq: For financial applications, it is of little interest to have a process with a single possible jump size. The compound Poisson process is a generalization where the waiting times between jumps are exponential but the jump sizes can have an arbitrary distribution.

  – *Merton*
    dW represents the source of 'diffusion' uncertainty and the last term represents is the source of 'jump' uncertainty (compound Poisson process with Gaussian jumps) in the price history of the stock. dW and ν represent the source of 'diffusion' uncertainty and the last term represents is the source of 'jump' uncertainty (compound Poisson process with Gaussian jumps) in the price history of the stock.
    It mixes Black-Scholes model Compounded Poisson process :
    $dS = \mu S dt + \sigma S dW + \sum_{n=1}^{N_t} Y_i$
  – *Bates*
    dW and ν represent the source of 'diffusion' uncertainty and the last term represents is the source of 'jump' uncertainty (compound Poisson process with Gaussian jumps) in the price history of the stock.
    It mixes Black-Scholes model Compounded Poisson process : It mixes Merton and Heston models:

    $$dS = \mu S dt + \sqrt{\nu} S dW + \sum_{n=1}^{N_t} Y_i$$

    $$d\nu = \kappa(\theta - \nu)dt + \xi\sqrt{\nu} dW$$
- **Pure jump models** have no diffusion, just a random process :

  – *CGMY*
  – *NIG*

1.3. Greeks

Greeks are the quantities representing the sensitivity of the price of options to a change in underlying parameters. We list some of them, futher in the project we will study them:

- **Delta** measures the rate of change of the theoretical option value with respect to changes in the underlying asset's price. **Delta** is the first derivative of the value V of the option with respect to the underlying instrument's price S.
- **Vega** measures sensitivity to volatility. **Vega** is the derivative of the option value with respect to the volatility of the underlying asset.
- **Theta** measures the sensitivity of the value of the derivative to the passage of time : the "time decay."
- **Rho** measures sensitivity to the interest rate: it is the derivative of the option value with respect to the risk free interest rate (for the relevant outstanding term).

2. Stochastics : basics

2.1. Stochastic process

A **stochastic process** in discrete time, denoted by $(X_t)_{t\in\mathcal{T}}$, is a sequence of random variables $X_0, X-1, ..X_T$ defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with values in $\mathbb{R}$. For a given $\omega \in \Omega$, the sequence $X_t(w), t \in \mathcal{T}$ is called a **path** of the process (X_t), where $\mathcal{T}$ stands for time series (dates) : $\mathcal{T} = \{0, 1, ..T\}$

2.2. Filtrations

A **filtration** $\mathcal{F}$ on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ is a sequence of σ-algebras $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq ... \subseteq \mathcal{F}_T \subseteq \mathcal{A}$. A quadruple $(\Omega, \mathcal{A}, \mathbb{P}, \mathcal{F})$ is called a **filtered probability space**.
A stochastic process $(X_t)_{t\in\mathcal{T}}$ is sais to be **adapted to a filtration** $\mathcal{F}$ if for all $t \in \mathcal{T}$, the random variable X_t is $\mathcal{F}_t$ - measurable.

2.3. Martingales

Let $(\Omega, \mathcal{A}, \mathbb{P}, \mathcal{F}))$ be a filtered probability space. A stochastic process $X = (X_t)_{t\in\mathcal{T}}$ is called a **martingale** if

i. X is adapted to $\mathcal{F}$
ii. $\mathbb{E}[|X_t|] < \inf \forall \in \mathcal{T}$
iii. $\mathbb{E}[X_t|\mathcal{F}_s] = X_s \forall t \geq s$

Note, the following brownian martingale is used in the Black-Scholes theory:

$$(e^{\alpha B_t - \frac{\alpha^2}{2}t}) \tag{1}$$

and

$$\mathbb{E}[e^{\alpha B_t - \frac{\alpha^2}{2}t}|\mathcal{F}_f = e^{\alpha B_s - \frac{\alpha^2}{2}s}, 0 \leq s < t \tag{2}$$

2.4. Itô Calculus

The **Itô integral** is the central concept of Itô calculus. The integrand and the integrator are stochastic processes:

$$Y_t = \int_0^t X_s dB_s, \tag{3}$$

where X is a locally square integrable process adapted to the filtration generated by B, which is a Wiener process or Brownian motion. The result of the integration is another process.

If we assume that $\{\pi_n\}$ is a sequence of partitions of $[0, t]$ with min mesh size, then the integration by parts for Itô integral is given by :

$\int_0^t XdB = \lim_{n\to\infty} \sum_{t_{i-1}, t_i \in \pi_n} X_{t_{i-1}}(B_{t_i} - B_{t_{i-1}})$.

X is square integrable, then its integral w.r.t. B can be defined and X is said to be B-integrable. As $\int_0^t (X - X_n)^2 ds \to 0$ in probability, the Itô integral becomes:

$\int_0^t XdB = \lim_{n\to\infty} \int_0^t X_n dB$

where we faal again on convergence in probability. This leads us to a very important property: **Itô isometry**:

$$\mathbb{E}[(\int_0^t X_s dB_s)^2] = \mathbb{E}[\int_0^t X_s^2 ds] \tag{4}$$

The most crucial notion for solving stochastic PDE's is the **Itô's formula (lemma)**.

Given a process X_t described by an SDE, the Itô formula tells us how another process $Y_t = f(t, X_t)$ that is given in terms of t and X_t is itself described by an SDE. The formula is so useful because it can be used to transform an SDE that is hard to solve (integrate) into another SDE that can be solved, and then transforming the solution back into a solution for the original equation.
Let X_t be a process given by the SDE:

$dX_t = udt + vdB_t$

Let $f(t,x) \in C^2$ and $Y_t = f(t, X_t)$. Then Y_t follows the SDE:

$$dY_t = \frac{\partial f}{\partial t}(t, X_t)dX_t + \frac{\partial f}{\partial x}(t, X_t)dX_t + \frac{1\partial^2 f}{2\partial x^2}(t, X_t)(dX_t)^2 \tag{5}$$

To calculate $(dX_t)^2$ we must keep in mind following rules:

$$dt * dt = dt * dB_t = dB_t * dt = 0, dB_t * dB_t = dt \tag{6}$$

A good way to remember Itô's formula is look at it as a Taylor's formula for a function with two variables together with the rules given above.

Let's solve the Black-Scholes SDE:

$$dX_t = \mu X_t dt + \sigma X_t dB_t, X_0 > 0 \tag{7}$$

We take $f(t,x) = \ln x$ and $Y_t = \ln X_t$ and with **Itô's lemma** we get:

$$dY_t = \frac{1}{X_t}dX_t + \frac{1}{2}(-\frac{1}{X_t^2})(dX_t)^2 = \mu dt + \sigma dB_t - \frac{1}{2}\frac{1}{X_t^2}\sigma^2 X_t^2 dt = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dB_t. \tag{8}$$

Now lets integrate it:

$Y_t - Y_0 = \int_0^T dY_t = (\mu - \frac{1}{2}\sigma^2)\int_0^T dt + \sigma \int_0^t dB_t = (\mu - \frac{1}{2}\sigma^2)T + \sigma B_t$

And we finally get the solution:

$$X_T = e^{Y_T} = e^{Y_0 + (\mu - \frac{1}{2}\sigma^2)T + \sigma dB_t} = X_0 e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma dB_t} \tag{9}$$

3. Looking closer to Black-Scholes model

3.1. Theoretical assumptions

The Black-Scholes model is one of the most important concepts in modern option pricing theory. Before introducing the underlying idea, we list the model's assumptions.

- The stock price follows the Geometric Brownian motion with μ and σ constant.
- The short selling of securities with full use of proceeds is permitted.(*No time limits or conditions on transaction*).
- No transaction costs or taxes. All securities are perfectly divisible.
- There are no dividends during the life of derivative. No arbitrage.
- Security trading is continuous.
- Risk-free rate is constant and the same of all maturities.

3.2. Geometric Brownian Motion

The Black-Scholes SDE describing the process for a stock price S_t is given by:

$$dS_t = \mu S_t dt + \sigma S_t dB_t \tag{10}$$

This process is called **Geometric Brownian Motion** - a continuous-time stochastic process in which the logarithm of the randomly varying quantity follows a Brownian motion with drift.
We have already seen that its solution is given by:

$$S_T = S_0 e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma d B_t} \tag{11}$$

where the stock price S_t is log-normally distributed.
Recall : μ is a drift rate of the diffusion and σ is the volatility.
So why to use GBM and not the Brownian motion itself?

- The expected returns of GBM are independent of the value of the process (stock price), which agrees with what we would expect in reality.
- A GBM process only assumes positive values, just like real stock prices.
- A GBM process shows the same kind of 'roughness' in its paths as we see in real stock prices.
- Calculations with GBM processes are relatively easy.

However, GBM has no jumps in it and the volatility is constant, which is not the case for a stock price. That's why we will take a look on Heston and Merton models.
For further computations we need to find mean and variance.Lets first find $\mathbb{E}[S_t]$ using martingales theory:

$$\mathbb{E}[S_t] = S_0 e^{\mu T} \mathbb{E}[e^{\sigma B_t - \frac{1}{2}\sigma^2 T}] = S_0 e^{T}. \tag{12}$$

since $\mathbb{E}[e^{\sigma B_T - \frac{1}{2}\sigma^2 T}] = e^{\sigma B_0} = 1$.
Now, lets find $\mathbb{E}[S_t]$ with moment generating function.We know that for a normally distributed variable $X \sim \mathcal{N}(\mu, \sigma^2)$ the moment generating function is given by $M_X(t) := \mathbb{E}[e^{tX}] = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$.
When we apply this to:

$X = \sigma B_T - \frac{1}{2}\sigma^2 T \sim \mathcal{N}(-\frac{1}{2}\sigma^2 T, \sigma^2 T)$

we get:

- $\mathbb{E}[e^X] = M_X(1) = e^{-\frac{1}{2}\sigma^2 T + \frac{1}{2}\sigma T} = 1$
- $\mathbb{E}[(e^X)^2] = \mathbb{E}[e^{2X}] = M_X(2) = e^{-\sigma^2 T + 2\sigma T}$

For the second moment we get:

$\mathbb{E}[S_t^2] = S_0^2 e^{2\mu T} \mathbb{E}[e^{2X}] = S_0^2 e^{2\mu T} e^{\sigma^2 T}$

And finally the variance:

$\mathbb{V}[S_T] = \mathbb{E}[S_T^2] - (\mathbb{E}[S_T])^2 = S_0^2 e^{2\mu T}(e^{\sigma^2 T} - 1)$

3.3. Black-Scholes formulas

Lets find the discounted expectation $e^{-rT}\mathbb{E}[max(S_T - K, 0)]$ for a Vanilla Call option C. Please note, that we need to take in account that $\mu = r - d$, where d corresponds to a dividend yield.

$C = e^{-rT}\mathbb{E}[max(S_0 e^{(r-d-\frac{\sigma^2}{2})T + \sigma B_T} - K, 0)]$

$= \frac{1}{\sqrt{2\pi}} \int_{\frac{\ln(\frac{K}{S_0}) - (r-d-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}}^{\infty} (S_0 e^{\sigma\sqrt{T}x - (d + \frac{\sigma^2}{2}T)} - e^{-rT}K) e^{-\frac{x^2}{2}} dx$

$= e^{-dT} S_0 \frac{1}{\sqrt{2\pi}} \int_{-d_2}^{\infty} e^{-\frac{(x-\sigma\sqrt{T})^2}{2}} dx - e^{-rT} K \frac{1}{\sqrt{2\pi}} \int_{-d_2}^{\infty} e^{-\frac{x^2}{2}} dx$

$= e^{-dT} S_0 (1 - \mathcal{N}(-d_2 - \sigma\sqrt{T})) - e^{-rT} K (1 - \mathcal{N}(-d_2))$

$= e^{-dT} S_0 \mathcal{N}(d_1) - e^{-rT} K \mathcal{N}(d_2))$
with

- $d_1 = \ln(\frac{S_0}{K}) + (r - d + \frac{\sigma^2}{2}) T \sigma\sqrt{T}$

- $d_2 = \ln(\frac{S_0}{K}) + (r - d - \frac{\sigma^2}{2})T\sigma\sqrt{T} = d_1 - \sigma\sqrt{T}$

With similar computation we get a formula for Put option:

$P = e^{-rT}K\mathcal{N}(-d_2)) - e^{-dT}S_0\mathcal{N}(-d_1)$