Scientific Computing

Understanding and solving stochastic PDE

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Notations

- S - Stock price, also called Spot price (or any underlying asset)
- $V(S,t)$ - value of an option, depending on time and spot price
- K - Strike price
- r - risk-free rate
- d - dividend yield
- μ - drift rate of S - the rate at which the average of S changes
- σ - volatility of the stock, standard deviation of $log(S)$ - return on stock
- T_0, T - initial and final time
- θ - long variance : as t tends to infinity, the expected value of ν tends to θ
- κ - the rate at which ν reverts to θ
- ξ - the volatility of volatility

Financial mathematics

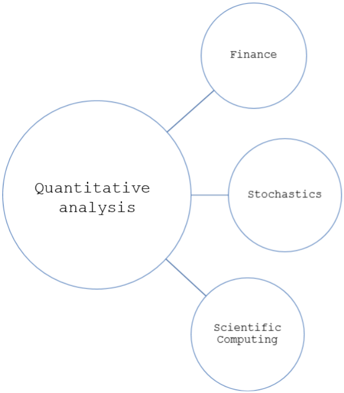


Figure : Financial mathematics

What is an option ?

An option - a contract that gives the buyer the right, **but not the obligation**, to buy or sell an underlying asset (a stock, a bond, gold, other option) at a specific price, called Strike price, on a certain date, called maturity.

What right is proposed ?

- The right to buy – **Call** option
- The right to sell – **Put** option

What are option parameters?

Parameters to fix at t_0 :

- Who buys (long), who sells (short)?
- What is the underlying asset ?
- What is the maturity T of the contract ?
- Does the contract give the right to buy (**call option**) or to sell(**put option**) ?
- What is the Strike price K ?
- What is the price of the option itself, i.e. premium?

Long Call Payoff

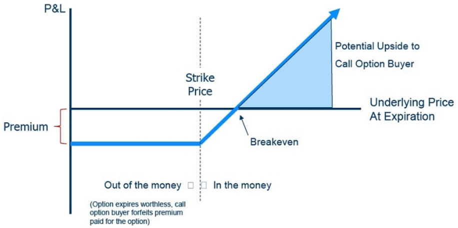


Figure : Options Payoffs

Buying Call Option brings profit (*in the money*), if S at maturity T is higher than the Breakeven point. So the payoff of this option is: (premium ignored):

$$Payoff = \begin{cases} S - K & \text{if } S(T) > K \\ 0 & \text{if not} \end{cases}$$

Other Payoffs

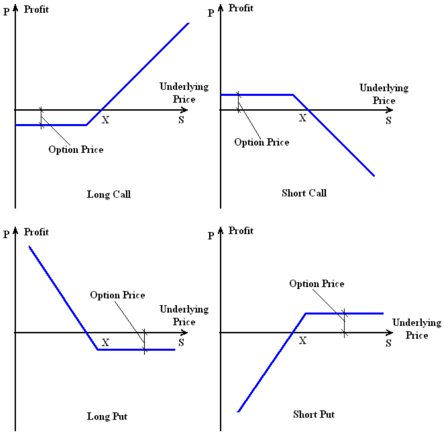


Figure : Other Payoffs

Vanilla VS Exotics

Vanilla Option

- European option

Exotic Option

- American option (Bermudian)
- Barrier option (Paris)
- Asian option
- Lookback option (Russian)
- Binary option
- Cliquer option
- etc, etc..

Vanilla put

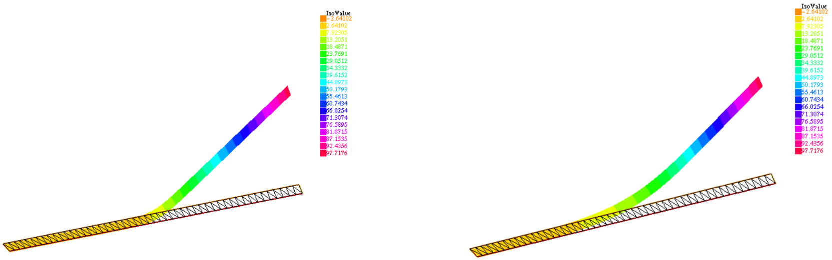


Figure : $\sigma = 0.1$ and $\sigma = 0.3$

Barrier put : 30, 90, 100 with K=100

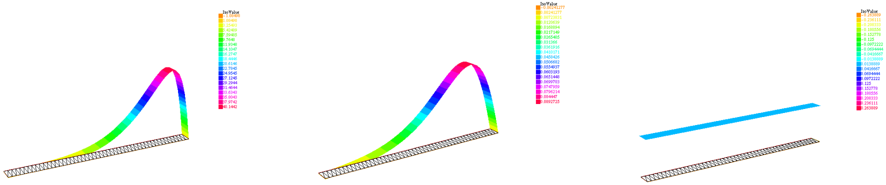


Figure : barriers $= 30, 90, 100$

Asian put

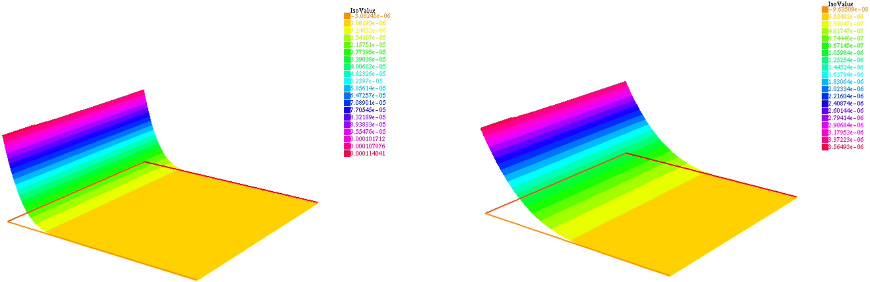


Figure : $\mu = 0.1$ and $\mu = 0.3$

Greeks

Delta - the rate of change of the option price w.r.t. the price of the underlying asset: $\delta = \frac{\partial C}{\partial S}$

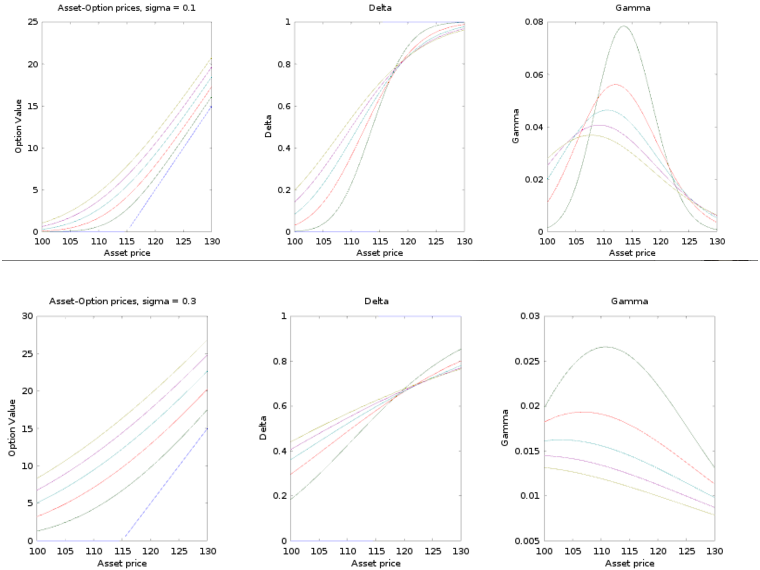
Gamma - the rate of change of the δ w.r.t. the price of the underlying asset: $\Gamma = \frac{\partial^2 \Pi}{\partial S^2}$

Theta - the rate of change of the portfolio price w.r.t. the time: $\theta = \frac{\partial \Pi}{\partial t}$

Vega - the rate of change of the portfolio price w.r.t. the volatility of the underlying asset: $\upsilon = \frac{\partial \Pi}{\partial \sigma}$

Rho - the rate of change of the portfolio price w.r.t. the interest rate:$\upsilon = \frac{\partial \Pi}{\partial r}$

Delta and Gamma for $\sigma = 0.1, 0.3$



Delta for $\sigma = 0.1$

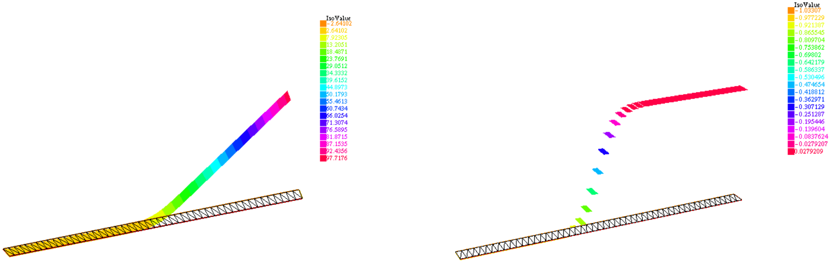


Figure : $\sigma = 0.1$ and $\sigma = 0.3$

Delta for $\sigma = 0.3$

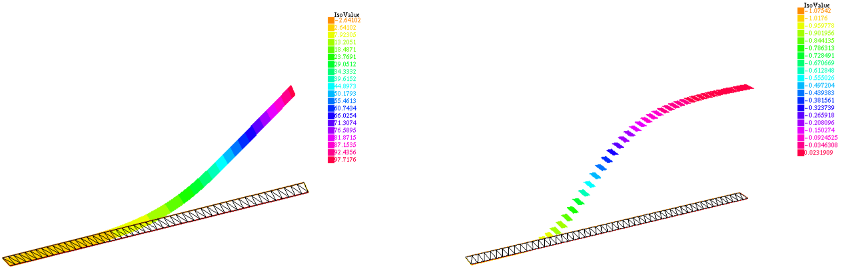
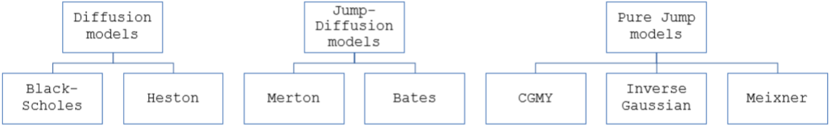
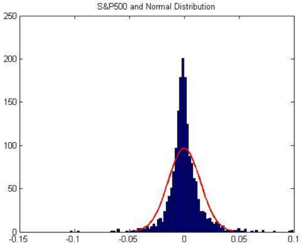


Figure : Delta for $\sigma = 0.1$ and $\sigma = 0.3$

Pricing models



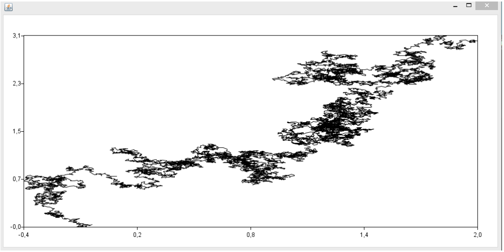


What is a Wiener process

- Lévy processes: independent, stationary increments
- Markov processes: "memoryless"

BUT: no jumps ? $\rightarrow$ add Compound Poisson process!

Figure : Wiener and Wiener-Poisson process



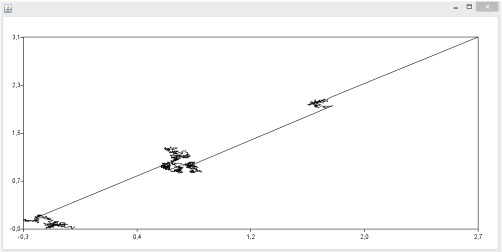


Figure : Black Scholes and Merton

Ito's lemma and Black-Scholes

Ito's lemma :

X_t given by: $dX_t = udt + vdB_t$, $f(t,x) \in C^2$, $Y_t = f(t, X_t)$.

$$dY_t = \frac{\partial f}{\partial t}(t, X_t)dt + \frac{\partial f}{\partial x}(t, X_t)dB_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, X_t)(dB_t)^2 \qquad (1)$$

Applying to Black-Scholes:

$$dS_t = \mu S_t dt + \sigma S_t dB_t, X_0 > 0 \qquad (2)$$

with $f(t,x) = lnx, f \in C^2$ and $Y_t = lnS_t$, gives:

$$\int_0^T dY_t = (\mu - \frac{1}{2}\sigma^2)T + \sigma B_t \qquad (3)$$

Or

$$S_T = e^{Y_T} = S_0 e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma dB_t} \qquad (4)$$

Technical aspects

- Numerical methods:
  - Monte-Carlo (Box-Muller Algorithm)
  - Tree methods (Cox-Ross-Rubenstein model)
  - **Solving PDE**
- S_t follows Geometric Brownian motion: $dS_t = S_t \mu dt + S_t \sigma dB_t$
- Put-Call Parity:

$$C + Ke^{-rT} = P + S_0 \tag{5}$$

Why use the Geometric Brownian motion?

- The expected returns of GBM are independent of the value of the process (stock price), which agrees with what we would expect in reality.
- A GBM process only assumes positive values, just like real stock prices.
- A GBM process shows the same kind of 'roughness' in its paths as we see in real stock prices.
- Calculations with GBM processes are relatively easy.

GBM path in C++ with gnuplot

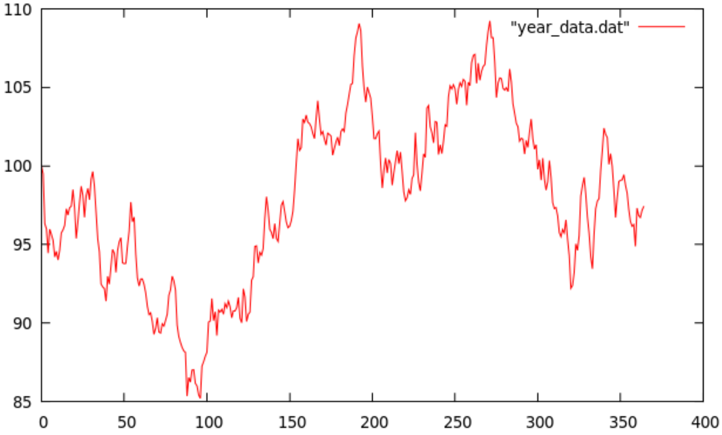


Figure : GBM path

Black-Scholes formulas

With Ito's lemma we find :

$$S_T = e^{Y_T} = e^{Y_0 + (\mu - \frac{1}{2}\sigma^2)T + \sigma dB_t} = S_0 e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma dB_t} \tag{6}$$

and

$$C = e^{-dT} S_0 \mathcal{N}(d_1) - e^{-rT} K \mathcal{N}(d_2)) \tag{7}$$

$$P = e^{-rT} K \mathcal{N}(-d_2)) - e^{-dT} S_0 \mathcal{N}(-d_1) \tag{8}$$

with

$$d_1 = \frac{\ln(\frac{S_0}{K}) + (r - d + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(\frac{S_0}{K}) + (r - d - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

Black-Scholes PDE - Call and Put

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S(t)^2} + rS(t)\frac{\partial C}{\partial S(t)} - rC = 0 \qquad (9)$$

$C(0, t) = 0$
$C(S, t) = e^{-dT} S_0 - e^{-rT} K$ when $S \longrightarrow \infty$
$C(S, T) = max(S - K, 0)$

$$\frac{\partial P}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S(t)^2} - rS(t)\frac{\partial P}{\partial S(t)} + rP = 0 \qquad (10)$$

$P(0, t) = e^{-rT} K$
$P(S, t) = 0$ when $S \longrightarrow \infty$
$P(S, T) = max(K - S, 0)$

Black-Scholes PDE terms

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S(t)^2} + rS(t)\frac{\partial C}{\partial S(t)} - rC = 0 \tag{11}$$

- Diffusion term: $\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S(t)^2}$
- Convection term: $rS(t)\frac{\partial C}{\partial S(t)}$

Peclet number : $\frac{diff}{conv} = \frac{rS}{\frac{1}{2}\sigma^2 S^2} = \frac{2rS}{\sigma^2 S^2} = \frac{2r}{\sigma^2 S}$

Remarque : $Pe < \frac{r}{\sigma^2}$, if not, then small σ is not compensated by a small r.

Weak Formulation

Linear parabolic PDE with non-constant coefficients and non-homogenous boundary conditions and, possibly, non-differentiable or discontinuous final conditions:

- Choosing space : Weighted Sobolev
  $\forall u \in V, V = \{v \in L^2(\mathbb{R}_+) : S\frac{\partial v}{\partial S} \in L^2(\mathbb{R}_+)\}$
- Bilinear form a:

$$a(u, w) = \int_0^\infty \frac{\partial u}{\partial S}\frac{\sigma^2}{2}\frac{S^2 w}{\partial S} - rSw\frac{\partial u}{\partial S} + ruw \tag{12}$$

- Weak formulation (for put):

$$(\frac{\partial u}{\partial t}, w) + a(u, w) = 0, \forall w \in V \tag{13}$$

Existence and Uniqueness

Martingales + Filtration + Ito Calculus

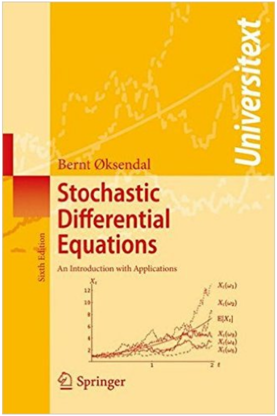


Figure : Oksendal - SDE (6ed)

Discretization

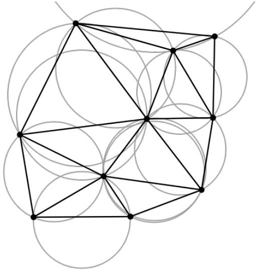
ADD!!!!!!!!!!!!!!!

Mesh adaptation and Delaunay Triangulation

Delanay algorithm keeps the error of interpolation bounded by:

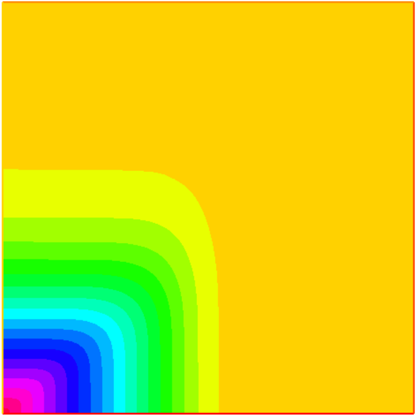
$$||u - u_h|| < C||\nabla(\nabla u)h^2 \quad (14)$$

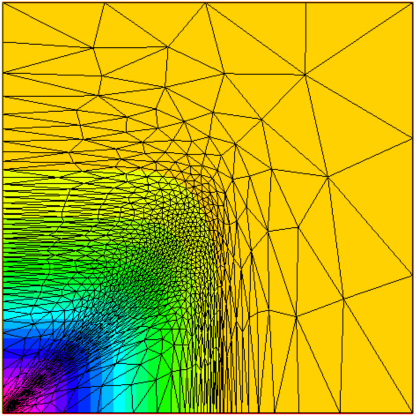
Delaunay triangulation helps to create a "good" mesh : no obtuse triangles, neighbor triangles have more or less the same size.
In other words, the Delaunay triangulation create a mesh where for each edge the circle circumscribing one triangle does not contain the fourth vertex.



Vanilla 2D

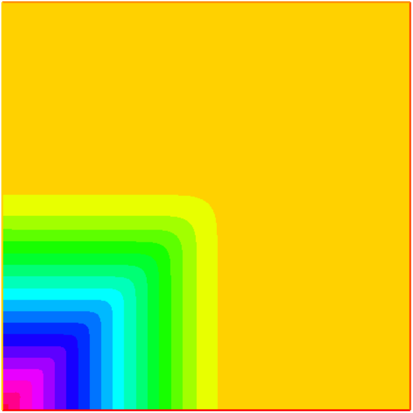
Classic asymmetric data

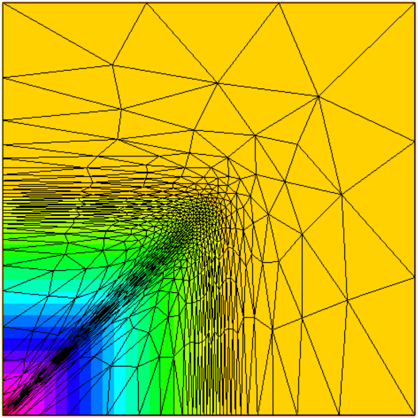




Vanilla 2D

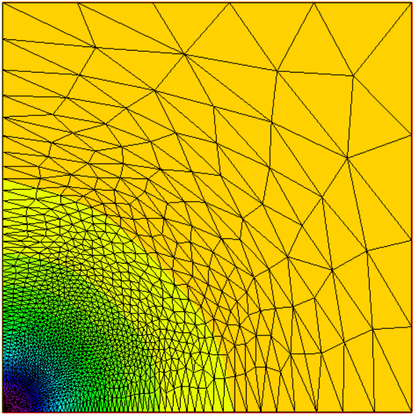
Low volatility with high correlation





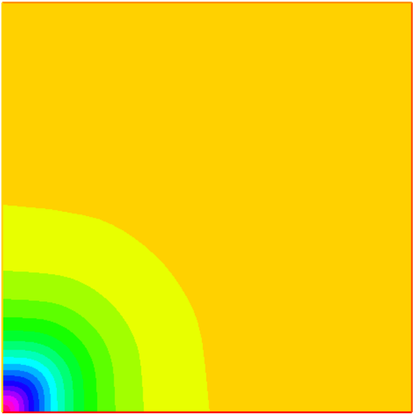
Vanilla 2D

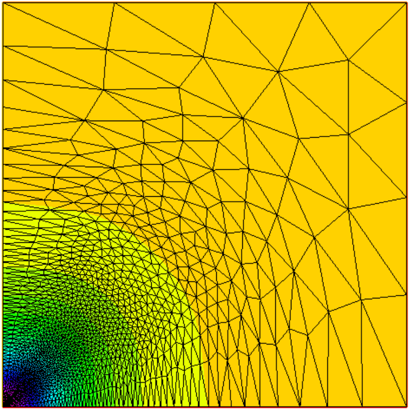
High volatility but low correlation



Vanilla 2D

High volatility with high correlation





Possible classifications