1. Financial Market : basics

Computational finance is an interdisciplinary field which joins financial mathematics, stochastics, numerics and scientific computing. Its task is to estimate as accurately and efficiently as possible the risks that financial instruments generate. In this project we will focus on options.

1.1. Option - financial instrument

An **Option** is a contract that gives the buyer the right, *but not the obligation*, to buy or sell an underlying asset (a stock, a bond, gold, other option, etc) at a specific price, called **Strike** price, on or before a certain date. An option is a security, just as stockes or bonds, it has its own price called **Premium**.

Every option contract has several parameters to be pre-set at $t = T_0$:

- What is the underlying asset ?
- What is the maturity T of the contract ?
- Does the contract give the right to buy (**call option**) or to sell(**put option**) ?
- What is the Strike price K ?
- What is the price of the option itself, i.e. premium?
  Rq: note that premium is ignored in following computations

Obviously, the asymmetry of the option contract leads us to the question if we buy this contract it or we sell it. A buyer of an option contract is said to be in **long** positions, the seller - in **short** position.

Let's take a look on potential gain diagram for Long Call Option:

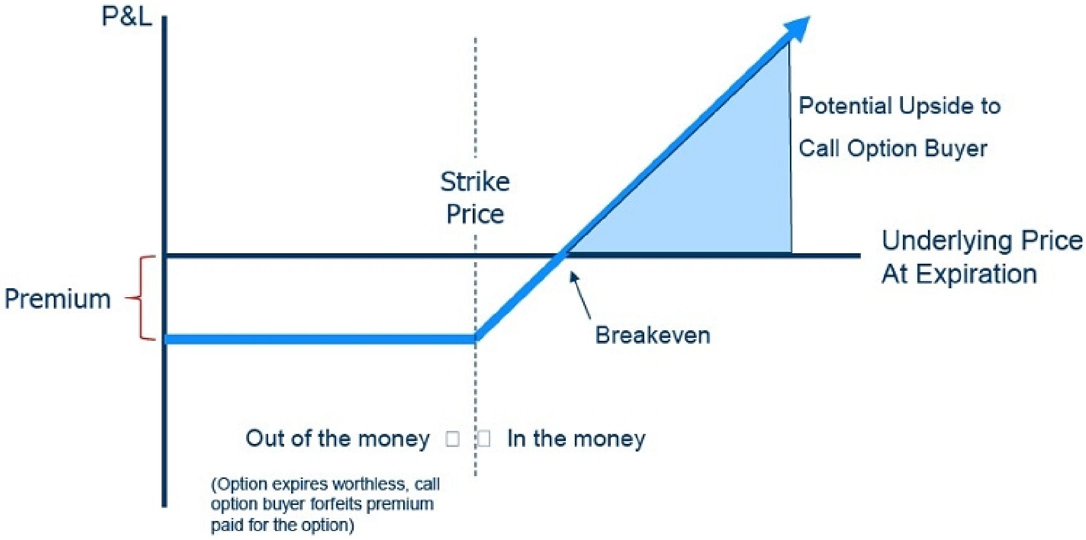

Figure 1: Long Call Option

As it comes from the figure 1, Long Call Option brings profit (« in the money ») , if S at maturity T is higher than the Breakeven point. So the payoff of this option is:

- $S - (K + Premium)$ if $S(T) > (K + Premium)$
- 0 if $S(T) < (K + Premium)$

This type of of options is said to be **Vanilla** : the simpliest version of all, without any optional extras, by analogy with the default ice cream flavour, vanilla. There exist a rich family of, so called **Exotic** options. Here we list only several of them:

- American option (Bermudian)
  can be executed not only at T,but on any time of life of the option (T_0, T);
  Bermudian option can be executed on a specific period during the life of the option, i.e. every second Monday, June, etc.

- Barrier option (Paris)
  can be activated for be executed only if the asset price touches (or not) a specific barrier;
  Paris barrier option can be activated for be executed only if the asset price satisfy the barrier condition for a certain period of time (i.e. 15min, 1 day, 30
- Asian option
  its payoff is determined by the average underlying price over some pre-set period of time.
- Lookback option (Russian)
  its payoff depends not at S at final time T, $S(T)$, but on $max(S)$ over the life of the option.
  Russian lookback option is a special case of lookback : it has no pre-set expiration time, it's up to buyer of the option when to execute it. It's also called 'no regret' option.

In this project we will focus on Barrier options, so let's look at its payoffs more closely:

- Up-and-in barrier call option
  Standard European call option with strike K when its maximum lies above the barrier H, while it is worthless otherwise.
- Up-and-out barrier call option
  Standard European call option with strike K when its maximum lies below the barrier H, while it is worthless otherwise.
- Down-and-in barrier
  Standard European call option with strike K when its minimum lies below the barrier H,while it is worthless otherwise.
- Down-and-out barrier
  Standard European call with strike K when its lies above some barrier H, while it is worthless otherwise.

1.2. Option Pricing models

1.2.1. General notations and definitions

- S - Stock price (or any underlying asset)
- $V(S,T)$ - Value of an option, depending on time and stock price
- K - Strike price
- r - risk-free rate
- μ - drift rate of S - the rate at which the average of S changes
- σ - volatility of the stock, standard deviation of $log(S)$ - return on stock
- T_0,T - initial and final time
- θ - long variance : as t tends to infinity, the expected value of ν tends to θ
- κ - the rate at which ν reverts to θ
- ξ - the volatility of volatility

1.2.2. Pricing models

- **Diffusion models** have only a diffusion component, given by Wiener process : *Black-Scholes, Heston*;

  - *Black-Scholes*
    W, and consequently its increment dW,represents the only source of 'diffusion' uncertainty in the price history of the stock :
    $dS = \mu S dt + \sigma S dW$
  - *Heston*
    W, and consequently its increment dW, and ν, and consequently root from it, represent two sources of 'diffusion' uncertainty in the price history of the stock :
    $dS = \mu S dt + \sqrt{\nu} S dW$
    $d\nu = \kappa(\theta - \nu)dt + \xi\sqrt{\nu} dW$

- **Jump diffusion models** have both diffusion component, given by Wiener process, and jump component, given by compounded Poisson process :

  Rq: For financial applications, it is of little interest to have a process with a single possible jump size. The compound Poisson process is a generalization where the waiting times between jumps are exponential but the jump sizes can have an arbitrary distribution.

- *Merton*
  dW represents the source of 'diffusion' uncertainty and the last term represents is the source of 'jump' uncertainty (compound Poisson process with Gaussian jumps) in the price history of the stock. dW and ν represent the source of 'diffusion' uncertainty and the last term represents is the source of 'jump' uncertainty (compound Poisson process with Gaussian jumps) in the price history of the stock.
  It mixes Black-Scholes model Compounded Poisson process :
  $dS = \mu S dt + \sigma S dW + \sum_{n=1}^{N_t} Y_i$
- *Bates*
  dW and ν represent the source of 'diffusion' uncertainty and the last term represents is the source of 'jump' uncertainty (compound Poisson process with Gaussian jumps) in the price history of the stock.
  It mixes Black-Scholes model Compounded Poisson process : It mixes Merton and Heston models:

  $dS = \mu S dt + \sqrt{\nu} S dW + \sum_{n=1}^{N_t} Y_i$

  $d\nu = \kappa(\theta - \nu)dt + \xi\sqrt{\nu} dW$

• **Pure jump models** have no diffusion, just a random process :

- *CGMY*
- *NIG*

1.3. Greeks

Greeks are the quantities representing the sensitivity of the price of options to a change in underlying parameters. We list some of them, futher in the project we will study them:

- **Delta** measures the rate of change of the theoretical option value with respect to changes in the underlying asset's price. **Delta** is the first derivative of the value V of the option with respect to the underlying instrument's price S.
- **Vega** measures sensitivity to volatility. **Vega** is the derivative of the option value with respect to the volatility of the underlying asset.
- **Theta** measures the sensitivity of the value of the derivative to the passage of time : the "time decay."
- **Rho** measures sensitivity to the interest rate: it is the derivative of the option value with respect to the risk free interest rate (for the relevant outstanding term).

2. Stochastics : basics

2.1. Stochastic process

A **stochastic process** in discrete time, denoted by $(X_t)_{t\in\mathcal{T}}$, is a sequence of random variables $X_0, X-1, ..X_T$ defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with values in $\mathbb{R}$. For a given $\omega \in \Omega$, the sequence $X_t(w), t \in \mathcal{T}$ is called a **path** of the process (X_t), where $\mathcal{T}$ stands for time series (dates) : $\mathcal{T} = \{0, 1, ..T\}$

2.2. Filtrations

A **filtration** $\mathcal{F}$ on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ is a sequence of σ-algebras $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq ... \subseteq \mathcal{F}_T \subseteq \mathcal{A}$. A quadruple $(\Omega, \mathcal{A}, \mathbb{P}, \mathcal{F})$ is called a **filtered probability space**.
A stochastic process $(X_t)_{t\in\mathcal{T}}$ is sais to be **adapted to a filtration** $\mathcal{F}$ if for all $t \in \mathcal{T}$, the random variable X_t is $\mathcal{F}_t$ - measurable.

2.3. Martingales

Let $(\Omega, \mathcal{A}, \mathbb{P}, \mathcal{F}))$ be a filtered probability space. A stochastic process $X = (X_t)_{t\in\mathcal{T}}$ is called a **martingale** if

i. X is adapted to $\mathcal{F}$
ii. $\mathbb{E}[|X_t|] < \inf \forall \in \mathcal{T}$
iii. $\mathbb{E}[X_t|\mathcal{F}_s] = X_s \forall t \geq s$

2.4. Itô Calculus

The **Itô integral** helps to deal with integration over stochastic processes.

$T_t = \int_0^t H_s dW_s,$

where H is a locally square integrable process adapted to the filtration generated by W, which is a Wiener process. The result of the integration is another process.

If we assume that H is *cadlag* and properly adapted and $\{\pi_n\}$ is a sequence of partitions of $[0,t]$ with min mesh size, then the integration by parts for Itô integral is given by :

$\int_0^t H dW = \lim_{n\to\infty} \sum_{t_{i-1},t_i \in \pi_n} H_{t_{i-1}}(W_{t_i} - W_{t_{i-1}}).$

H is square integrable, then its integral w.r.t. W can be defined and H is said to be W-integrable. As $\int_0^t (H - H_n)^2 ds \to 0$ in probability, the Itô integral becomes:
$\int_0^t H dW = \lim_{n\to\infty} \int_0^t H_n dW$
This leads as to the **Itô isometry**:
$\mathbb{E}[(\int_0^t H_s dW_s)^2] = \mathbb{E}[\int_0^t H_s^2 ds]$