In [1]:

```
import numpy as np
from numpy.typing import ArrayLike
```

In [2]:

```
def dot(A:ArrayLike, B:ArrayLike)->ArrayLike:
    '''
    Parameters
    ----------
    A: numpy.typing.ArrayLike
        Numpy Array for multiplication

    B: numpy.typing.ArrayLike
        Numpy Array for multiplication

    Returns
    ----------
    res: numpy.typing.ArrayLike
        Numpy Array representing the dot product of A and B
    '''

    # extracting the rows and columns from the matrices
    r1, c1 = A.shape
    r2, c2 = B.shape

    # validating the operation
    assert c1 == r2, \
    f"Invalid operation: Num cols of A and Num rows of B are not equal for {A.shape}, {B.s

    # creating an array of zeros
    res = np.zeros((r1,c2), dtype=A.dtype)

    # populating res
    for i in range(len(A)):
        for j in range(len(B[0])):
            for k in range(len(B)):
                res[i][j] += int(A[i][k]) * int(B[k][j])
    return res
```

In [3]:

```
# a fun little list comprehension solution

def dot_comprehension(A:ArrayLike, B:ArrayLike)->ArrayLike:
    '''
    Parameters
    ----------
    A: numpy.typing.ArrayLike
        Numpy Array for multiplication

    B: numpy.typing.ArrayLike
        Numpy Array for multiplication

    Returns
    ----------
    res: numpy.typing.ArrayLike
        Numpy Array representing the dot product of A and B
    '''

    # extracting the rows and columns from the matrices
    r1, c1 = A.shape
    r2, c2 = B.shape

    assert c1 == r2, \
    f"Invalid operation: Num cols of A and Num rows of B are not equal for {A.shape}, {B.s
```

```
    res = np.array([[sum(a*b for a,b in zip(A_row,B_col)) for B_col in zip(*B)] for A_row
    return res
```

In [4]:

```
A = np.array([[-4,-3,-2], [6,0,-1], [2,1,3]])
B = np.array([[5,4], [6,7],[-4,-3]])

X = np.array([[1,0],[0,1]])
Y = np.array([[1,2,3,4],[5,6,7,8]])
```

In [5]:

```
print(f"Dot:\n{dot(A,B)}\n\nNumpy Dot:\n{np.dot(A,B)}")
```

```
Dot:
[[-30 -31]
 [ 34  27]
 [  4   6]]

Numpy Dot:
[[-30 -31]
 [ 34  27]
 [  4   6]]
```

In [6]:

```
print(f"Dot:\n{dot(X,Y)}\n\nNumpy Dot:\n{np.dot(X,Y)}")
```

```
Dot:
[[1 2 3 4]
 [5 6 7 8]]

Numpy Dot:
[[1 2 3 4]
 [5 6 7 8]]
```

In [ ]:

DSCC 265
Assignment 1
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1. Attached .ipynb file

2. (a) i. Non-negativity: By definition, the absolute value is always non-negative. Since, $|x_i - y_i| \geq 0 \; \forall \; x, y \in \mathbb{R}$, the sum over all i must be non-negative as well.

   ii. Definiteness: If, $x_i = y_i$, $|x_i - y_i| = |x_i - x_i| = 0$ and by symmetry this is true for $y_i = x_i$. Since, this is true for any general x_i and y_i this must be true for the sum over all i.
   If $L_1(x, y) = 0 \Rightarrow 0 = \sum_{i=1}^{d} |x_i - y_i| \Rightarrow |x_i - y_i| = 0 \Rightarrow |x_i| - |y_i| = 0 \Rightarrow x_i = y_i$. As a result, $\sum_{i=1}^{d} |x_i - y_i| = 0$ iff $x_i = y_i$.

   iii. Homogeneity:

$$L_1(tx, ty) = \sum_{i=1}^{d} |tx_i - ty_i| = \sum_{i=1}^{d} |tx_i - ty_i| = \sum_{i=1}^{d} |t||x_i - y_i| = |t| \sum_{i=1}^{d} |x_i - y_i| = |t| L_1(x, y)$$

   iv. Triangle Inequality:

$$\begin{aligned} d(x, y) &= \sum_{i=1}^{d} |x_i - y_i| \\ &= \sum_{i=1}^{d} |(x_i - z_i) + (z_i - y_i)| \\ &\leq \sum_{i=1}^{d} |x_i - z_i| + \sum_{i=1}^{d} |z_i - y_i| \\ &= d(x, z) + d(z, y) \; \forall \; x, y, z \in \mathbb{R} \end{aligned}$$

   (b) i. Non-negativity: $(x_i - y_i)^2 \geq 0 \; \forall \; x, y \in \mathbb{R} \Rightarrow \sqrt{(x_i - y_i)^2} \geq 0 \; \forall \; x, y \in \mathbb{R}$. Therefore, the sum over all i must be non-negative as well.

   ii. Definiteness: If, $x_i = y_i$, $\sqrt{(x_i - y_i)^2} = \sqrt{(x_i - x_i)^2} = 0$ and by symmetry this is true for $x_i = y_i$. Since, this is true for any general x_i and y_i this must be true for the sum over all i.
   If $L_2(x, y) = 0 \Rightarrow 0 = \sqrt{\sum_{i=1}^{d} (x_i - y_i)^2} \Rightarrow 0 = \sum_{i=1}^{d} (x_i - y_i)^2 \Rightarrow (x_i - y_i)^2 = 0 \Rightarrow x_i - y_i = 0 \Rightarrow x_i = y_i$. As a result, $\sqrt{\sum_{i=1}^{d} (x_i - y_i)^2} = 0$ iff $x_i = y_i$.

   iii. Homogeneity:

$$L_2(tx, ty) = \sqrt{\sum_{i=1}^{d} (tx_i - ty_i)^2} = \sqrt{\sum_{i=1}^{d} (tx_i - ty_i)^2} = \sqrt{\sum_{i=1}^{d} t^2 (x_i - ty_i)^2} =$$

$$\sqrt{(t)^2} \sqrt{\sum_{i=1}^{d} (x_i - y_i)^2} = |t| L_2(x, y)$$

   iv. Triangle Inequality:

$$
\begin{aligned}
[L_2(x,y)]^2 &= \left(\sqrt{\sum_{i=1}^{d}(x_i-y_i)^2}\right)^2 \\
&= \sum_{i=1}^{d}(x_i-y_i)^2 \\
&= \sum_{i=1}^{d}[(x_i-z_i)+(z_i-y_i)]^2 \\
&= \sum_{i=1}^{d}(x_i-z_i)^2+\sum_{i=1}^{d}(z_i-y_i)^2+2\sum_{i=1}^{d}(x_i-z_i)(z_i-y_i)
\end{aligned}
$$

$$
\begin{aligned}
[L_2(x,z)+L_2(z,y)]^2 &= \left(\sqrt{\sum_{i=1}^{d}(x_i-z_i)^2}+\sqrt{\sum_{i=1}^{d}(z_i-y_i)^2}\right)^2 \\
&= \sum_{i=1}^{d}(x_i-z_i)^2+\sum_{i=1}^{d}(z_i-y_i)^2+2\sqrt{\sum_{i=1}^{d}(x_i-z_i)^2}\sqrt{\sum_{i=1}^{d}(z_i-y_i)^2}
\end{aligned}
$$

By the Cauchy–Schwarz inequality in n-dimensional Euclidean space,

$$
\begin{gathered}
2\sum_{i=1}^{d}(x_i-z_i)(z_i-y_i) \leq 2\sqrt{\sum_{i=1}^{d}(x_i-z_i)^2}\sqrt{\sum_{i=1}^{d}(z_i-y_i)^2} \\
\therefore [L_2(x,y)]^2 \leq [L_2(x,z)+L_2(z,y)]^2 \\
\Rightarrow L_2(x,y) \leq L_2(x,z)+L_2(z,y) \\
\Rightarrow d(x,y) \leq d(x,z)+d(z,y)
\end{gathered}
$$

(c) i. Homogeneity:

$$
L_2^2(tx,ty)=\sum_{i=1}^{d}(tx_i-ty_i)^2=\sum_{i=1}^{d}(tx_i-ty_i)^2=\sum_{i=1}^{d}t^2(x_i-y_i)^2=(t)^2\sum_{i=1}^{d}(x_i-y_i)^2 \neq |t|L_2^2(x,y)
$$

$\therefore$ NOT A METRIC

3.

$$
A-\lambda I=\begin{bmatrix} 4-\lambda & 4 & 4 \\ -2 & -3-\lambda & -6 \\ 1 & 3 & 6-\lambda \end{bmatrix}
$$

Characteristic Polynomial:

$$\begin{aligned}
det(A-\lambda I) &= (4-\lambda)[(-3-\lambda)(6-\lambda)+18] - 4[-2(6-\lambda)+6] + 4[-6-(-3-\lambda)] \\
&= (4-\lambda)(\lambda^2-3\lambda) - 4(2\lambda-6) = 4(\lambda-3) \\
&= (\lambda-3)(\lambda(4-\lambda)-8+4) \\
&= (\lambda-3)(-\lambda^2+4\lambda-4) \\
&= -(\lambda-3)(\lambda^2-4\lambda+4) \\
&= -(\lambda^3-7\lambda^2+16\lambda-12) \\
&= -\lambda^3+7\lambda^2-16\lambda+12
\end{aligned}$$

Eigenvalues:

$$\begin{aligned}
-\lambda^3+7\lambda^2-16\lambda+12 &= 0 \\
(\lambda-3)(\lambda^2-4\lambda+4) &= 0 \\
(\lambda-3)(\lambda-2)^2 &= 0 \\
\therefore \lambda = 2(\text{multiplicity } = 2), 3
\end{aligned}$$

Eigenvectors:

For, $\lambda = 3$,

$$\left[\begin{array}{ccc|c} 1 & 4 & 4 & 0 \\ -2 & -6 & -6 & 0 \\ 1 & 3 & 3 & 0 \end{array}\right] \xrightarrow{2r_1+r_2 \to r_2} \left[\begin{array}{ccc|c} 1 & 4 & 4 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 3 & 3 & 0 \end{array}\right] \xrightarrow{r_3-r_1 \to r_3} \left[\begin{array}{ccc|c} 1 & 4 & 4 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{r_2/2 \to r_2} \left[\begin{array}{ccc|c} 1 & 4 & 4 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right] \xrightarrow{r_3+r_2 \to r_3} \left[\begin{array}{ccc|c} 1 & 4 & 4 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{r_1-4r_2 \to r_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned}
\therefore x_1 &= 0 \\
x_2 + x_3 = 0 &\Rightarrow x_2 = -x_3 \\
\Rightarrow X &= \begin{pmatrix} 0 \\ -x_3 \\ x_3 \end{pmatrix} \Rightarrow \left\{ x_3 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\}
\end{aligned}$$

Let, $x_3 = 1$,

$$v_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

Similarly, for $\lambda = 2$

$$\left[\begin{array}{ccc|c} 2 & 4 & 4 & 0 \\ -2 & -5 & -6 & 0 \\ 1 & 3 & 4 & 0 \end{array}\right] \xrightarrow{r_1/2 \to r_1} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ -2 & -5 & -6 & 0 \\ 1 & 3 & 4 & 0 \end{array}\right] \xrightarrow{r_2+2r_1 \to r_2} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & -1 & -2 & 0 \\ 1 & 3 & 4 & 0 \end{array}\right]$$

$$\xrightarrow{r_3-r_1\to r_3}\begin{bmatrix}1&2&2&|&0\\0&-1&-2&|&0\\0&1&2&|&0\end{bmatrix}\xrightarrow{-r_2\to r_2}\begin{bmatrix}1&2&2&|&0\\0&1&2&|&0\\0&1&2&|&0\end{bmatrix}\xrightarrow{r_3-r_2\to r_3}\begin{bmatrix}1&0&0&|&0\\0&1&2&|&0\\0&0&0&|&0\end{bmatrix}\xrightarrow{r_1-2r_2\to r_1}\begin{bmatrix}1&0&-2&|&0\\0&1&2&|&0\\0&0&0&|&0\end{bmatrix}$$

$$\begin{aligned}\therefore x_1 - 2x_3 &= 0\\ x_2 + 2x_3 &= 0\\ \Rightarrow X = \begin{pmatrix}2x_3\\-2x_3\\x_3\end{pmatrix} &\Rightarrow \left\{x_3\begin{pmatrix}2\\-2\\1\end{pmatrix}\right\}\end{aligned}$$

Let, $x_3 = 1$,

$$v_2 = \begin{pmatrix}2\\-2\\1\end{pmatrix}$$

4. (a)

$$\begin{aligned}Tr(ABC) &= \sum_{i=1}(ABC)_{ii}\\ &= \sum_{i=1}\sum_{j=1}A_{ij}(BC)_{ji}\\ &= \sum_{i=1}\sum_{j=1}\sum_{k=1}A_{ij}B_{jk}C_{ki}\end{aligned}$$

Since, A_{ij}, B_{jk}, C_{ki} are all scalars, they are commutative.

$$\begin{aligned}\therefore \sum_{i=1}\sum_{j=1}\sum_{k=1}A_{ij}B_{jk}C_{ki} &= \sum_{i=1}\sum_{j=1}\sum_{k=1}C_{ki}A_{ij}B_{jk}\\ &= \sum_{i=1}\sum_{k=1}C_{ki}(AB)_{ik}\\ &= \sum_{k=1}(CAB)_{kk}\\ &= Tr(CAB)\end{aligned}$$

Similarly,

$$\begin{aligned}\therefore \sum_{i=1}\sum_{j=1}\sum_{k=1}A_{ij}B_{jk}C_{ki} &= \sum_{i=1}\sum_{j=1}\sum_{k=1}B_{jk}C_{ki}A_{ij}\\ &= \sum_{j=1}\sum_{k=1}B_{jk}(CA)_{kj}\\ &= \sum_{j=1}(BCA)_{jj}\\ &= Tr(BCA)\end{aligned}$$

So, we can conclude that,

$$Tr(ABC) = \sum_{i=1}\sum_{j=1}\sum_{k=1}A_{ij}B_{jk}C_{ki} = Tr(CAB) = Tr(BCA)$$

(b) From our proof in (a) it is already obvious that Tr(BAC) is not always equal to Tr(ABC) due to the dimensions of the two matrices. Let, $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$$ABC = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\therefore Tr(ABC) = 1$$

Now,

$$BAC = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\therefore Tr(BAC) = 0$$

So, $Tr(ABC) \neq Tr(BAC)$ for all A, B, C.

5. (a) Let, $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$

$$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$BA = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$\therefore$ if $AB = 0$, $BA = 0$ is not always true

(b) Not true. Counterexample is the same as (a). AB = 0 but neither A = 0 nor B = 0

(c) A is invertible, since, $det(A) \neq 0$. So,

$$AB = 0$$
$$A^{-1}AB = 0$$
$$\therefore B = 0$$

(d) $det(AB) = det(A)det(B) = det(B)det(A) = det(BA) = 0$. Let, $BA = C$, so C is a singular matrix. $\therefore$ $C\boldsymbol{v} = \mathbf{0}$ which defines the nullspace of C, $\{\boldsymbol{v} \in \mathbb{R}^n : C\boldsymbol{v} = \mathbf{0}\}$ can be nontrivial.