## Question 1

A restaurant is testing a new menu and wants to know whether customers are likely to recommend it to friends. They survey 10 randomly selected customers after the meal. Let $Y_1$ be the number of customers who say they would recommend the restaurant. Assume each customer responds independently and with the same probability $\pi$. The management believes $\pi$ could be one of four values: 0.10, 0.30, 0.50, or 0.70, all equally likely a priori. They observe that 5 out of 10 customers said they would recommend the restaurant.

1. Use the discrete prior model to compute the posterior distribution of $\pi$. **Complete the table. All final numerical values must be rounded to 5 decimal places.**

| $\pi$ | Prior $P(\pi)$ | Likelihood $P(Y_1 = 5 \mid \pi)$ | Unnormalized Posterior | Posterior $P(\pi \mid Y_1 = 5)$ |
|---|---|---|---|---|
| 0.10 | | | | |
| 0.30 | | | | |
| 0.50 | | | | |
| 0.70 | | | | |

2. Which value of $\pi$ is most plausible based on the posterior distribution? Briefly explain.
3. Suppose the restaurant can allocate a budget to improve the menu only if the probability that more than 50% of customers will recommend it is at least 75%. Based on your posterior distribution, should they invest in improvements? Justify your answer using the posterior probabilities.
4. Compute the posterior expected value of $\pi$ and interpret it in the context of customer recommendations.

## Question 2

An online education platform tracks whether students complete a new short course. Each student either completes the course (success) or does not (failure). Let $Y_2$ be the number of students who completed the course out of 8 enrolled students. Assume the same completion probability $\pi$ for each student. Based on past data, $\pi$ could be 0.25, 0.50, 0.75, or 0.90, all equally likely a priori. They observe that 6 out of 8 students completed the course.

1. Use the discrete prior model to compute the posterior distribution of $\pi$. **Complete the table. Round to 5 decimal places.**

| $\pi$ | Prior $P(\pi)$ | Likelihood $P(Y_2 = 6 \mid \pi)$ | Unnormalized Posterior | Posterior $P(\pi \mid Y_2 = 6)$ |
|---|---|---|---|---|
| 0.25 | | | | |
| 0.50 | | | | |
| 0.75 | | | | |
| 0.90 | | | | |

2. Which value of $\pi$ is most plausible based on the posterior distribution? Briefly explain.

3. Based on the posterior probabilities, how confident can the platform be that the course is highly engaging (i.e., $\pi \geq 0.75$)? What decision would you recommend about promoting this course on the homepage?

4. Calculate the posterior predictive probability that exactly 7 out of the next 8 students will complete the course.

## Question 3

The university health officer wants to estimate the proportion $\pi$ of students who regularly attend fitness sessions organized on campus. Based on prior observation, it is believed that roughly 30% of students do so, but this estimate comes with some uncertainty. A prior mean of 0.30 and a prior standard deviation of 0.10 are used.

1. Determine the parameters $a$ and $b$ of the $\text{Beta}(a, b)$ prior that reflects this belief.

2. Calculate the equivalent sample size of this prior.

3. A survey of 100 students reveals that 38 of them regularly attend fitness sessions. Find the posterior distribution of $\pi$.

4. What is the posterior mean of $\pi$? Interpret it in the context of the university's decision on expanding the program.

5. Should the university continue investing in fitness infrastructure if the expected attendance exceeds 35%? Justify your decision based on the posterior mean.

## Question 4

A startup is testing a new electric bike rental system. The project manager wants to estimate the proportion $\pi$ of users who would recommend the bikes after their first use. Based on past feedback and pilot trials, the prior belief is that about 70% recommend it, with moderate uncertainty. The prior mean is 0.7 and prior standard deviation is 0.12.

1. Determine the Beta prior parameters $a$, $b$ that match these beliefs.

2. What is the equivalent sample size of this prior?

3. A follow-up survey of 40 users finds 28 would recommend the service. Determine the posterior distribution of $\pi$.

4. Compute the posterior mean and interpret it in the context of advertising the service more widely.

5. What is the posterior probability that more than 80% of users would recommend the bike?

## Question 5

A logistics officer is analyzing the error rate in the automated baggage handling system at an international airport. Each error is defined as a bag misrouted or delayed beyond 15 minutes. These events are modeled as occurring randomly with a Poisson distribution. On average, 1,000 bags are handled per hour. A sample of 200 hours of operation shows a total of 94 errors.

1. The officer believes, based on experience, that the average error rate is about 0.5 errors per hour with a standard deviation of 0.3. Determine the parameters of a Gamma$(r, v)$ prior that match this belief.
2. Determine the posterior distribution of $\lambda$, the rate of baggage handling errors per hour.
3. Use the normal approximation to the Gamma distribution to calculate a 95% Bayesian credible interval for $\lambda$.
4. Interpret your credible interval. Should the airport consider system improvements if the target error rate is below 0.4 per hour?
5. What is the posterior probability that $\lambda > 0.6$?

## Question 6

An engineer monitors a new robotic arm used in a car manufacturing plant. The defects made by the arm are modeled as a Poisson process per 10,000 parts. Over the course of inspecting 120,000 parts, a total of 42 defects are found.

1. The engineer's prior belief is that the defect rate $\lambda$ has mean 3 and standard deviation 1. Determine the parameters of the Gamma$(r, v)$ prior.
2. Determine the posterior distribution of $\lambda$, the defect rate per 10,000 parts.
3. Use the normal approximation to compute a 95% credible interval for $\lambda$.
4. Suppose the quality control target is to keep defects below 3.5 per 10,000 parts. Should the process be flagged for review?
5. Compute the posterior probability that the defect rate is greater than 3.5.

## Question 7

A digital marketing agency is testing the success of a new email campaign. They are interested in the proportion $\pi$ of users who open the email. Their prior belief about $\pi$ follows a Beta(2, 1.5) distribution. After conducting an A/B test, their posterior distribution becomes Beta(5, 3.5).

They wish to test:

$$H_0 : \pi \leq 0.5 \quad \text{versus} \quad H_1 : \pi > 0.5$$

*Note: Use the Beta function $B(a, b)$ to represent normalizing constants. Leave answers in integral form where appropriate.*

1. Write the expression for the posterior probability of the alternative hypothesis $H_1$, as a definite integral in terms of the posterior density.
2. Using your result from part (1), write the formula for the posterior odds in favor of $H_1$ over $H_0$.
3. Write the expression for the prior probability of the alternative hypothesis $H_1$, and the formula for the prior odds in favor of $H_1$.
4. Using parts (2) and (3), write the expression for the Bayes Factor in favor of $H_1$.
5. Briefly describe how the Bayes Factor and posterior odds assist in making a decision between these two hypotheses.
6. Suppose your decision threshold for recommending the campaign is if the posterior probability of $H_1$ exceeds 80%. Based on your setup, discuss what more you would need to calculate to determine if this condition is met.