Control and Automation Engineering

Introduction to Nonlinear Control Systems

THE FURUTA PENDULUM

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Content Table

1 Introduction

- *Furuta Pendulum System*
  - *-Direct-drive motor*
  - *-Pendulum (attached to rotating shaft)*
- *Developed by K. Furuta at Tokyo Institute of Technology*
- *Underactuated mechanical system*
  - *-Angular acceleration cannot be controlled*
- *Conventional inverted pendulum*
  - *-movement limitation*

Figure 1. Furuta Pendulum

2 Modeling of the System

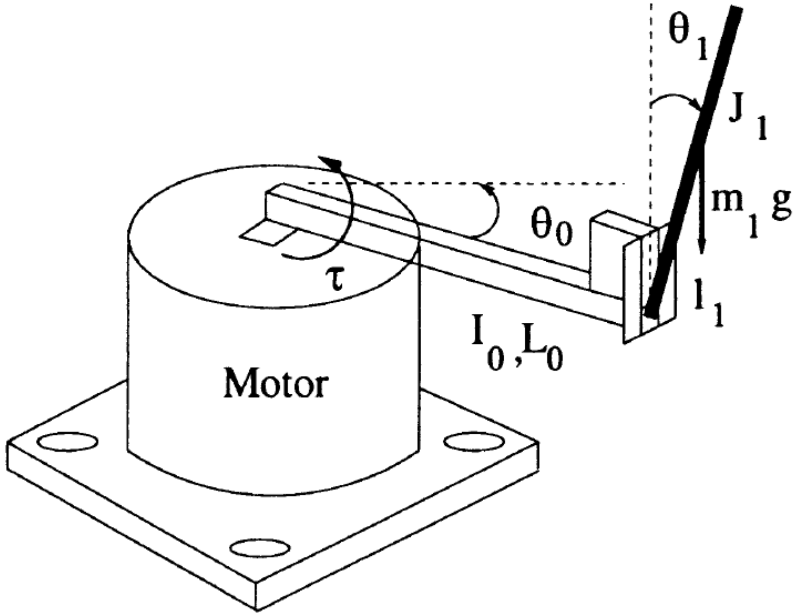


Figure 2. Furuta Pendulum System

I_0	*Inertia of the arm*
L_0	*Total length of the arm*
m_1	*Mass of the pendulum*
l_1	*Distance to the center of gravity of the pendulum*
J_1	*Inertia of the pendulum around its center of gravity*
θ_0	*Rotational angle of the arm*
θ_1	*Rotational angle of the pendulum*
τ	*Input torque applied on the arm*

Table 1. Notations

2.1 Energy of the system

The arm

The kinetic energy of the arm is given by: $K_0 = \frac{1}{2} I_0 \dot{\theta}_0^{\,2}$

The pendulum

The kinetic energy of the pendulum is given by:

$$
\begin{aligned}
K_1 \;&=\; \overbrace{\boxed{\tfrac{1}{2} J_1 \dot{\theta}_1^{\,2}}}^{\text{Kinetic energy}} + \tfrac{1}{2} m_1 \Big[\Big\{ \overbrace{\boxed{\tfrac{d}{dt}(L_0 \sin\theta_0 + l_1 \sin\theta_1 \cos\theta_0)}}^{\text{Tangential Velocity}} \Big\}^2 \\
&\quad + \Big\{ \underbrace{\boxed{\tfrac{d}{dt}(L_0 \cos\theta_0 - l_1 \sin\theta_1 \sin\theta_0)}}_{\text{Radial Velocity}} \Big\}^2 + \Big\{ \underbrace{\boxed{\tfrac{d}{dt}(l_1 \cos\theta_1)}}_{\text{Vertical Velocity}} \Big\}^2 \Big]
\end{aligned}
$$

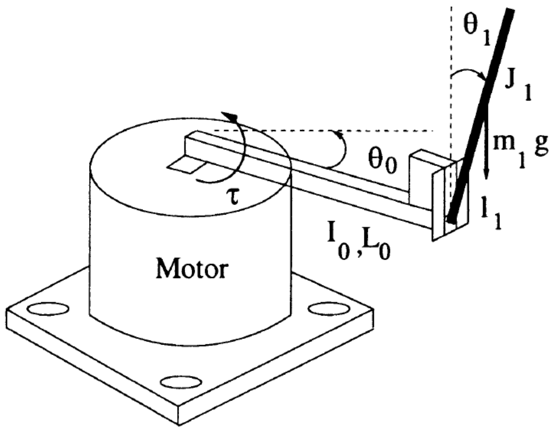
$$
\begin{aligned}
K_1 \;&=\; \tfrac{1}{2} J_1 \dot{\theta}_1^{\,2} + \tfrac{1}{2} m_1 L_0^2 \dot{\theta}_0^{\,2} + \tfrac{1}{2} m_1 l_1^2 \dot{\theta}_1^{\,2} \\
&\quad + \tfrac{1}{2} m_1 l_1^2 \sin^2\theta_1 \dot{\theta}_0^{\,2} + m_1 L_0 l_1 \cos\theta_1 \dot{\theta}_0 \dot{\theta}_1
\end{aligned}
$$

$$P_1 = m_1 g l_1 (\cos\theta_1 - 1)$$

Table 1. Notations

I_0	*Inertia of the arm*
L_0	*Total length of the arm*
m_1	*Mass of the pendulum*
l_1	*Distance to the center of gravity of the pendulum*
J_1	*Inertia of the pendulum around its center of gravity*
θ_0	*Rotational angle of the arm*
θ_1	*Rotational angle of the pendulum*
τ	*Input torque applied on the arm*

Figure 2. Furuta Pendulum System



2.2 Euler – Lagrange dynamic equations

Euler-Lagrange formulation: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) - \frac{\partial L}{\partial \theta_i} = F_i$ *where* $L = K - P$, $K = K_0 - K_1$ and $P = P_1$

$$\left(\frac{\partial L}{\partial \dot{\theta}_0}\right) = \left[I_0 + m_1(L_0^2 + l_1^2 \sin^2\theta_1)\right]\dot{\theta}_0 + m_1 l_1 L_0 \cos\theta_1 \dot{\theta}_1$$

$$\left(\frac{\partial L}{\partial \theta_0}\right) = 0$$

$$\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = m_1 l_1 L_0 \cos\theta_1 \dot{\theta}_0 + \left[J_1 + m_1 l_1^2\right]\dot{\theta}_1$$

$$\left(\frac{\partial L}{\partial \theta_1}\right) = m_1 l_1^2 \sin\theta_1 \cos\theta_1 \dot{\theta}_0^{\,2} - m_1 l_1 L_0 \sin\theta_1 \dot{\theta}_1 \dot{\theta}_0 + m_1 g l_1 \sin\theta_1$$

the system is given by

$$\begin{aligned}
\tau &= \left[I_0 + m_1(L_0^2 + l_1^2 \sin^2\theta_1)\right]\ddot{\theta}_0 + m_1 l_1 L_0 \cos\theta_1 \ddot{\theta}_1 \\
&\quad + m_1 l_1^2 \sin(2\theta_1)\dot{\theta}_0\dot{\theta}_1 - m_1 l_1 L_0 \sin\theta_1 \dot{\theta}_1^{\,2} \\
0 &= m_1 l_1 L_0 \cos\theta_1 \ddot{\theta}_0 + \left[J_1 + m_1 l_1^2\right]\ddot{\theta}_1 \\
&\quad - m_1 l_1^2 \sin\theta_1 \cos\theta_1 \dot{\theta}_0^{\,2} - m_1 g l_1 \sin\theta_1
\end{aligned}$$

2.2 Euler – Lagrange dynamic equations

$$\tau = \left[I_0 + m_1(L_0^2 + l_1^2\sin^2\theta_1)\right]\ddot{\theta}_0 + m_1 l_1 L_0\cos\theta_1\ddot{\theta}_1 + m_1 l_1^2\sin(2\theta_1)\dot{\theta}_0\dot{\theta}_1 - m_1 l_1 L_0\sin\theta_1\dot{\theta}_1^{\,2}$$

$$0 = m_1 l_1 L_0\cos\theta_1\ddot{\theta}_0 + \left[J_1 + m_1 l_1^2\right]\ddot{\theta}_1 - m_1 l_1^2\sin\theta_1\cos\theta_1\dot{\theta}_0^{\,2} - m_1 g l_1\sin\theta_1$$

Can be written as $D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = F$

$$q = \begin{bmatrix}\theta_0\\ \theta_1\end{bmatrix} \qquad D(q) = \begin{bmatrix} I_0 + m_1(L_0^2 + l_1^2\sin^2\theta_1) & m_1 l_1 L_0\cos\theta_1 \\ m_1 l_1 L_0\cos\theta_1 & J_1 + m_1 l_1^2\end{bmatrix} \qquad g(q) = \begin{bmatrix}0\\ -m_1 g l_1\sin\theta_1\end{bmatrix}$$

$$C(q,\dot{q}) = \begin{bmatrix} \frac{1}{2}m_1 l_1^2\sin(2\theta_1)\dot{\theta}_1 & -m_1 l_1 L_0\sin\theta_1\dot{\theta}_1 + \frac{1}{2}m_1 l_1^2\sin(2\theta_1)\dot{\theta}_0 \\ -\frac{1}{2}m_1 l_1^2\sin(2\theta_1)\dot{\theta}_0 & 0\end{bmatrix} \qquad F = \begin{bmatrix}\tau\\ 0\end{bmatrix}$$

$$d_{11} = I_0 + m_1(L_0^2 + l_1^2\sin^2\theta_1) \geq I_0 + m_1 L_0^2 > 0$$

$$\det(D(q)) = (I_0 + m_1(L_0^2 + l_1^2\sin^2\theta_1))(J_1 + m_1 l_1^2) - m_1^2 l_1^2 L_0^2\cos^2\theta_1$$
$$= (I_0 + m_1 l_1^2\sin^2\theta_1)(J_1 + m_1 l_1^2) + J_1 m_1 L_0^2 + m_1^2 l_1^2 L_0^2\sin^2\theta_1 > 0$$

$D(q)$: *symmetric and positive definite for all q*

2.2 Euler – Lagrange dynamic equations

$$D(q) = \begin{bmatrix} I_0 + m_1(L_0^2 + l_1^2 \sin^2\theta_1) & m_1 l_1 L_0 \cos\theta_1 \\ m_1 l_1 L_0 \cos\theta_1 & J_1 + m_1 l_1^2 \end{bmatrix} \quad C(q,\dot{q}) = \begin{bmatrix} \frac{1}{2} m_1 l_1^2 \sin(2\theta_1)\dot{\theta}_1 & -m_1 l_1 L_0 \sin\theta_1 \dot{\theta}_1 + \frac{1}{2} m_1 l_1^2 \sin(2\theta_1)\dot{\theta}_0 \\ -\frac{1}{2} m_1 l_1^2 \sin(2\theta_1)\dot{\theta}_0 & 0 \end{bmatrix}$$

$$\dot{D}(q) - 2C(q,\dot{q}) = m_1 l_1 (l_1 \sin(2\theta_1)\dot{\theta}_0 - L_0 \sin\theta_1 \dot{\theta}_1) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$ *skew-symmetric matrix*

Passivity property of the Furuta Pendulum $z^T(\dot{D}(q) - 2C(q,\dot{q}))z = 0 \qquad \forall z$

Potential energy of the system: $P = m_1 g l_1 (\cos\theta_1 - 1)$

$$g(q) = \frac{\partial P}{\partial q} = \begin{bmatrix} 0 \\ -m_1 g l_1 \sin\theta_1 \end{bmatrix}$$

2.3 Passivity properties of the Furuta Pendulum

The total energy of the system:

$$E = K(q,\dot{q}) + P(q) = \tfrac{1}{2}\dot{q}^T D(q)\dot{q} + m_1 g l_1(\cos\theta_1 - 1)$$

Using $g(q) = \frac{\partial P}{\partial q} = \begin{bmatrix} 0 \\ -m_1 g l_1 \sin\theta_1 \end{bmatrix}$ $\dot{E}$ *can be found as:*

$$\dot{E} = \dot{q}^T D(q)\ddot{q} + \tfrac{1}{2}\dot{q}^T \dot{D}(q)\dot{q} + \dot{q}^T g(q)$$

Substitute $D(q)$ *from* $D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = F$

$$= \dot{q}^T(-C(q,\dot{q})\dot{q} - g(q) + F) + \tfrac{1}{2}\dot{q}^T \dot{D}(q)\dot{q} + \dot{q}^T g(q)$$

Substitute $\dot{D}(q)$ *from* $\dot{D}(q) - 2C(q,\dot{q}) = m_1 l_1(l_1 \sin(2\theta_1)\dot{\theta}_0 - L_0 \sin\theta_1 \dot{\theta}_1)\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$\dot{E} = \dot{q}^T F = \dot{\theta}_0 \tau$$

2.3 Passivity properties of the Furuta Pendulum

$\dot{E} = \dot{q}^T F = \dot{\theta}_0 \tau$ *Integrating both sides, we obtain;* $\int_0^t \dot{\theta}_0 \tau dt = E(t) - E(0) \geq -2m_1 g l_1 - E(0)$

τ: Input
$\dot{\theta}_0$: Output

$\theta_0 \in [0, 2\pi[$

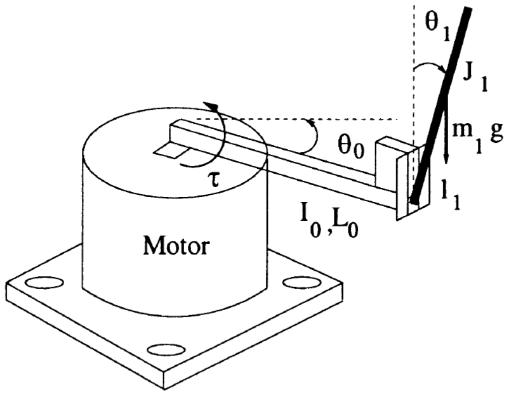
Subset of the system $D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = F$

$(\theta_0, \dot{\theta}_0, \theta_1, \dot{\theta}_1) = (*, 0, 0, 0)$ *: Unstable equilibrium set points (Total energy:* $E(q,\dot{q}) = 0$ *)*

$(\theta_0, \dot{\theta}_0, \theta_1, \dot{\theta}_1) = (*, 0, \pi, 0)$*: Stable equilibrium set points (Total energy:* $E(q,\dot{q}) = -2m_1 g l_1$*)*

Control objective: Stabilize the system around ***unstable equilibrium*** *point to achieve*

$\theta_0 = 0$	*Rotational angle of the arm*
$\theta_1 = 0$	*Rotational angle of the pendulum*



Bring ***pendulum to its upper position*** *and the* ***arm angle to zero*** *simultaneously.*

3 Controllability of the linearized model

When pendulum is in ***neighborhood*** *of* ***top unstable equilibrium*** *position* ***a linear controller*** *can* ***stabilize*** *the pendulum. In order to implement a* ***balancing linear controller***; ***linearize the system*** *about the* ***top equilibrium position***

$$\begin{aligned}\tau &= \left[I_0 + m_1(L_0^2 + l_1^2 \sin^2\theta_1)\right]\ddot{\theta}_0 + m_1 l_1 L_0 \cos\theta_1 \ddot{\theta}_1 \\ &\quad + m_1 l_1^2 \sin(2\theta_1)\dot{\theta}_0\dot{\theta}_1 - m_1 l_1 L_0 \sin\theta_1 \dot{\theta}_1^{\,2}\end{aligned}$$

$$\begin{aligned}0 &= m_1 l_1 L_0 \cos\theta_1 \ddot{\theta}_0 + \left[J_1 + m_1 l_1^2\right]\ddot{\theta}_1 \\ &\quad - m_1 l_1^2 \sin\theta_1 \cos\theta_1 \dot{\theta}_0^{\,2} - m_1 g l_1 \sin\theta_1\end{aligned}$$

The general non-linear equations are rewritten.

$$\begin{aligned}\ddot{\theta}_0 &= \frac{1}{\det(D(q))}\Big[(J_1 + m_1 l_1^2)\tau - (J_1 + m_1 l_1^2) m_1 l_1^2 \sin(2\theta_1)\dot{\theta}_0\dot{\theta}_1 \\ &\quad - \frac{1}{2} m_1^2 l_1^3 L_0 \cos\theta_1 \sin(2\theta_1)\dot{\theta}_0^{\,2} + (J_1 + m_1 l_1^2) m_1 l_1 L_0 \sin\theta_1 \dot{\theta}_1^{\,2} \\ &\quad - m_1^2 l_1^2 L_0 g \cos\theta_1 \sin\theta_1\Big]\end{aligned}$$

$$\begin{aligned}\ddot{\theta}_1 &= \frac{1}{\det(D(q))}\Big[-(m_1 l_1 L_0 \cos\theta_1)\tau - m_1^2 l_1^2 L_0^2 \sin\theta_1 \cos\theta_1 \dot{\theta}_1^{\,2} \\ &\quad + m_1 l_1^2 \sin(2\theta_1)\dot{\theta}_0 \left[m_1 l_1 L_0 \cos\theta_1 \dot{\theta}_1 + \frac{1}{2}(I_0 + m_1 L_0^2 + l_1^2 \sin^2\theta_1)\dot{\theta}_0\right] \\ &\quad + (I_0 + m_1 L_0^2 + l_1^2 \sin^2\theta_1) m_1 l_1 g \sin\theta_1\Big]\end{aligned}$$

3 Controllability of the linearized model

Linearizing the non-linear equations about the top unstable equilibrium point;

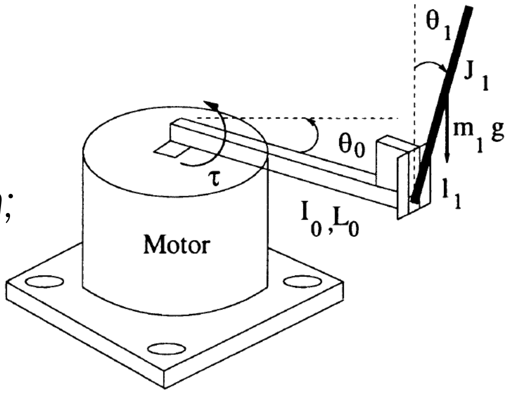
$$\frac{d}{dt}\begin{bmatrix}\theta_0\\ \dot\theta_0\\ \theta_1\\ \dot\theta_1\end{bmatrix} = \underbrace{\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & \frac{-m_1^2 l_1^2 L_0 g}{I_0(J_1+m_1l_1^2)+J_1m_1L_0^2} & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & \frac{(I_0+m_1L_0^2)m_1l_1g}{I_0(J_1+m_1l_1^2)+J_1m_1L_0^2} & 0\end{bmatrix}}_{A}\begin{bmatrix}\theta_0\\ \dot\theta_0\\ \theta_1\\ \dot\theta_1\end{bmatrix} + \underbrace{\begin{bmatrix}0\\ \frac{J_1+m_1l_1^2}{I_0(J_1+m_1l_1^2)+J_1m_1L_0^2}\\ 0\\ \frac{-m_1l_1L_0}{I_0(J_1+m_1l_1^2)+J_1m_1L_0^2}\end{bmatrix}}_{B}\tau = AX + B\tau$$

$$B = \begin{bmatrix}0\\ \frac{J_1+m_1l_1^2}{I_0(J_1+m_1l_1^2)+J_1m_1L_0^2}\\ 0\\ \frac{-m_1l_1L_0}{I_0(J_1+m_1l_1^2)+J_1m_1L_0^2}\end{bmatrix}\quad AB = \begin{bmatrix}\frac{J_1+m_1l_1^2}{I_0(J_1+m_1l_1^2)+J_1m_1L_0^2}\\ 0\\ \frac{-m_1l_1L_0}{I_0(J_1+m_1l_1^2)+J_1m_1L_0^2}\\ 0\end{bmatrix}\quad A^2B = \begin{bmatrix}0\\ \frac{m_1^3l_1^3L_0^2g}{(I_0(J_1+m_1l_1^2)+J_1m_1L_0^2)^2}\\ 0\\ \frac{-m_1^2l_1^2L_0g(I_0+m_1L_0^2)}{(I_0(J_1+m_1l_1^2)+J_1m_1L_0^2)^2}\end{bmatrix}\quad A^3B = \begin{bmatrix}\frac{m_1^3l_1^3L_0^2g}{(I_0(J_1+m_1l_1^2)+J_1m_1L_0^2)^2}\\ 0\\ \frac{-m_1^2l_1^2L_0g(I_0+m_1L_0^2)}{(I_0(J_1+m_1l_1^2)+J_1m_1L_0^2)^2}\\ 0\end{bmatrix}$$

$$\det\left(B|AB|A^2B|A^3B\right) = \frac{m_1^4l_1^4L_0^2g^2}{(I_0(J_1+m_1l_1^2)+J_1m_1L_0^2)^4} > 0$$ *The linearized system is controllable*

Full state feedback can successfully stabilize the system in a neighborhood of its unstable equilibrium point $(f = -K^TX)$

4 Stabilization algorithm



$$E = K(q,\dot q) + P(q) = \tfrac{1}{2}\dot q^T D(q)\dot q + m_1 g l_1(\cos\theta_1 - 1)$$

If: $\dot\theta_0 = 0$ *and* $E(q,\dot q) = 0$ *then;*

$$\frac{1}{2}(J_1 + m_1 l_1^2)\dot\theta_1^2 = m_1 g l_1(1-\cos\theta_1)$$

$\dot\theta_1 = 0$ *only when* $\theta_1 = 0$

Means the pendulum angular position moves clockwise or counter-clockwise until it reaches the equilibrium point $(\theta_1, \dot\theta_1) = (0,0)$ *which solves "swinging up" the pendulum. Control must eventually be switched to a controller that guarantees (local) asymptotic stability of this equilibrium.*

Lyapunov function candidate to achieve $\dot\theta_0 = 0,\ \theta_0 = 0\ and\ E(q,\dot q) = 0$ $\quad V(q,\dot q) = \frac{k_E}{2}E(q,\dot q)^2 + \frac{k_\omega}{2}\dot\theta_0^2 + \frac{k_\theta}{2}\theta_0^2$

$V(q,\dot q)$ *is a positive semi-definite. Using* $\dot E = \dot q^T F = \dot\theta_0 \tau$

$$\dot V = k_E E\dot E + k_\omega \dot\theta_0\ddot\theta_0 + k_\theta\theta_0\dot\theta_0 = k_E E\dot\theta_0\tau + k_\omega\dot\theta_0\ddot\theta_0 + k_\theta\theta_0\dot\theta_0 = \dot\theta_0(k_E E\tau + k_\omega\ddot\theta_0 + k_\theta\theta_0)$$

$$D^{-1}(q) = \frac{1}{[\det(D(q))]}\begin{bmatrix} J_1 + m_1 l_1^2 & -m_1 l_1 L_0\cos\theta_1 \\ -m_1 l_1 L_0 \cos\theta_1 & I_0 + m_1(L_0^2 + l_1^2\sin^2\theta_1)\end{bmatrix}$$

4 Stabilization algorithm

$$\dot{V} = k_E E\dot{E} + k_\omega \dot{\theta}_0\ddot{\theta}_0 + k_\theta \theta_0\dot{\theta}_0 = k_E E\dot{\theta}_0\tau + k_\omega \dot{\theta}_0\ddot{\theta}_0 + k_\theta \theta_0\dot{\theta}_0 = \dot{\theta}_0(k_E E\tau + k_\omega \ddot{\theta}_0 + k_\theta \theta_0)$$

$$D^{-1}(q) = \frac{1}{[\det(D(q))]}\begin{bmatrix} J_1 + m_1 l_1^2 & -m_1 l_1 L_0 \cos\theta_1 \\ -m_1 l_1 L_0 \cos\theta_1 & I_0 + m_1(L_0^2 + l_1^2 \sin^2\theta_1) \end{bmatrix}$$

$$\det(D(q)) = (I_0 + m_1 l_1^2 \sin^2\theta_1)(J_1 + m_1 l_1^2) + J_1 m_1 L_0^2 + m_1^2 l_1^2 L_0^2 \sin^2\theta_1$$

Using

$$\ddot{\theta}_0 = \frac{1}{\det(D(q))}\Big[(J_1 + m_1 l_1^2)\tau - (J_1 + m_1 l_1^2) m_1 l_1^2 \sin(2\theta_1)\dot{\theta}_0\dot{\theta}_1 - \frac{1}{2} m_1^2 l_1^3 L_0 \cos\theta_1 \sin(2\theta_1)\dot{\theta}_0^{\,2} + (J_1 + m_1 l_1^2) m_1 l_1 L_0 \sin\theta_1 \dot{\theta}_1^{\,2} - m_1^2 l_1^2 L_0 g \cos\theta_1 \sin\theta_1\Big]$$

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = F$$

$$F(q,\dot{q}) = \Big[-(J_1 + m_1 l_1^2) m_1 l_1^2 \sin(2\theta_1)\dot{\theta}_0\dot{\theta}_1 - \frac{1}{2} m_1^2 l_1^3 L_0 \cos\theta_1 \sin(2\theta_1)\dot{\theta}_0^{\,2} + (J_1 + m_1 l_1^2) m_1 l_1 L_0 \sin\theta_1 \dot{\theta}_1^{\,2} - m_1^2 l_1^2 L_0 g \cos\theta_1 \sin\theta_1\Big]$$

4 Stabilization algorithm

$$\ddot{\theta}_0 = \frac{1}{\det(D(q))}\left[(J_1 + m_1 l_1^2)\tau + F(q,\dot{q})\right]$$ *Substituting on derivative of Lyapunov Function* $$\dot{V} = \dot{\theta}_0(k_E E\tau + k_\omega \ddot{\theta}_0 + k_\theta \theta_0)$$

$$\dot{V} = \dot{\theta}_0\left[\tau\left(k_E E + \frac{k_\omega (J_1 + m_1 l_1^2)}{\det(D(q))}\right) + \frac{k_\omega F(q,\dot{q})}{\det(D(q))} + k_\theta \theta_0\right]$$ *Propose a control law such that;*

$$\tau\left(k_E E + \frac{k_\omega (J_1 + m_1 l_1^2)}{\det(D(q))}\right) + \frac{k_\omega F(q,\dot{q})}{\det(D(q))} + k_\theta \theta_0 = -k_\delta \dot{\theta}_0$$

Which will lead to $\dot{V} = -k_\delta \dot{\theta}_0^2$

$$\left(k_E E + \frac{k_\omega (J_1 + m_1 l_1^2)}{\det(D(q))}\right) \neq 0$$ *No singularity* $E \geq -2m_1 g l_1$ $$\frac{k_\omega (J_1 + m_1 l_1^2)}{\max_{\theta_1}(\det(D(q)))} > k_E(2m_1 g l_1)$$

$$\frac{k_\omega}{k_E} > 2m_1 g l_1 (I_0 + m_1 l_1^2 + m_1 L_0^2)$$

The pendulum can still get stuck at the (lower) stable equilibrium point, $(\theta_0, \dot{\theta}_0, \theta_1, \dot{\theta}_1) = (0, 0, \pi, 0)$ *for* $\tau = 0$ *occurs when* $E(q,\dot{q}) = -2m_1 g l_1$

$$|E| < 2m_1 g l_1$$

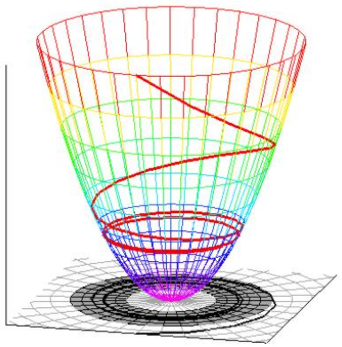
4 Stabilization algorithm

Since V is a non-increasing function $\dot{V} = -k_\delta \dot{\theta}_0^2$

$|E| < 2m_1 g l_1$ *will hold if the initial conditions are* $V(0) < 2k_E m_1^2 g^2 l_1^2$ *Which is region of attraction*

Control law can be written as;

$$\tau = \frac{-k_\omega F(q,\dot{q}) - \det(D(q))\left(k_\delta \dot{\theta}_0 + k_\theta \theta_0\right)}{\det(D(q)) k_E E + k_\omega (J_1 + m_1 l_1^2)}$$



5 Stability analysis

In order to apply LaSalle .s theorem, it is required to define a compact (closed and bounded) set Ω *with the property that every solution of system*

$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = F$ *that starts in* Ω *remains in* Ω *for all future time.*

$V(q,\dot{q}) = \frac{k_E}{2}E(q,\dot{q})^2 + \frac{k_\omega}{2}\dot{\theta}_0^2 + \frac{k_\theta}{2}\theta_0^2$ *non-increasing function* $\dot{V} = -k_\delta \dot{\theta}_0^2$ *Than,* $\theta_0, \dot{\theta}_0, \dot{\theta}_1$ *are bounded.*

Than, $cos\theta_0, sin\theta_0, cos\theta_1, sin\theta_1$ *are bounded.*

Define a state z of the closed loop system composed of: $\theta_0, \dot{\theta}_0, cos\theta_1, sin\theta_1, \dot{\theta}_1$ *so that solution of closed loop system* $\dot{z} = F(z)$ *remains inside a compact set* Ω.

LaSalle "s theorem ensures that every solution starting in Ω *approaches M (largest invariant set) as* $t \to \infty$

controllaw has been chosen $-k_\delta \dot{\theta}_0 = k_E E\tau + k_\omega \ddot{\theta}_0 + k_\theta \theta_0$ E_τ *is constant in set of all points in* Ω

Either $E = 0$ *or* $E \neq 0$

5 Stability analysis

$$-k_\delta \dot{\theta}_0 = k_E E\tau + k_\omega \ddot{\theta}_0 + k_\theta \theta_0$$

Case a: *If* $E = 0$, $\theta_0 = 0$, τ *is bounded.* $E = 0$ *means the trajectories are in the homoclinic orbit.* $\theta_0, \dot{\theta}_0$ *and* E *converge to zero.* τ *does not necessarily converge to zero.*

Case b: *If* $E \neq 0, E_\tau$ *is constant, then* τ *is also constant. If* $E \neq 0$ *then* $\tau = 0$ *in set of all points in* Ω

Theorem: *Consider the Furuta Pendulum system;*

$$\begin{aligned}
\tau &= \left[I_0 + m_1(L_0^2 + l_1^2 \sin^2\theta_1)\right]\ddot{\theta}_0 + m_1 l_1 L_0 \cos\theta_1 \ddot{\theta}_1 \\
&\quad + m_1 l_1^2 \sin(2\theta_1)\dot{\theta}_0\dot{\theta}_1 - m_1 l_1 L_0 \sin\theta_1 \dot{\theta}_1^{\,2} \\
0 &= m_1 l_1 L_0 \cos\theta_1 \ddot{\theta}_0 + \left[J_1 + m_1 l_1^2\right]\ddot{\theta}_1 \\
&\quad - m_1 l_1^2 \sin\theta_1 \cos\theta_1 \dot{\theta}_0^{\,2} - m_1 g l_1 \sin\theta_1
\end{aligned}$$

and controller (with strictly positive constant $k_E\ k_w\ k_\theta\ k_\delta$*)*

$$\tau = \frac{-k_\omega F(q,\dot{q}) - \det(D(q))\left(k_\delta\dot{\theta}_0 + k_\theta\theta_0\right)}{\det(D(q))k_E E + k_\omega(J_1 + m_1 l_1^2)}$$

The state initial conditions satisfy the inequality;

$V(0) < 2k_E m_1^2 g^2 l_1^2$ *Region of attraction*

Solution of the closed-loop system converges to the invariant set M given by the homoclinic orbit

$$\frac{1}{2}(J_1 + m_1 l_1^2)\dot{\theta}_1^2 = m_1 g l_1(1 - \cos\theta_1)$$

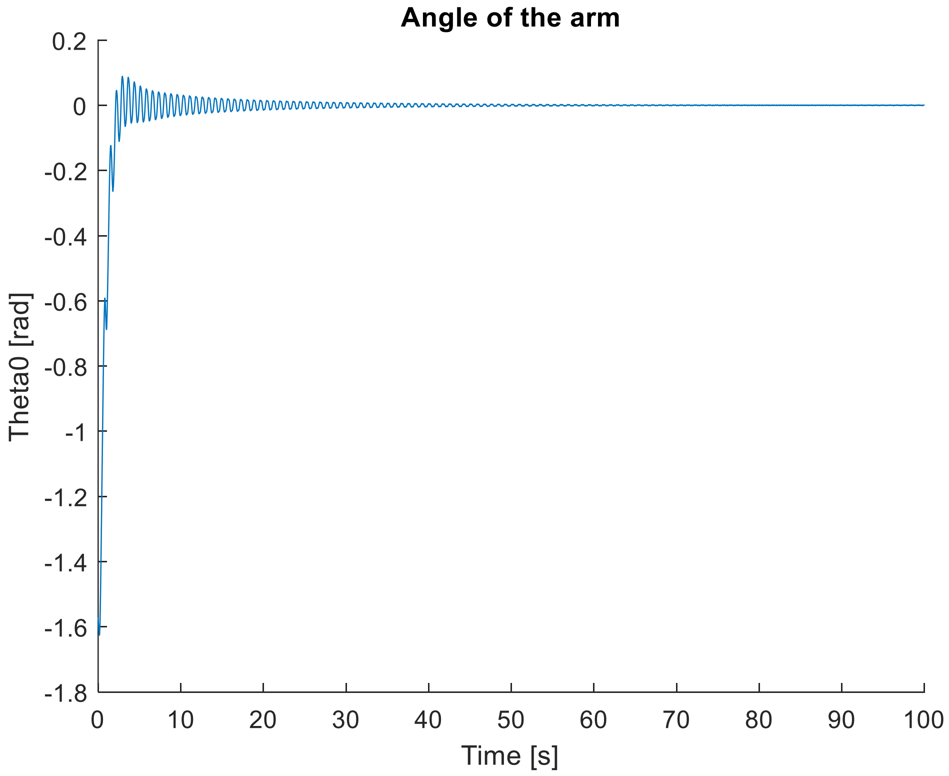
$(\theta_0, \dot{\theta}_0) = (0{,}0)$ and τ *does not necessarily converge to zero*

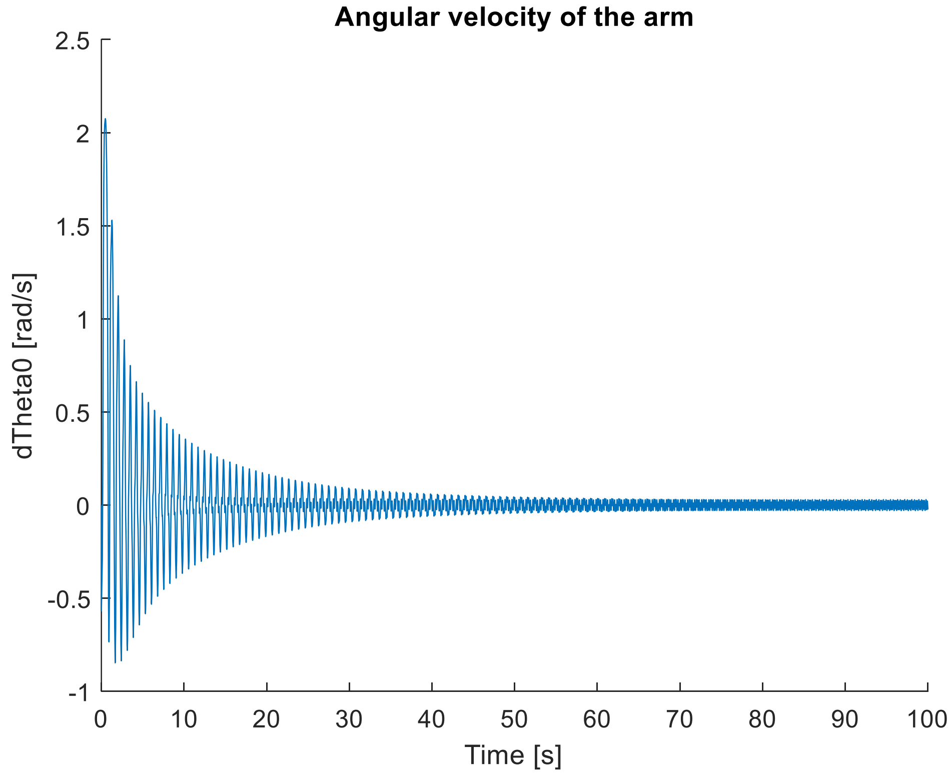
Initial position: $\theta_0 = -\frac{\pi}{2}$ $\dot{\theta}_0 = 0$ $\theta_1 = -\frac{2.5\pi}{3}$ $\dot{\theta}_1 = 0$

$k = 1$ $k_E = 480$ $k_w = 1$ $k_\theta = 1$ $k_\delta = 1$

6 Simulation Results

Swing-up





Initial position: $\theta_0 = -\frac{\pi}{2} \quad \dot{\theta}_0 = 0 \quad \theta_1 = -\frac{2.5\pi}{3} \quad \dot{\theta}_1 = 0$

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6 Simulation Results

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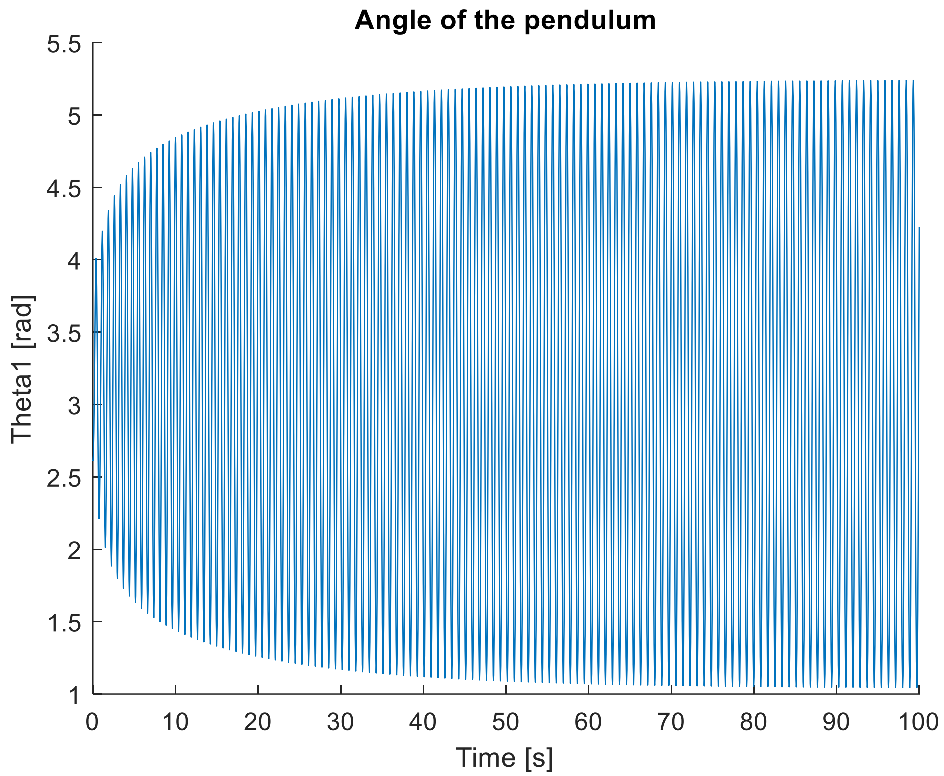

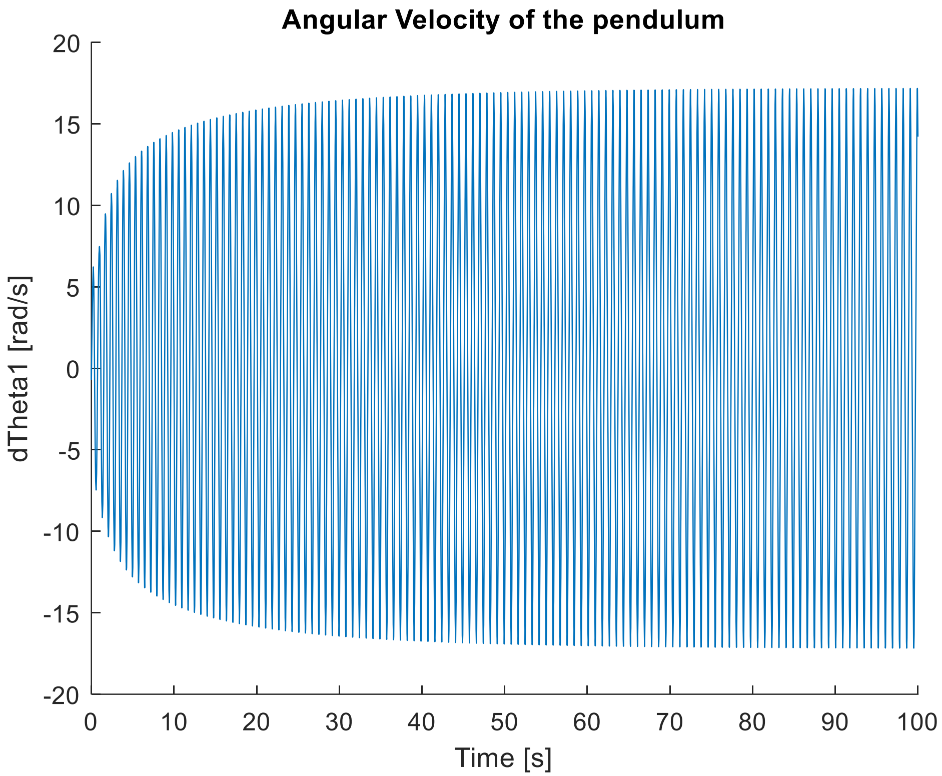

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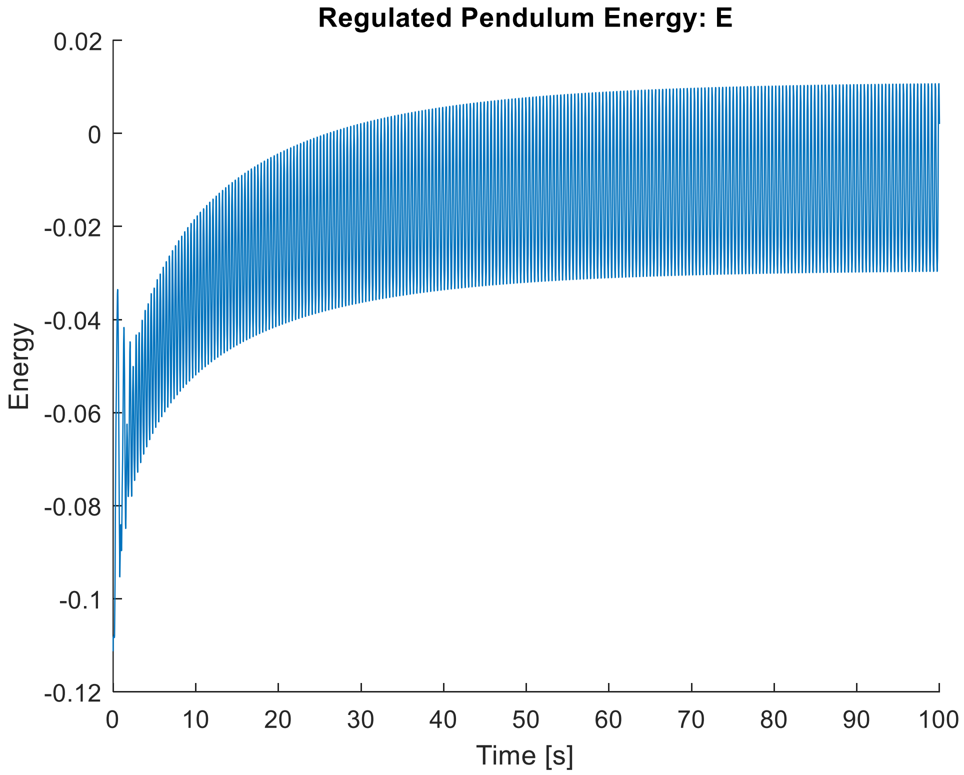

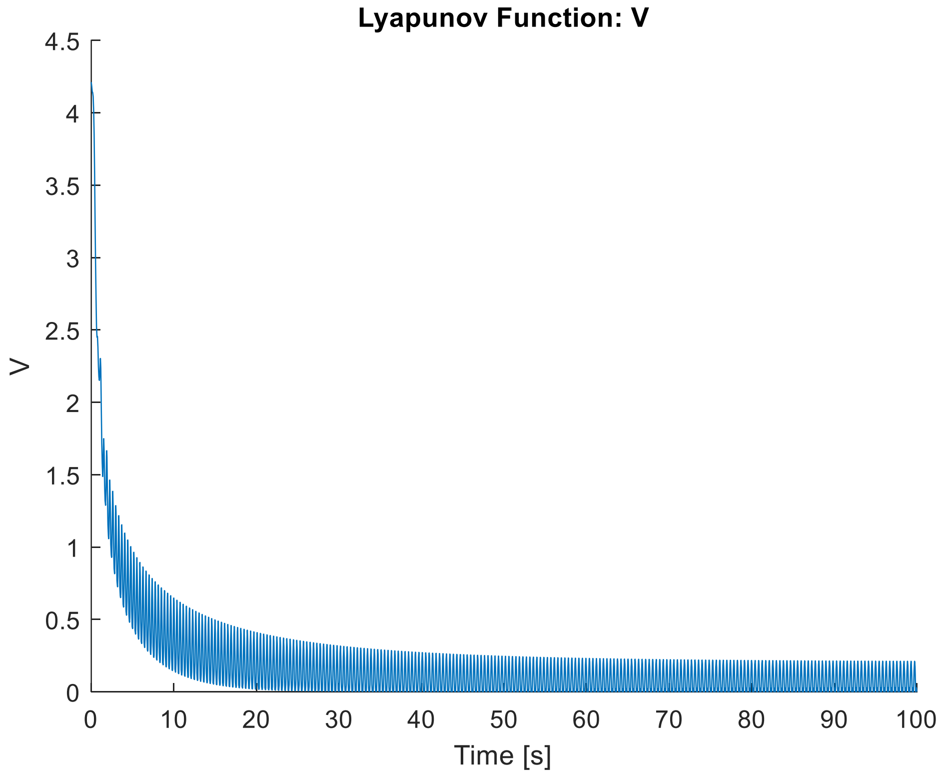

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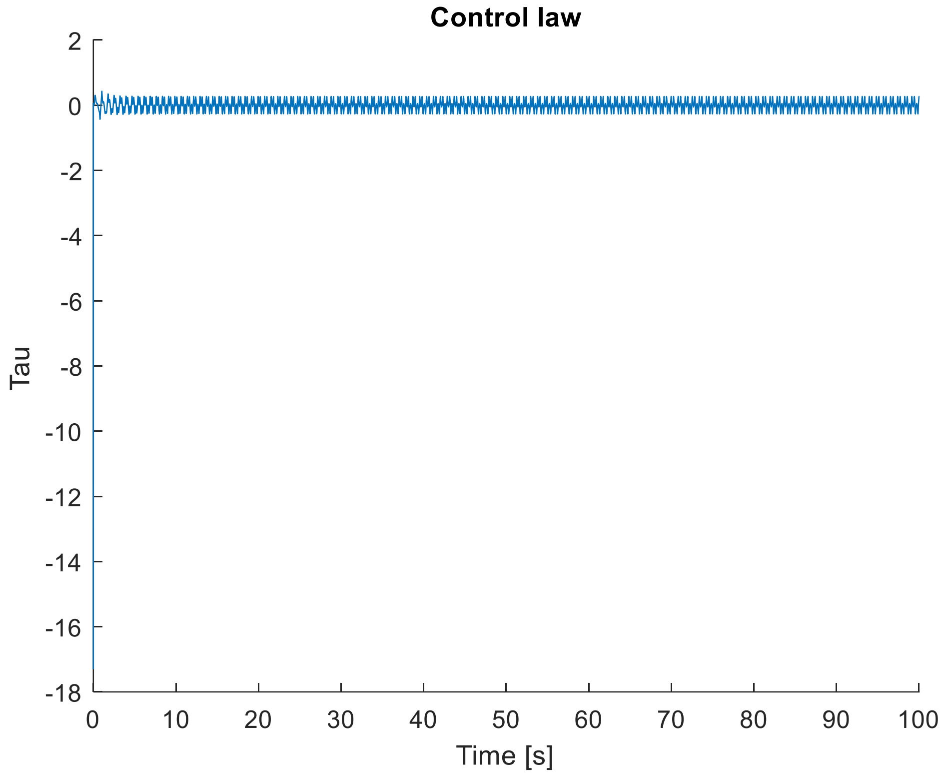
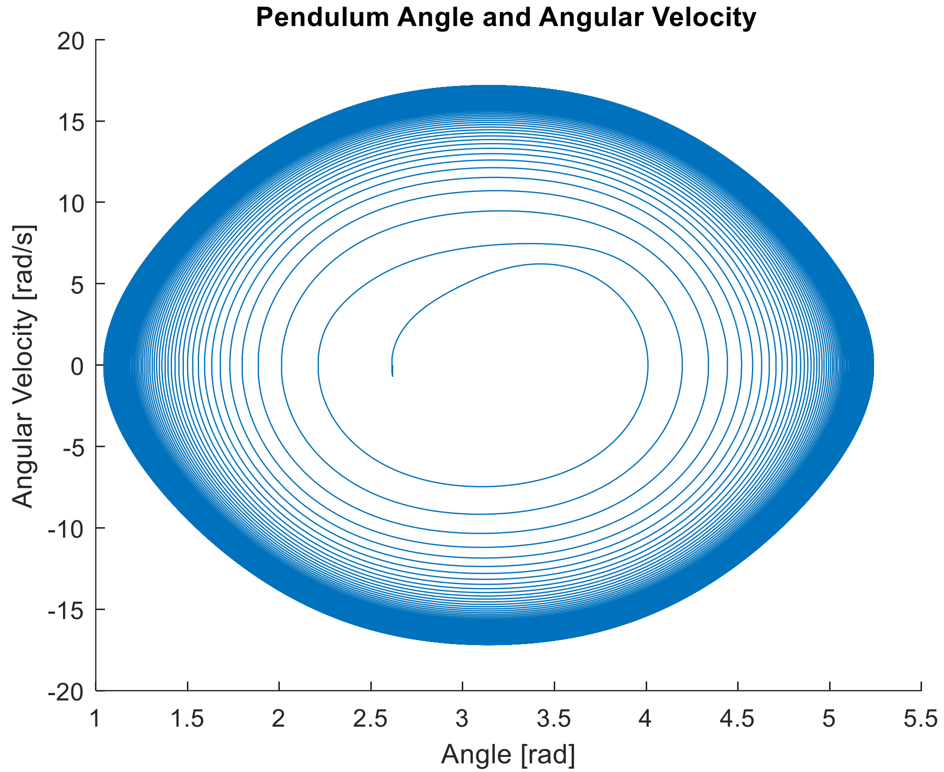
6 Simulation Results

Swing-up





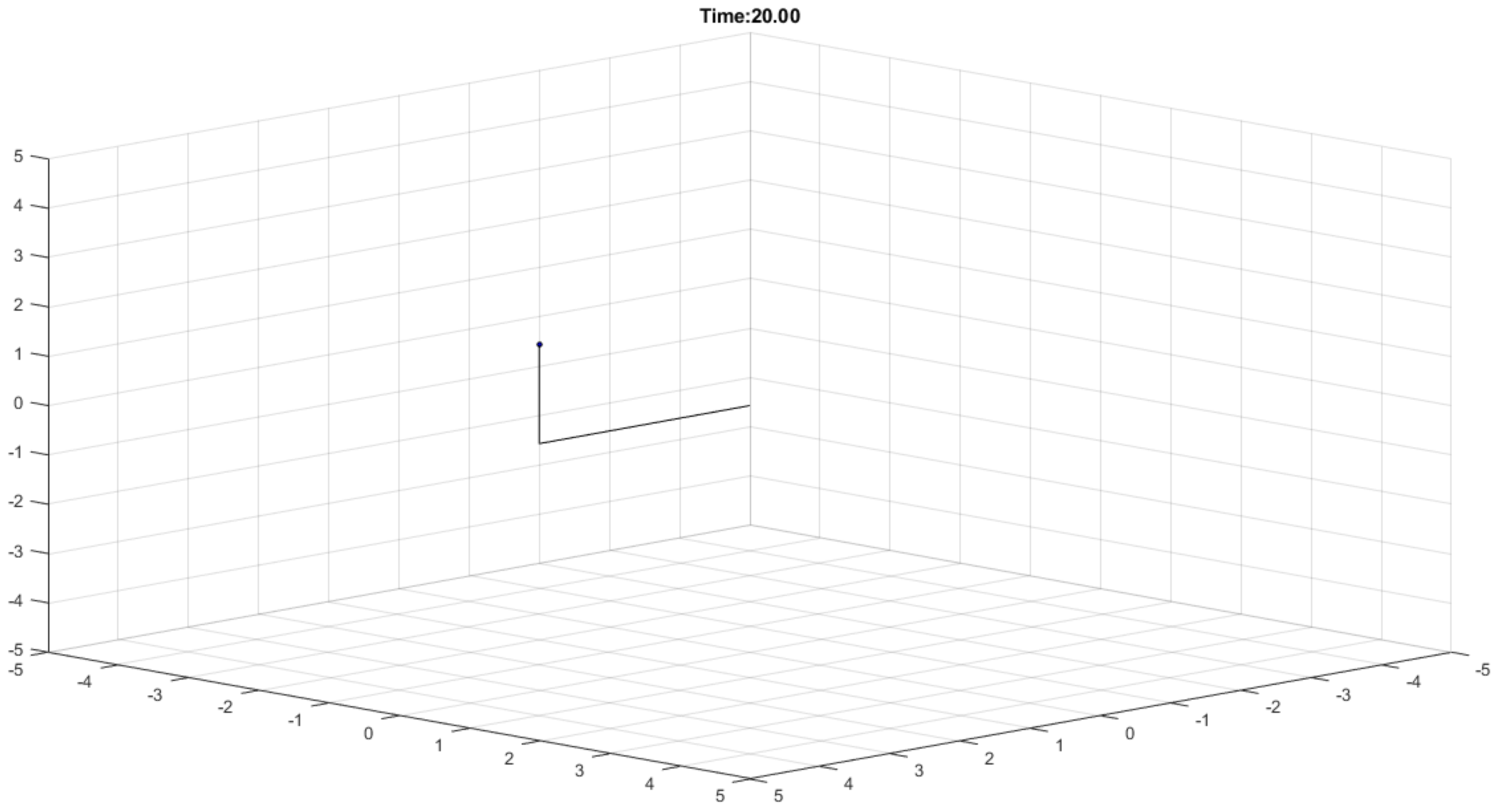
6 Simulation Results

Swing-up & Balancing

Initial position: $\theta_0 = -\frac{\pi}{2} \quad \dot{\theta}_0 = 0 \quad \theta_1 = -\frac{2.5\pi}{3} \quad \dot{\theta}_1 = 0$

$k = 6 \; k_E = 480 \quad k_w = 1 \quad k_\theta = 1 \quad k_\delta = 1$

K=[−0.1000 −0.1535 −6.0576 −1.1749]



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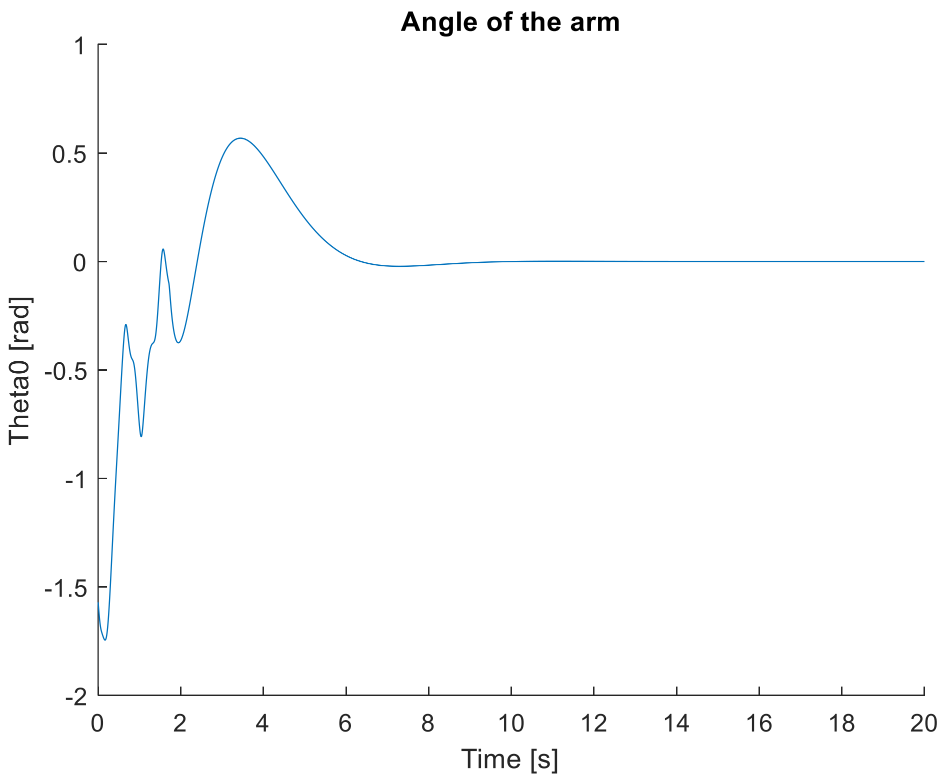
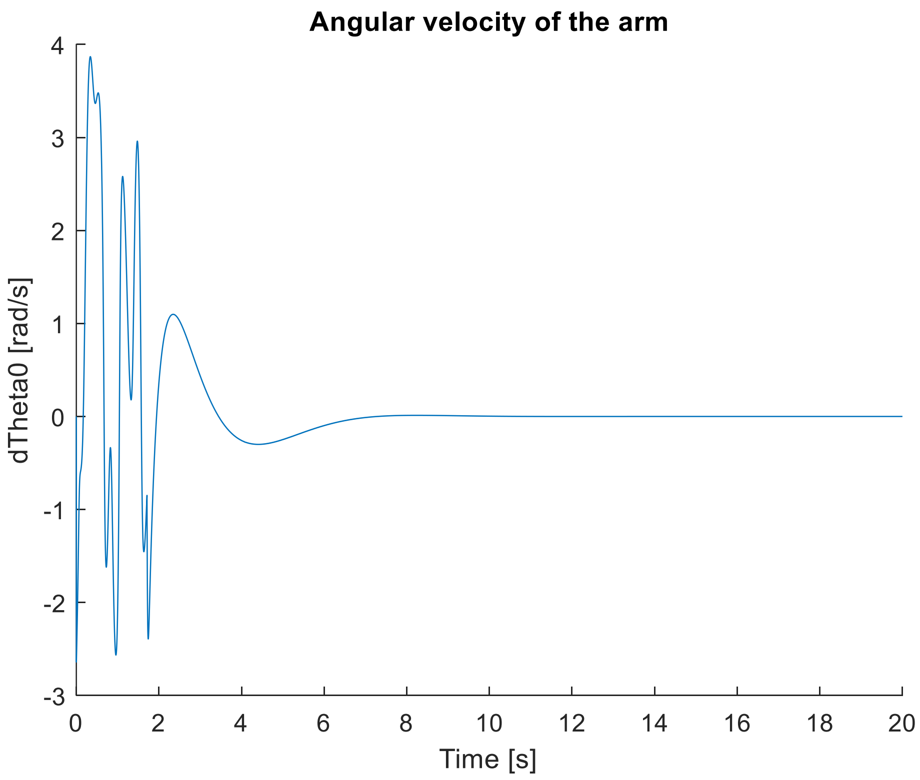
6 Simulation Results

Swing-up & Balancing





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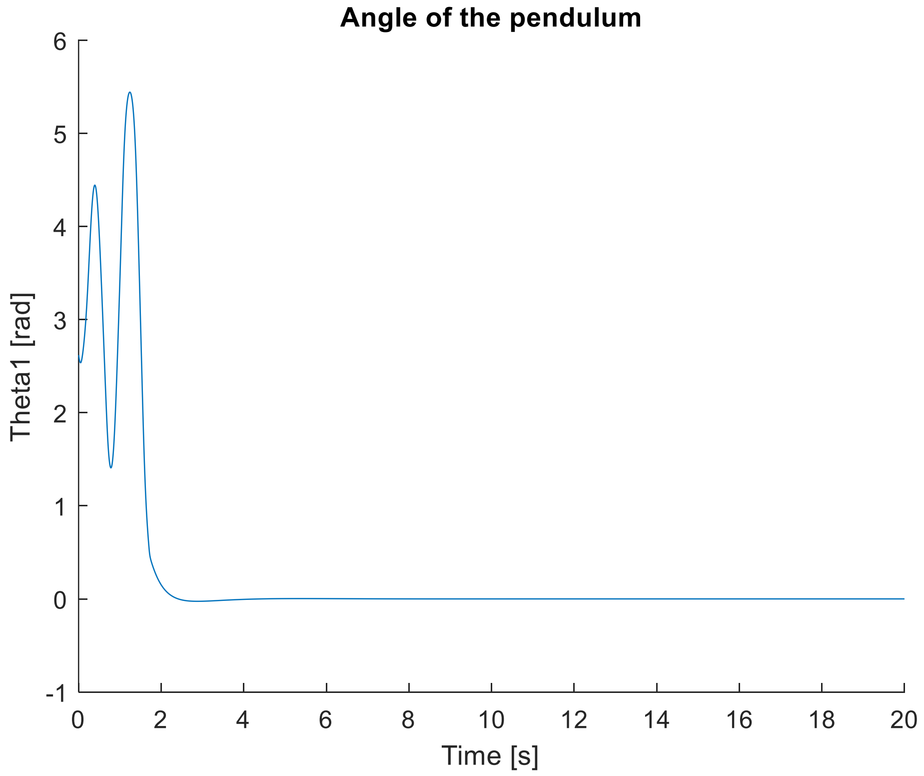

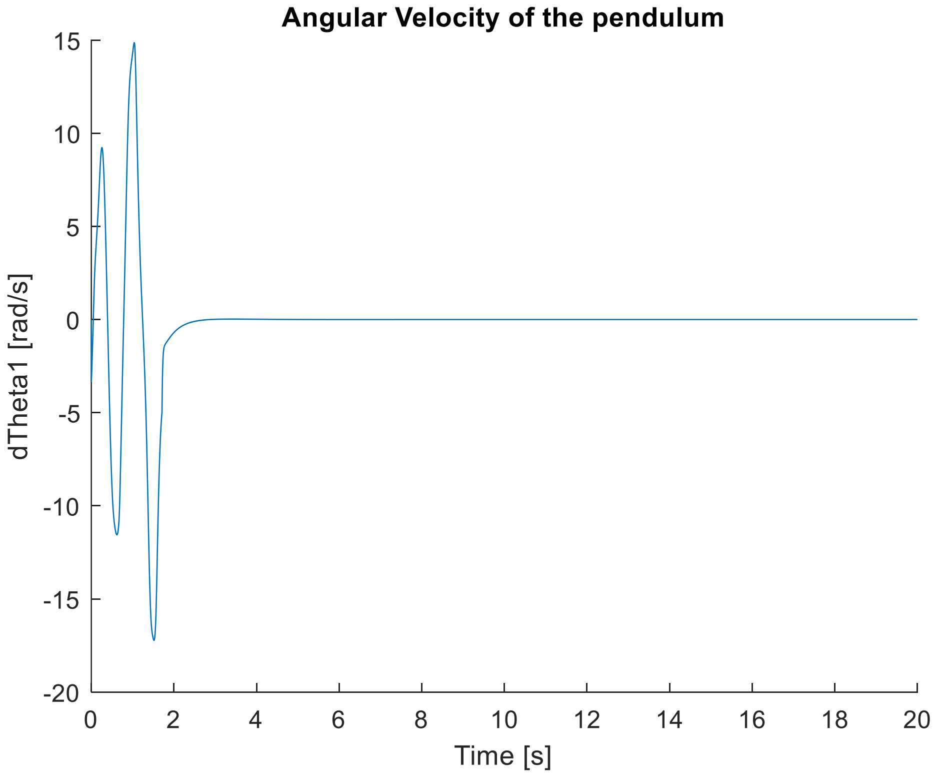

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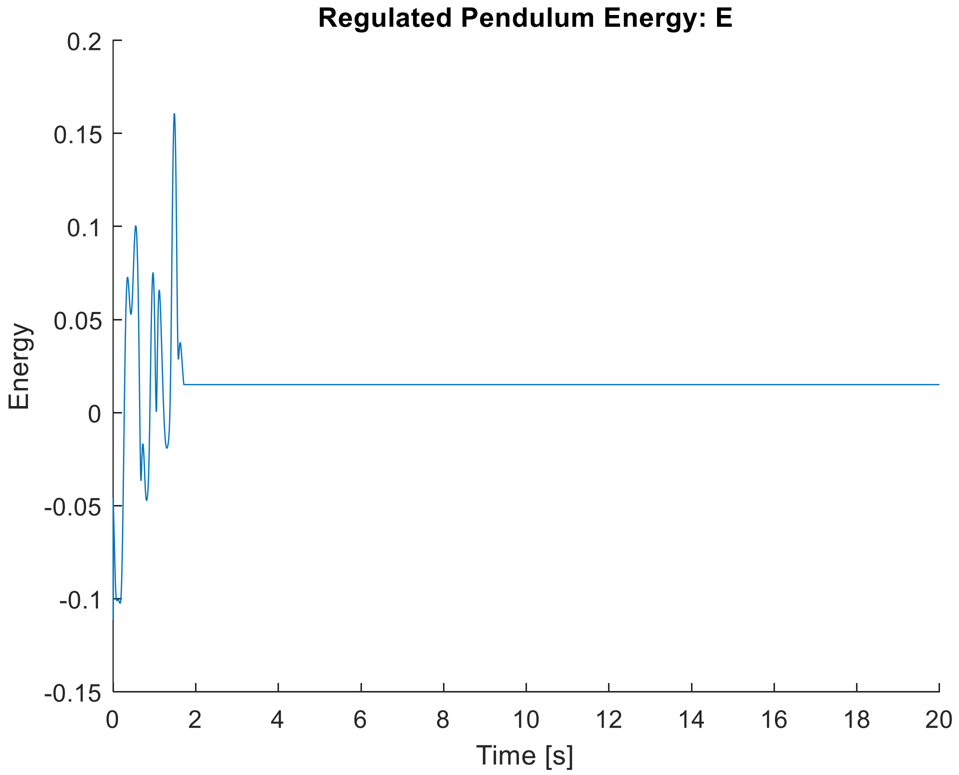

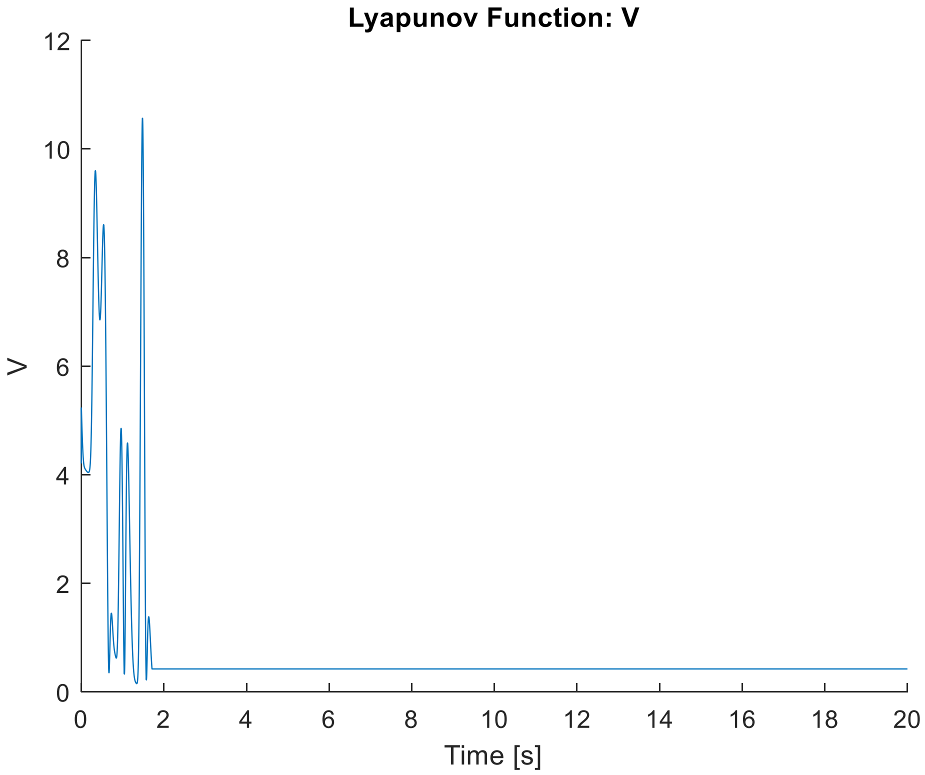

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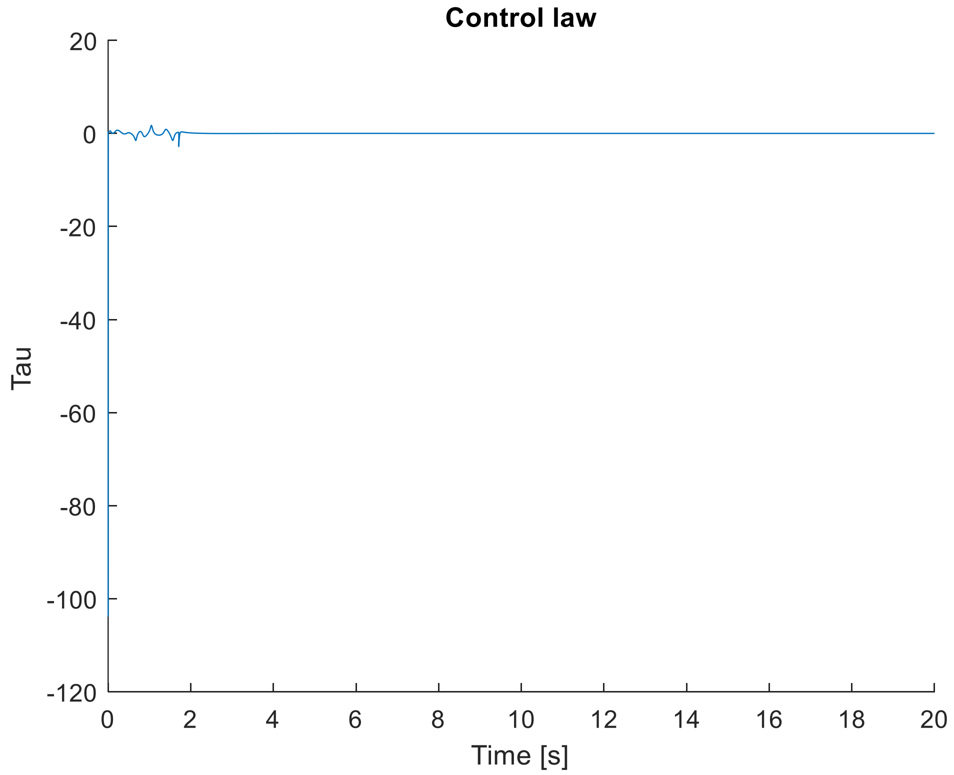

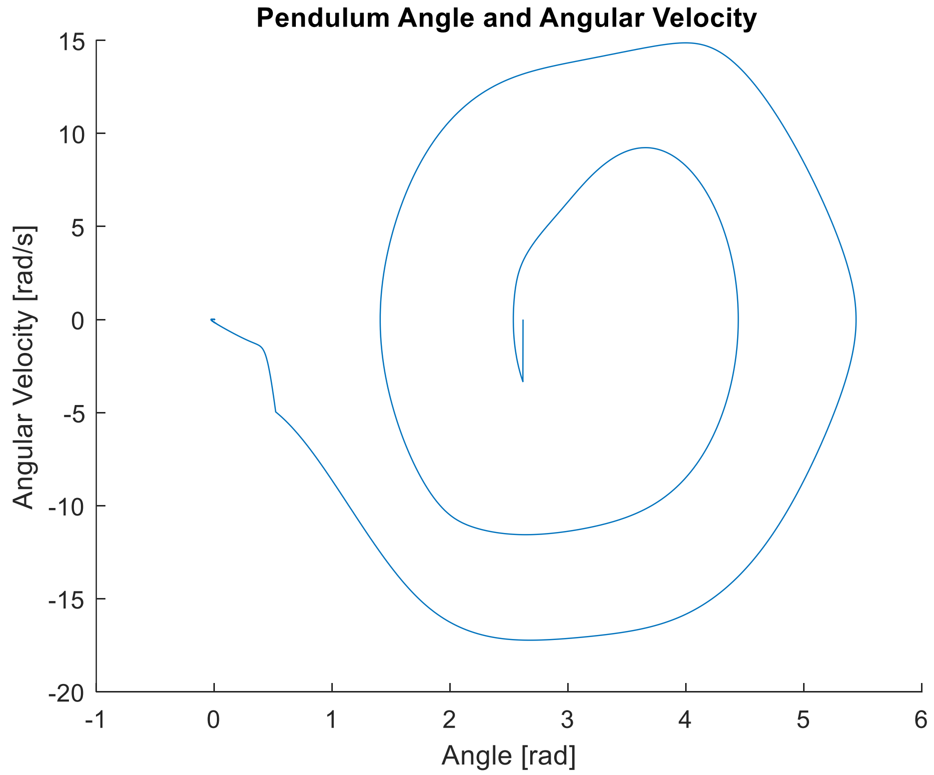

7 Conclusion

Proposed a control strategy to "swing up" and state-feedback implemented to balance the Furuta pendulum.

The control design is based on the passivity properties of this rotational inverted pendulum.

Convergence of the trajectories of the system to a homoclinic orbit has been proved by using LaSalle's invariance theorem.

8 References

Isabelle Fantoni and Rogelio Lozano. 2001. Non-Linear Control for Underactuated Mechanical Systems. Springer-Verlag, Berlin, Heidelberg.

THANKS

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