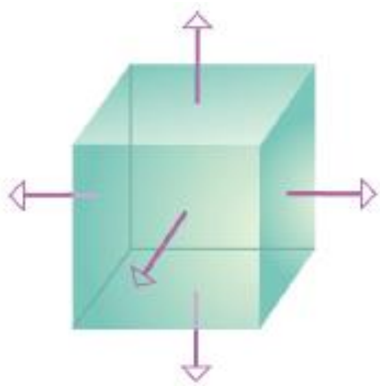
ORIENTATION OF PIECEWISE SMOOTH CLOSED SURFACES

In the last section we studied flux across general surfaces. Here we will be concerned exclusively with surfaces that are boundaries of finite solids—the surface of a solid sphere, the surface of a solid box, or the surface of a solid cylinder, for example. Such surfaces are said to be ***closed***. A closed surface may or may not be smooth, but most of the surfaces that arise in applications are ***piecewise smooth***; that is, they consist of finitely many smooth surfaces joined together at the edges (a box, for example). We will limit our discussion to piecewise smooth surfaces that can be assigned an ***inward orientation*** (toward the interior of the solid) and an ***outward orientation*** (away from the interior). It is difficult to make this concept mathematically precise, but the basic idea is that each piece of the surface is orientable, and oriented pieces fit together in such a way that the entire surface can be assigned an orientation (Figure 15.7.1).



Box with outward orientation

Figure 15.7.1

THE DIVERGENCE THEOREM:

In Section 15.1 we defined the divergence of a vector field

$$\mathbf{F}(x, y, z) = f(x, y, z)\mathbf{i} + g(x, y, z)\mathbf{j} + h(x, y, z)\mathbf{k}$$

as

$$\text{div } \mathbf{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

but we did not attempt to give a physical explanation of its meaning at that time. The following result, known as the ***Divergence Theorem*** or ***Gauss's Theorem***, will provide us with a physical interpretation of divergence in the context of fluid flow.

15.7.1 THEOREM (*The Divergence Theorem*) *Let G be a solid whose surface σ is oriented outward. If*

$$\mathbf{F}(x, y, z) = f(x, y, z)\mathbf{i} + g(x, y, z)\mathbf{j} + h(x, y, z)\mathbf{k}$$

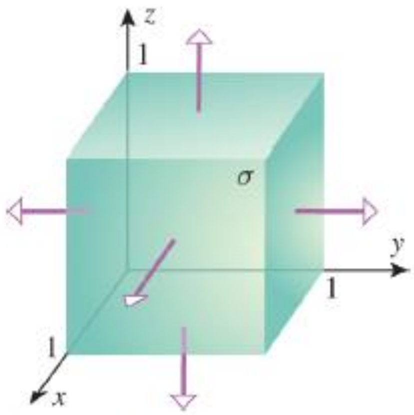
where f, g, and h have continuous first partial derivatives on some open set containing G, and if $\mathbf{n}$ is the outward unit normal on σ, then

$$\iint_{\sigma} \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_{G} \text{div } \mathbf{F}\, dV \qquad (1)$$

▶ Example 2 Use the Divergence Theorem to find the outward flux of the vector field

$$\mathbf{F}(x, y, z) = 2x\mathbf{i} + 3y\mathbf{j} + z^2\mathbf{k}$$

across the unit cube in Figure 15.7.3.



▲ Figure 15.7.3

Solution. Let σ denote the outward-oriented surface of the cube and G the region that it encloses. The divergence of the vector field is

$$\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(3y) + \frac{\partial}{\partial z}(z^2) = 5 + 2z$$

so from (1) the flux across σ is

$$\Phi = \iint_\sigma \mathbf{F}\cdot\mathbf{n}\, dS = \iiint_G (5+2z)\, dV = \int_0^1\int_0^1\int_0^1 (5+2z)\, dz\, dy\, dx$$

$$= \int_0^1\int_0^1 \left[5z + z^2\right]_{z=0}^{1} dy\, dx = \int_0^1\int_0^1 6\, dy\, dx = 6$$

EXERCISE SET 15.7 C CAS

1–4 Verify Formula (1) in the Divergence Theorem by evaluating the surface integral and the triple integral.

1. $\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$; σ is the surface of the cube bounded by the planes $x=0, x=1, y=0, y=1, z=0, z=1$.

3. $\mathbf{F}(x,y,z) = 2x\mathbf{i} - yz\mathbf{j} + z^2\mathbf{k}$; the surface σ is the paraboloid $z = x^2 + y^2$ capped by the disk $x^2 + y^2 \le 1$ in the plane $z = 1$.

4. $\mathbf{F}(x,y,z) = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$; σ is the surface of the cube bounded by the planes $x=0$, $x=2$, $y=0$, $y=2$, $z=0$, $z=2$.

Solution:

1. $\sigma_1 : x = 0, \mathbf{F}\cdot\mathbf{n} = -x = 0, \iint_{\sigma_1} (0)dA = 0,$ $\quad \sigma_2 : x = 1, \mathbf{F}\cdot\mathbf{n} = x = 1, \iint_{\sigma_2} (1)dA = 1,$

$\sigma_3 : y = 0, \mathbf{F}\cdot\mathbf{n} = -y = 0, \iint_{\sigma_3} (0)dA = 0,$ $\quad \sigma_4 : y = 1, \mathbf{F}\cdot\mathbf{n} = y = 1, \iint_{\sigma_4} (1)dA = 1,$

$\sigma_5 : z = 0, \mathbf{F}\cdot\mathbf{n} = -z = 0, \iint_{\sigma_5} (0)dA = 0,$ $\quad \sigma_6 : z = 1, \mathbf{F}\cdot\mathbf{n} = z = 1, \iint_{\sigma_6} (1)dA = 1.$

$$\iint_\sigma \mathbf{F}\cdot\mathbf{n} = 3; \quad \iiint_G \operatorname{div}\, \mathbf{F} dV = \iiint_G 3dV = 3.$$

3. $\sigma_1 : z = 1, \mathbf{n} = \mathbf{k}, \mathbf{F} \cdot \mathbf{n} = z^2 = 1, \iint_{\sigma_1} (1) dS = \pi$, $\sigma_2 : \mathbf{n} = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k}, \mathbf{F} \cdot \mathbf{n} = 4x^2 - 4x^2y^2 - x^4 - 3y^4$, $\iint_{\sigma_2} \mathbf{F} \cdot \mathbf{n}\, dS = \int_0^{2\pi} \int_0^1 \left[4r^2 \cos^2\theta - 4r^4 \cos^2\theta \sin^2\theta - r^4 \cos^4\theta - 3r^4 \sin^4\theta\right] r\, dr\, d\theta = \frac{\pi}{3}$;

$\iint_{\sigma} \mathbf{F} \cdot \mathbf{n}\, dS = \frac{4\pi}{3}$; $\iiint_G \operatorname{div} \mathbf{F} dV = \iiint_G (2+z) dV = \int_0^{2\pi} \int_0^1 \int_{r^2}^1 (2+z) dz\, r\, dr\, d\theta = 4\pi/3$.

4. $\sigma_1 : x = 0, \mathbf{F} \cdot \mathbf{n} = -xy = 0, \iint_{\sigma_1} (0) dA = 0,$ $\sigma_2 : x = 2, \mathbf{F} \cdot \mathbf{n} = xy = 2y, \iint_{\sigma_2} (2y) dA = 8,$

$\sigma_3 : y = 0, \mathbf{F} \cdot \mathbf{n} = -yz = 0, \iint_{\sigma_3} (0) dA = 0,$ $\sigma_4 : y = 2, \mathbf{F} \cdot \mathbf{n} = yz = 2z, \iint_{\sigma_4} (2z) dA = 8,$

$\sigma_5 : z = 0, \mathbf{F} \cdot \mathbf{n} = -xz = 0, \iint_{\sigma_5} (0) dA = 0,$ $\sigma_6 : z = 2, \mathbf{F} \cdot \mathbf{n} = xz = 2x, \iint_{\sigma_6} (2x) dA = 8.$