SURFACE INTEGRAL:

15.5.2 THEOREM *Let σ be a smooth parametric surface whose vector equation is*

$$\mathbf{r} = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

where (u, v) varies over a region R in the uv-plane. If $f(x, y, z)$ is continuous on σ, then

$$\iint_{\sigma} f(x, y, z)\, dS = \iint_{R} f(x(u, v), y(u, v), z(u, v)) \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| dA \qquad (6)$$

15.5.3 THEOREM

(a) *Let σ be a surface with equation $z = g(x, y)$ and let R be its projection on the xy-plane. If g has continuous first partial derivatives on R and $f(x, y, z)$ is continuous on σ, then*

$$\iint_{\sigma} f(x, y, z)\, dS = \iint_{R} f(x, y, g(x, y)) \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA \qquad (8)$$

(b) *Let σ be a surface with equation $y = g(x, z)$ and let R be its projection on the xz-plane. If g has continuous first partial derivatives on R and $f(x, y, z)$ is continuous on σ, then*

$$\iint_{\sigma} f(x, y, z)\, dS = \iint_{R} f(x, g(x, z), z) \sqrt{\left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + 1}\, dA \qquad (9)$$

(c) *Let σ be a surface with equation $x = g(y, z)$ and let R be its projection on the yz-plane. If g has continuous first partial derivatives on R and $f(x, y, z)$ is continuous on σ, then*

$$\iint_{\sigma} f(x, y, z)\, dS = \iint_{R} f(g(y, z), y, z) \sqrt{\left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2 + 1}\, dA \qquad (10)$$

▶ Example 2 Evaluate the surface integral

$$\iint_{\sigma} xz\, dS$$

where σ is the part of the plane $x + y + z = 1$ that lies in the first octant.

Solution. The equation of the plane can be written as

$$z = 1 - x - y$$

Consequently, we can apply Formula (8) with $z = g(x, y) = 1 - x - y$ and $f(x, y, z) = xz$. We have

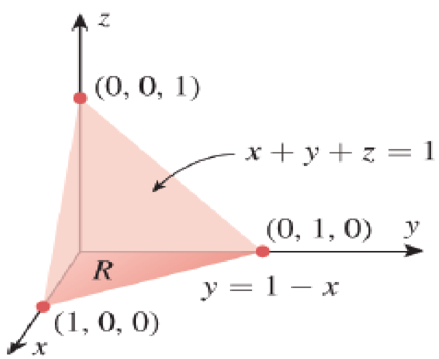
$$\frac{\partial z}{\partial x} = -1 \quad \text{and} \quad \frac{\partial z}{\partial y} = -1$$

so (8) becomes

$$\iint_{\sigma} xz\, dS = \iint_{R} x(1 - x - y)\sqrt{(-1)^2 + (-1)^2 + 1}\, dA \tag{11}$$

where R is the projection of σ on the xy-plane (Figure 15.5.3). Rewriting the double integral in (11) as an iterated integral yields

$$\begin{aligned}
\iint_{\sigma} xz\, dS &= \sqrt{3}\int_0^1 \int_0^{1-x} (x - x^2 - xy)\, dy\, dx \\
&= \sqrt{3}\int_0^1 \left[xy - x^2y - \frac{xy^2}{2}\right]_{y=0}^{1-x} dx \\
&= \sqrt{3}\int_0^1 \left(\frac{x}{2} - x^2 + \frac{x^3}{2}\right) dx \\
&= \sqrt{3}\left[\frac{x^2}{4} - \frac{x^3}{3} + \frac{x^4}{8}\right]_0^1 = \frac{\sqrt{3}}{24} \blacktriangleleft
\end{aligned}$$

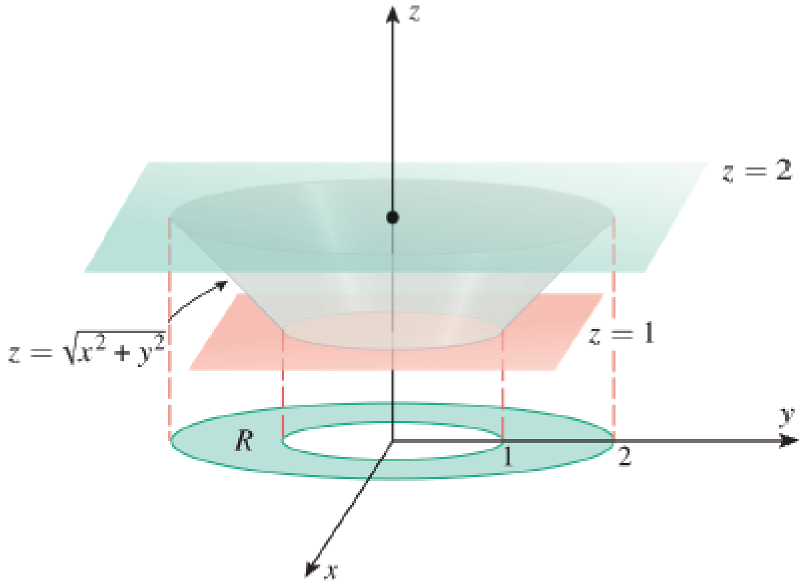


▲ **Figure 15.5.3**

▶ Example 3 Evaluate the surface integral

$$\iint_{\sigma} y^2 z^2 \, dS$$

where σ is the part of the cone $z = \sqrt{x^2 + y^2}$ that lies between the planes $z = 1$ and $z = 2$ (Figure 15.5.4).



▶ Figure 15.5.4

Solution. We will apply Formula (8) with

$$z = g(x, y) = \sqrt{x^2 + y^2} \quad \text{and} \quad f(x, y, z) = y^2 z^2$$

Thus,

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}$$

so

$$\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} = \sqrt{2}$$

(verify), and (8) yields

$$\iint_{\sigma} y^2 z^2 \, dS = \iint_{R} y^2 \left(\sqrt{x^2 + y^2}\right)^2 \sqrt{2} \, dA = \sqrt{2} \iint_{R} y^2 (x^2 + y^2) \, dA$$

where R is the annulus enclosed between $x^2+y^2=1$ and $x^2+y^2=4$ (Figure 15.5.4). Using polar coordinates to evaluate this double integral over the annulus R yields

$$\begin{aligned}
\iint_{\sigma} y^2z^2\,dS &= \sqrt{2}\int_0^{2\pi}\int_1^2 (r\sin\theta)^2(r^2)r\,dr\,d\theta \\
&= \sqrt{2}\int_0^{2\pi}\int_1^2 r^5\sin^2\theta\,dr\,d\theta \\
&= \sqrt{2}\int_0^{2\pi}\left[\frac{r^6}{6}\sin^2\theta\right]_{r=1}^2 d\theta = \frac{21}{\sqrt{2}}\int_0^{2\pi}\sin^2\theta\,d\theta \\
&= \frac{21}{\sqrt{2}}\left[\frac{1}{2}\theta-\frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{21\pi}{\sqrt{2}}
\end{aligned}$$

Formula (7), Section 7.3

EXERCISE SET 15.5 C CAS

1–8 Evaluate the surface integral

$$\iint_{\sigma} f(x,y,z)\,dS$$

1. $f(x,y,z)=z^2$; σ is the portion of the cone $z=\sqrt{x^2+y^2}$ between the planes $z=1$ and $z=2$.
2. $f(x,y,z)=xy$; σ is the portion of the plane $x+y+z=1$ lying in the first octant.
3. $f(x,y,z)=x^2y$; σ is the portion of the cylinder $x^2+z^2=1$ between the planes $y=0$, $y=1$, and above the xy-plane.

5. $f(x,y,z)=x-y-z$; σ is the portion of the plane $x+y=1$ in the first octant between $z=0$ and $z=1$.
6. $f(x,y,z)=x+y$; σ is the portion of the plane $z=6-2x-3y$ in the first octant.
7. $f(x,y,z)=x+y+z$; σ is the surface of the cube defined by the inequalities $0\le x\le 1, 0\le y\le 1, 0\le z\le 1$. [*Hint:* Integrate over each face separately.]