

**Theorem 1** (Boundedness). *For all  $t > 0$ , the risk mass  $\theta_t$  is contained within the scenario-defined limits:*

$$\theta_{\min} \leq \theta_t \leq \theta_{\max}.$$

*Proof.* **Step 1: Lower Bound Verification.**

The risk mass at  $t + 1$  is defined by the recursive fading operator:

$$\theta_{t+1} = \theta_{\min} + (\theta_t - \theta_{\min}) \exp(-\lambda_s d_t).$$

Since  $\exp(-\lambda_s d_t) \geq 0$  and  $(\theta_t - \theta_{\min}) \geq 0$ , their product is non-negative. Therefore:

$$\theta_{t+1} \geq \theta_{\min}.$$

**Step 2: Upper Bound Verification.**

From Theorem 1 (Monotonicity),  $\theta_{t+1} \leq \theta_t$  for all  $t$ . Since  $\exp(-\lambda_s d_t) \leq 1$ , we have:

$$\theta_{t+1} \leq \theta_t \leq \dots \leq \theta_{\max}.$$

**Step 3: Conclusion.**

Combining the bounds from Steps 1 and 2:

$$\theta_{\min} \leq \theta_t \leq \theta_{\max}.$$

□