Theorem 3: Inevitable Divergence Under Exponential Focus Decay

Theorem 1 (Focus-Decay Divergence Theorem)**.** *Let $D(t), A(t) \in \mathbb{R}^n$ be continuously differentiable.*

Define the misalignment vector

$$E(t) = D(t) - A(t).$$

Assume:

(1) The agent adaptation decays exponentially:

$$\dot{A}(t) = Be^{-\lambda t}, \quad \lambda > 0, \quad B \in \mathbb{R}^n.$$

(2) There exists $\varepsilon > 0$ and $T \geq 0$ such that for all $t \geq T$,

$$\frac{E(t)}{\|E(t)\|} \cdot \dot{D}(t) \geq \varepsilon.$$

Then there exists $T^* \geq T$ such that for all $t \geq T^*$,

$$\frac{d}{dt}\|E(t)\| \geq \frac{\varepsilon}{2},$$

and therefore

$$\|E(t)\| \to \infty \quad \textit{as } t \to \infty.$$

Proof. From Theorem 1 (Dual Failure Law),

$$\frac{d}{dt}\|E(t)\| = \frac{E(t)}{\|E(t)\|} \cdot \left(\dot{D}(t) - \dot{A}(t)\right).$$

Rewrite:

$$\frac{d}{dt}\|E(t)\| = \frac{E(t)}{\|E(t)\|} \cdot \dot{D}(t) - \frac{E(t)}{\|E(t)\|} \cdot \dot{A}(t).$$

By assumption (2),

$$\frac{E(t)}{\|E(t)\|} \cdot \dot{D}(t) \geq \varepsilon.$$

Now bound the adaptation term using Cauchy–Schwarz:

$$\left|\frac{E(t)}{\|E(t)\|} \cdot \dot{A}(t)\right| \leq \|\dot{A}(t)\|.$$

Since

$$\|\dot{A}(t)\| = \|B\|e^{-\lambda t},$$

we have

$$\|\dot{A}(t)\| \to 0 \quad \text{as } t \to \infty.$$

Thus there exists $T^* \geq T$ such that for all $t \geq T^*$,

$$\|\dot{A}(t)\| \leq \frac{\varepsilon}{2}.$$

Therefore, for all $t \geq T^*$,

$$\frac{d}{dt}\|E(t)\| \geq \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2}.$$

Integrating from T^* to t,

$$\|E(t)\| \geq \|E(T^*)\| + \frac{\varepsilon}{2}(t - T^*).$$

Hence $\|E(t)\|$ grows at least linearly for large t and diverges as $t \to \infty$.

$\square$