Theorem 1: Dual Failure Law

Theorem 1 (Dual Failure Law)**.** *Let $D(t), A(t) \in \mathbb{R}^n$ be continuously differentiable functions. Define the misalignment vector*

$$E(t) = D(t) - A(t).$$

Then for all t such that $E(t) \neq 0$,

$$\frac{d}{dt}\|E(t)\| = \frac{E(t)}{\|E(t)\|} \cdot \left(\dot{D}(t) - \dot{A}(t)\right).$$

Proof. **Step 1: Norm Representation**

The Euclidean norm satisfies

$$\|E(t)\| = \sqrt{E(t) \cdot E(t)}.$$

Define

$$\phi(t) = E(t) \cdot E(t).$$

Then

$$\|E(t)\| = \sqrt{\phi(t)}.$$

Step 2: Chain Rule

$$\frac{d}{dt}\|E(t)\| = \frac{1}{2\sqrt{\phi(t)}}\phi'(t).$$

Step 3: Differentiate the Inner Product

$$\phi'(t) = 2E(t) \cdot \dot{E}(t).$$

Thus,

$$\frac{d}{dt}\|E(t)\| = \frac{1}{2\|E(t)\|} \cdot 2E(t) \cdot \dot{E}(t) = \frac{E(t) \cdot \dot{E}(t)}{\|E(t)\|}.$$

Step 4: Substitute Error Dynamics

Since

$$E(t) = D(t) - A(t),$$

we have

$$\dot{E}(t) = \dot{D}(t) - \dot{A}(t).$$

Therefore,

$$\frac{d}{dt}\|E(t)\| = \frac{E(t)}{\|E(t)\|} \cdot \left(\dot{D}(t) - \dot{A}(t)\right).$$

□