Calculus & Analytical Geometry

COURSE CODE : MT1003

1

Functions

Function

A function f from a set D to a set Y is a rule that assigns a unique value y in Y to each x in D.

The set D is called the domain of f.

The set Y is called the codomain of f.

The range of f is the set of all y values as x varies throughout the domain.

Since the y value that we are getting depends upon the input x value that's why we say that x is independent variable and y is dependent variable and f is a function of x and write as $y = f(x)$.

If for a number α in D we have β in Y then we write $f(\alpha) = \beta$ and say that β is the value of f at α.

Representations of Functions

There are four possible ways to represent a function:

- verbally (by a description in words)
- numerically (by a table of values)
- visually (by a graph)
- algebraically (by an explicit formula)

Let’s represent one function in all these ways...

Verbally

Let $C(w)$ be the cost of mailing a large envelope with weight w. The cost is \$2 for up to 1kg, plus \$1.5 for each additional kg (or less) up to 5 kg.

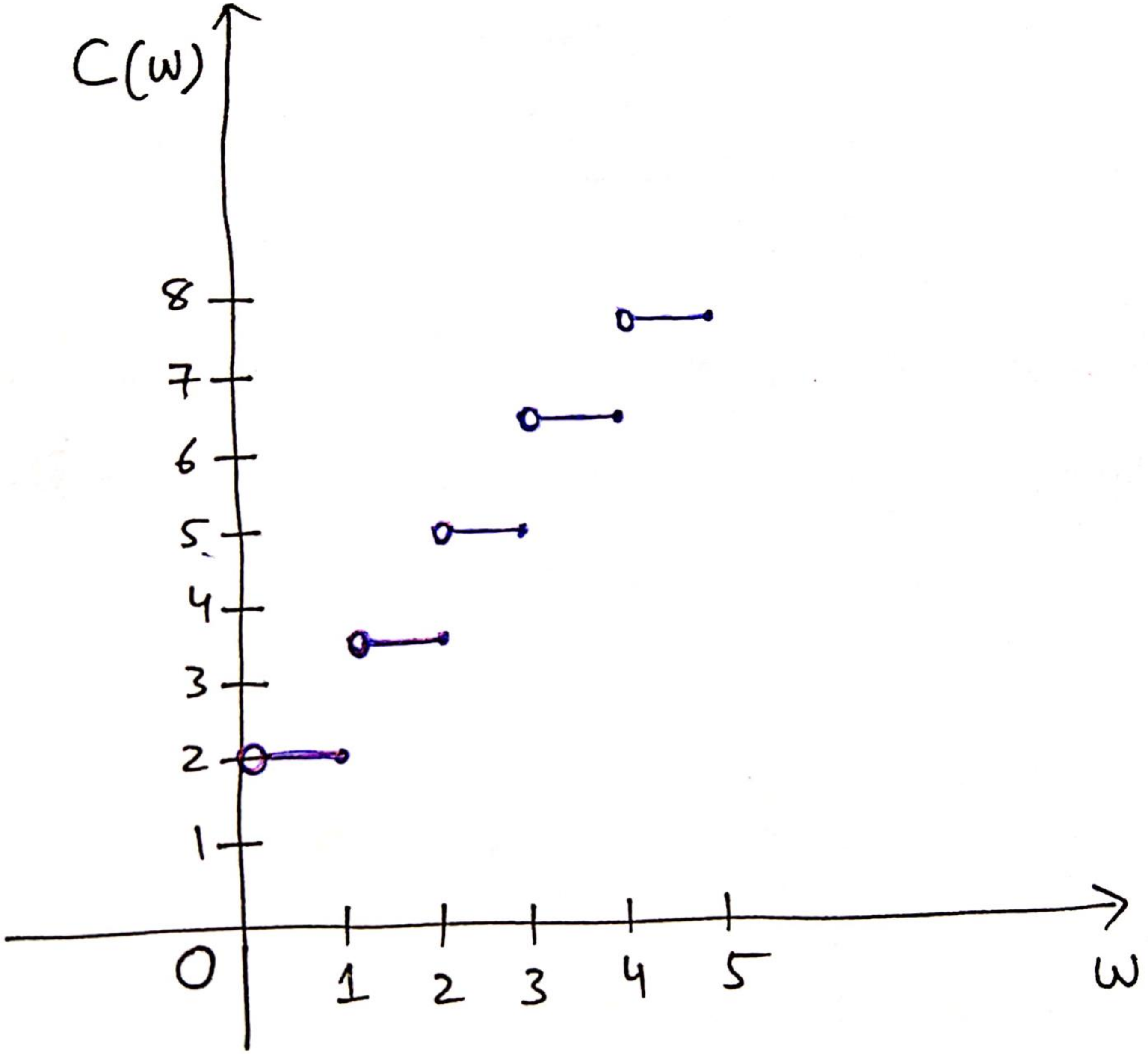
Numerically

Weight in Kg w	Cost in dollars $C(w)$
$0 < w \leq 1$	2
$1 < w \leq 2$	3.5
$2 < w \leq 3$	5
$3 < w \leq 4$	6.5
$4 < w \leq 5$	8

Visually

Graph of a Function

The graph of a function f consists of all points (x, y) in the coordinate plane such that $y = f(x)$ and x is in the domain of f.


C(w)
8
7
6
5
4
3
2
1
0
1
2
3
4
5
w

Algebraically

$$c(w) = \begin{cases} 2 & \text{if } 0 < w \le 1 \\ 3.5 & \text{if } 1 < w \le 2 \\ 5 & \text{if } 2 < w \le 3 \\ 6.5 & \text{if } 3 < w \le 4 \\ 8 & \text{if } 4 < w \le 5 \end{cases}$$

Domain Convention

If a function is given by a formula and the domain is not stated explicitly, the convention is that the domain is the set of all numbers for which the formula makes sense and defines a real number.

Problem

Find the domain of each function.

(a) $f(x) = \sqrt{x + 2}$ (b) $g(x) = \dfrac{1}{x^2 - x}$

Solution (a) $x + 2 \geq 0 \Rightarrow x \geq -2$

Thus $D(f) = [-2, \infty]$

Solution (b)

All real number except 0 and 1 is the domain of g.

or

$$D(g) = R \setminus \{0,1\}$$

or

$$D(g) = (-\infty, 0) \cup (0,1) \cup (1, \infty)$$

Practice Problem

Find the domain of the following functions

(i) $f(x) = \dfrac{1}{|x+2|-3}$

(ii) $g(x) = \dfrac{\sqrt{x^2+7x+12}}{x+2}$

(iii) $h(x) = \dfrac{\sqrt{x+1}}{x^2-4}$

(iv) $L(x) = \sqrt[3]{x^2-4}$

(v) $R(x) = \dfrac{1}{(x+1)(x-2)(x+4)}$

(vi) $P(x) = x^2 + x + 1$

(vii) $C(x) = \dfrac{x+2}{\sqrt{x^2+7x+12}}$

(viii) $H(x) = \dfrac{1}{|x+100| - |x-2|}$

(ix) $G(x) = \sqrt{(x-1)(x+2)(x-3}$

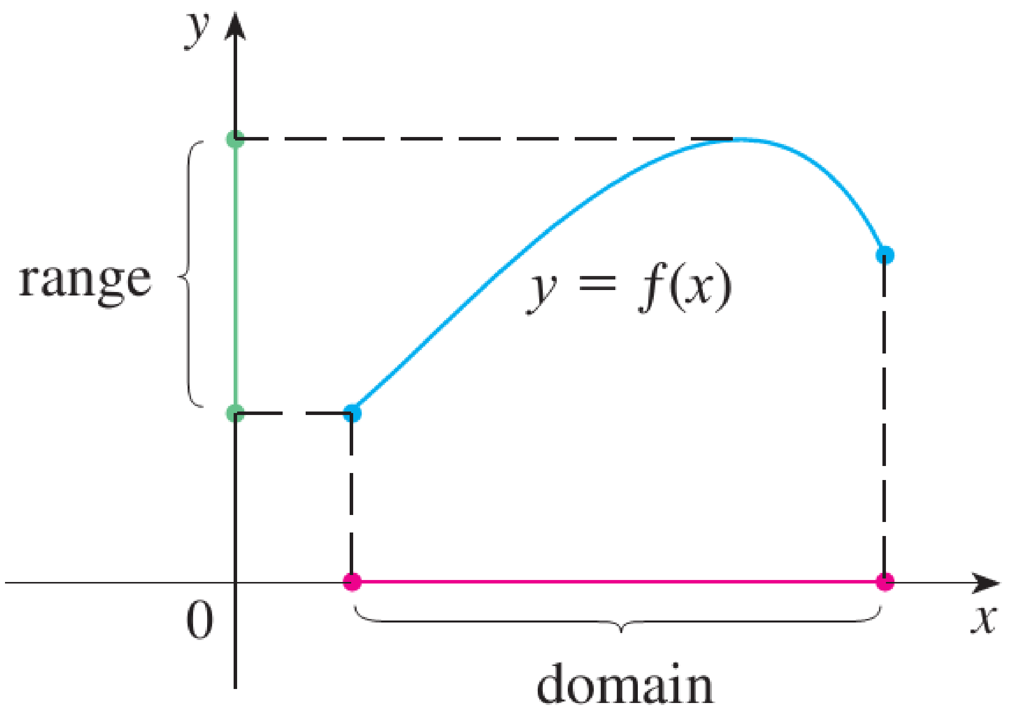
(x) $l(x) = \sqrt[7]{x^2+4}$

(xi) $r(x) = \dfrac{1}{x^2+1}$

(xii) $c(x) = \sqrt[8]{x^2+4}$

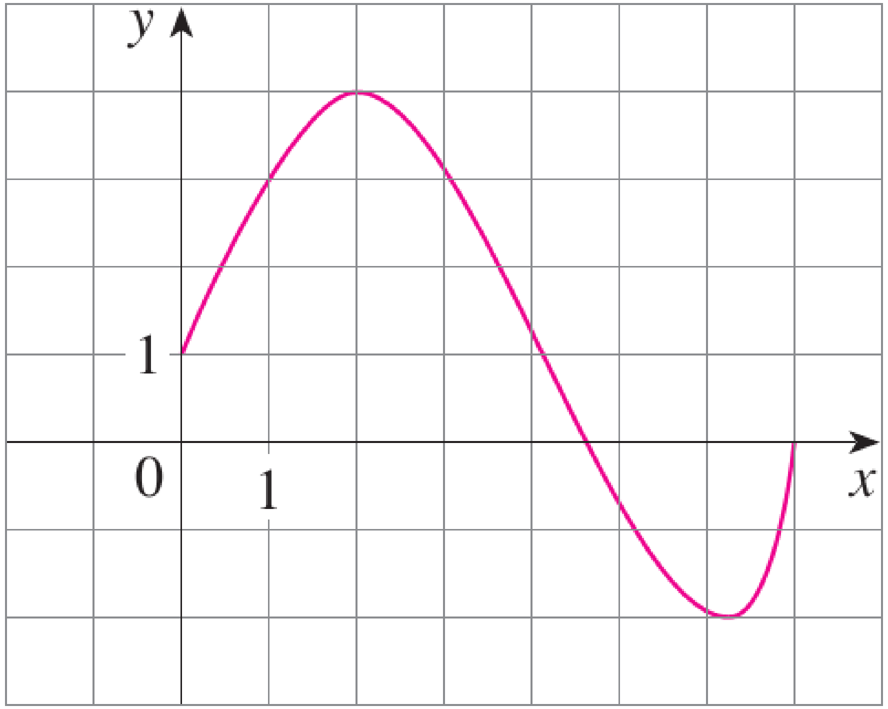
Finding Domain and Range from the graph

If the graph of a function is given then it is very easy to find its domain and range.



Problem

The graph of a function f is shown in Figure



(a) Find the values of $f(1)$ and $f(5)$.

(b) What are the domain and range of f?

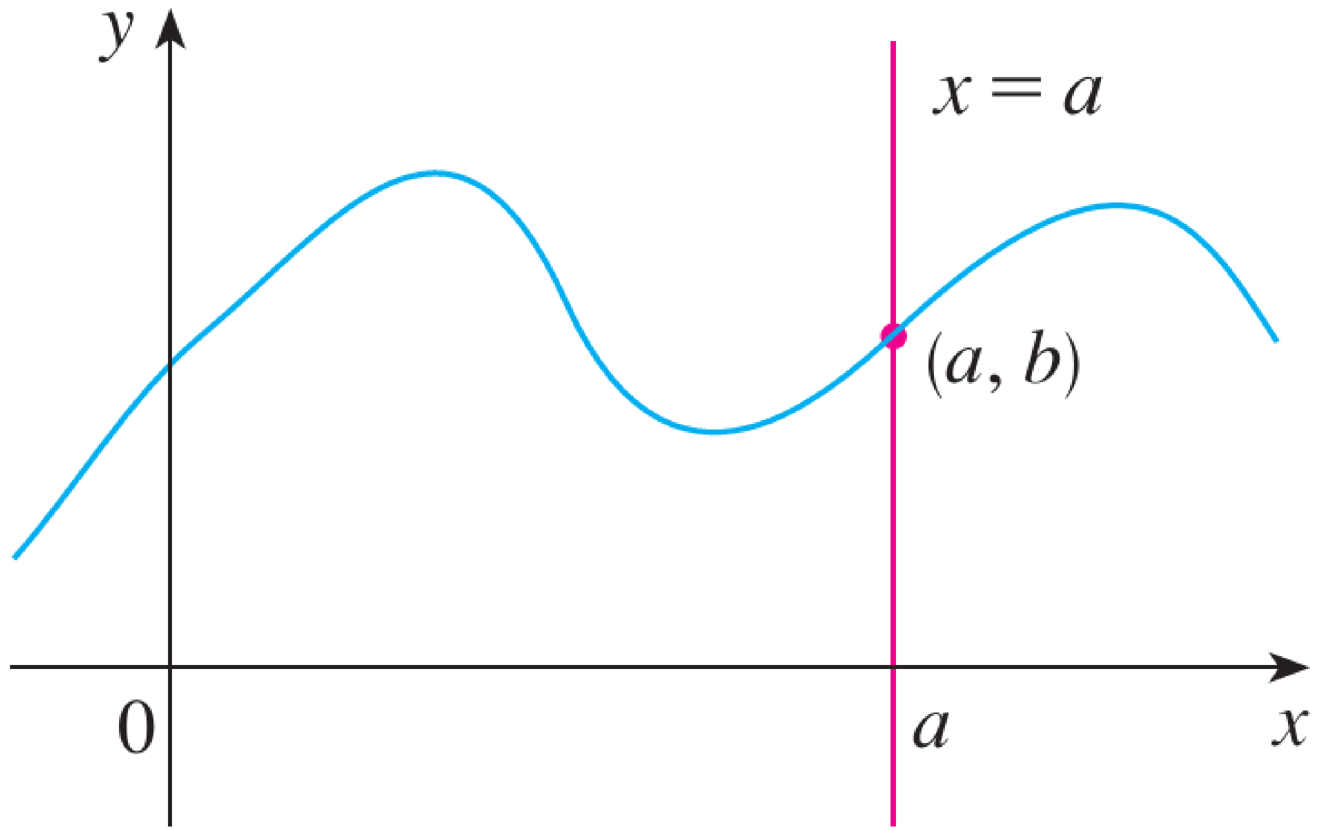
Practice Problem

Find the domain and range and sketch the graph of the function $h(x) = \sqrt{4 - x^2}$.

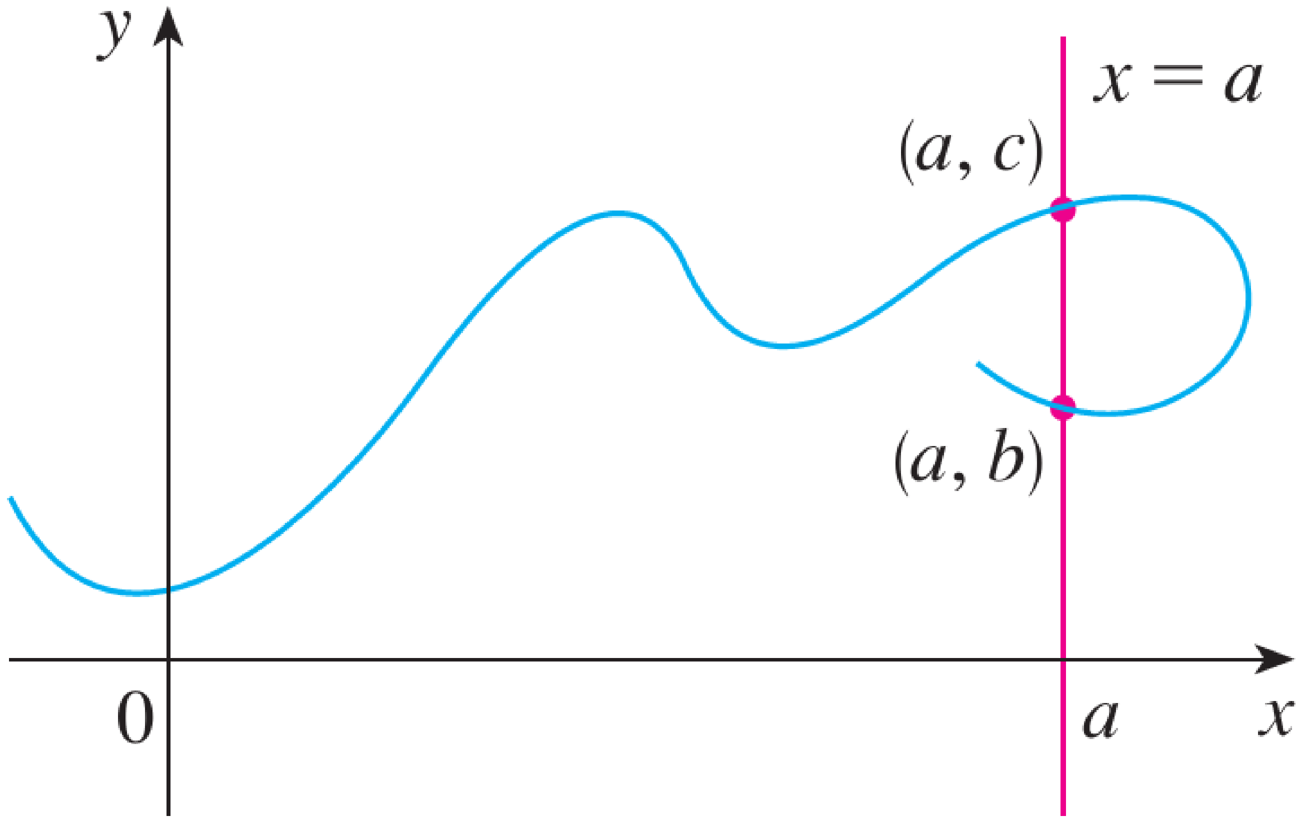
Vertical Line Test for checking whether a curve is the graph of a function

The graph of a function is a curve in the xy-plane. But the question arises: Which curves in the xy-plane are graphs of functions? This is answered by the following test.

The Vertical Line Test: A curve in the xy-plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.



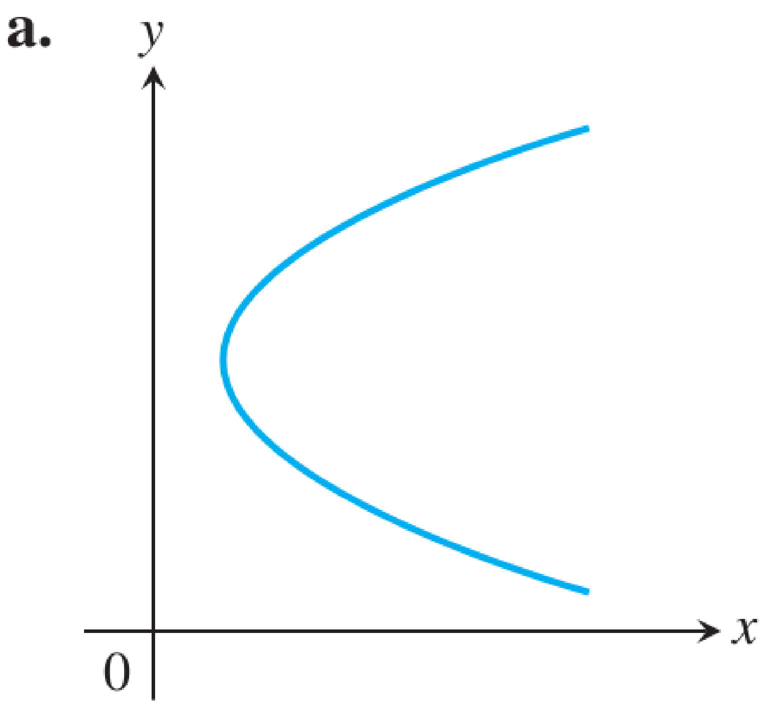
This curve represents a function

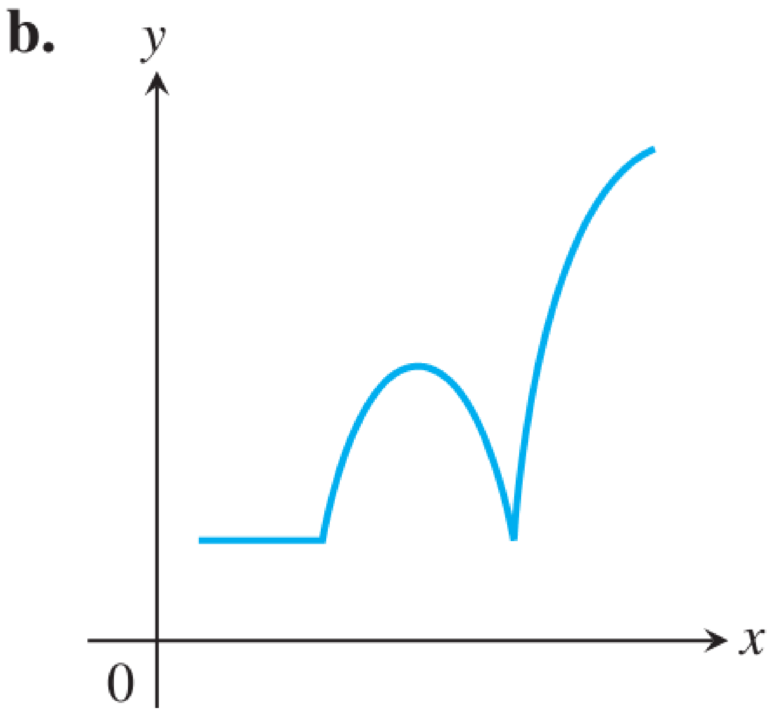


This curve doesn’t represent a function.

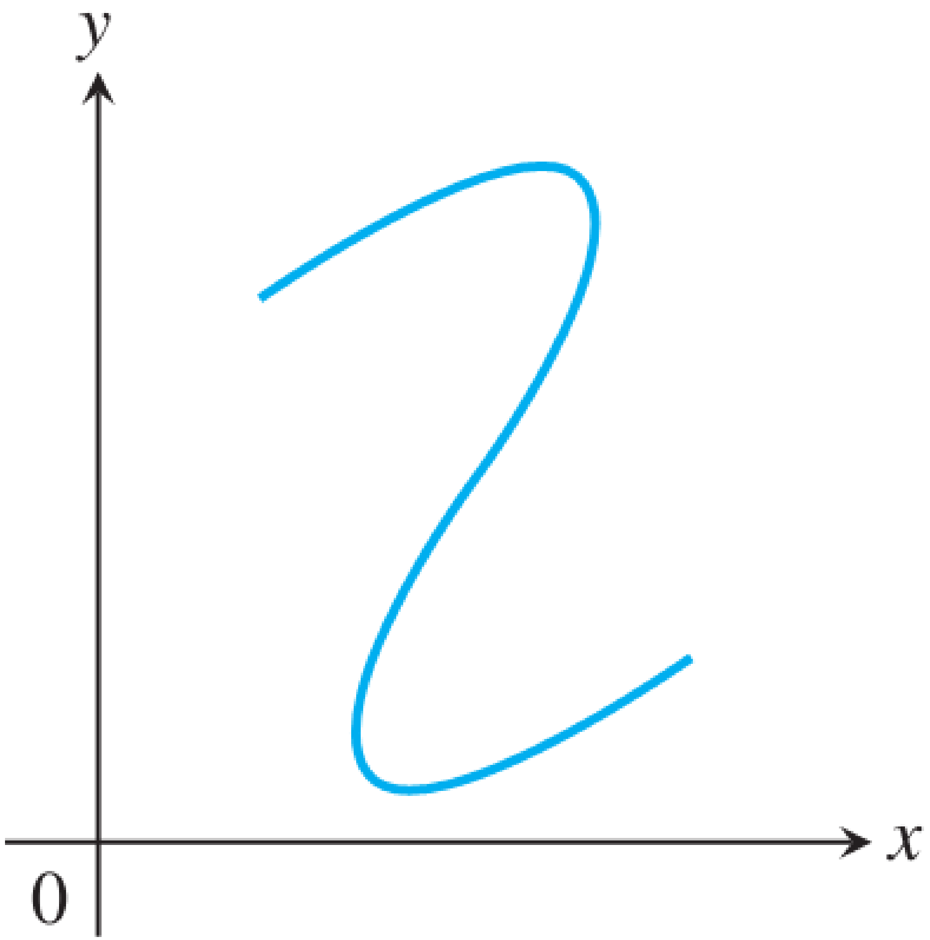
Practice Problem

Which of the graphs are graphs of functions of x, and which are not? Give reasons for your answers.

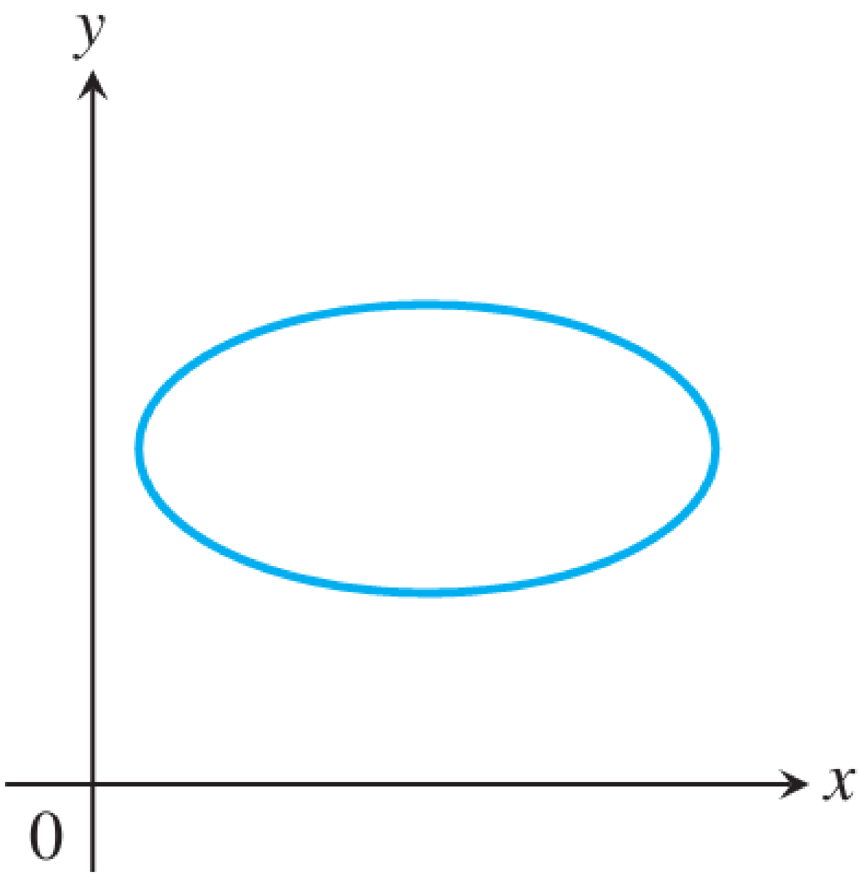




c.



d.



Piecewise Defined Functions

A piecewise defined function is a function that is described in pieces where we have different formulas for different parts of the domain.

Example of Piecewise defined function is

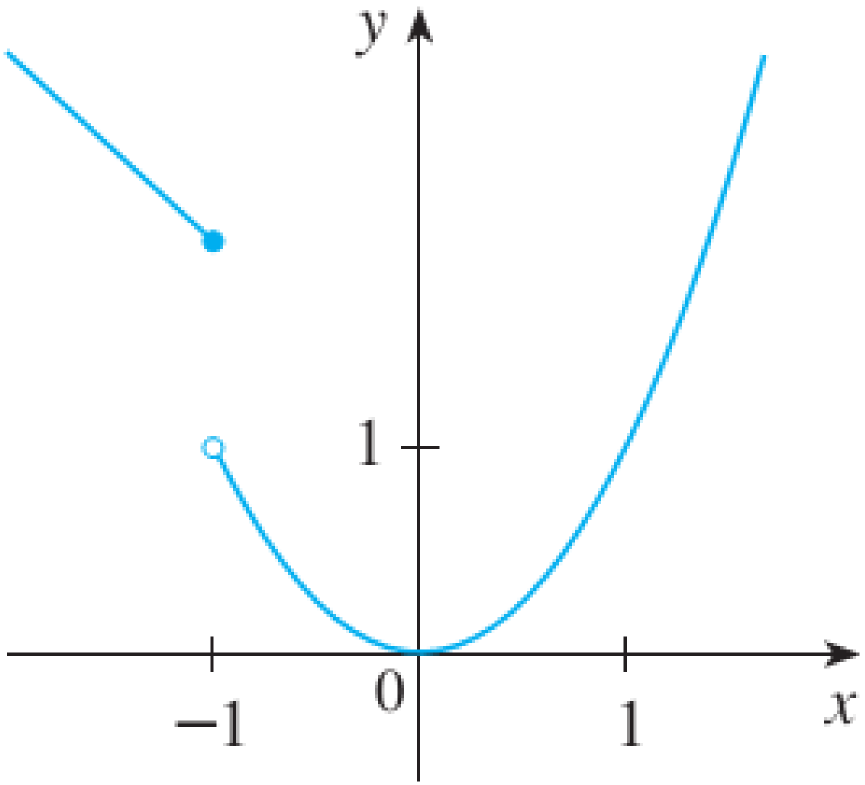
$$f(x) = \begin{cases} 1 - x & \text{if } x \leqslant -1 \\ x^2 & \text{if } x > -1 \end{cases}$$

Lets Evaluate $f(-2)$, $f(-1)$, and $f(0)$ and sketch the graph.

$$f(-2) = 1 - (-2) = 3. \qquad f(-1) = 1 - (-1) = 2.$$

$$f(0) = 0^2 = 0.$$

Graph

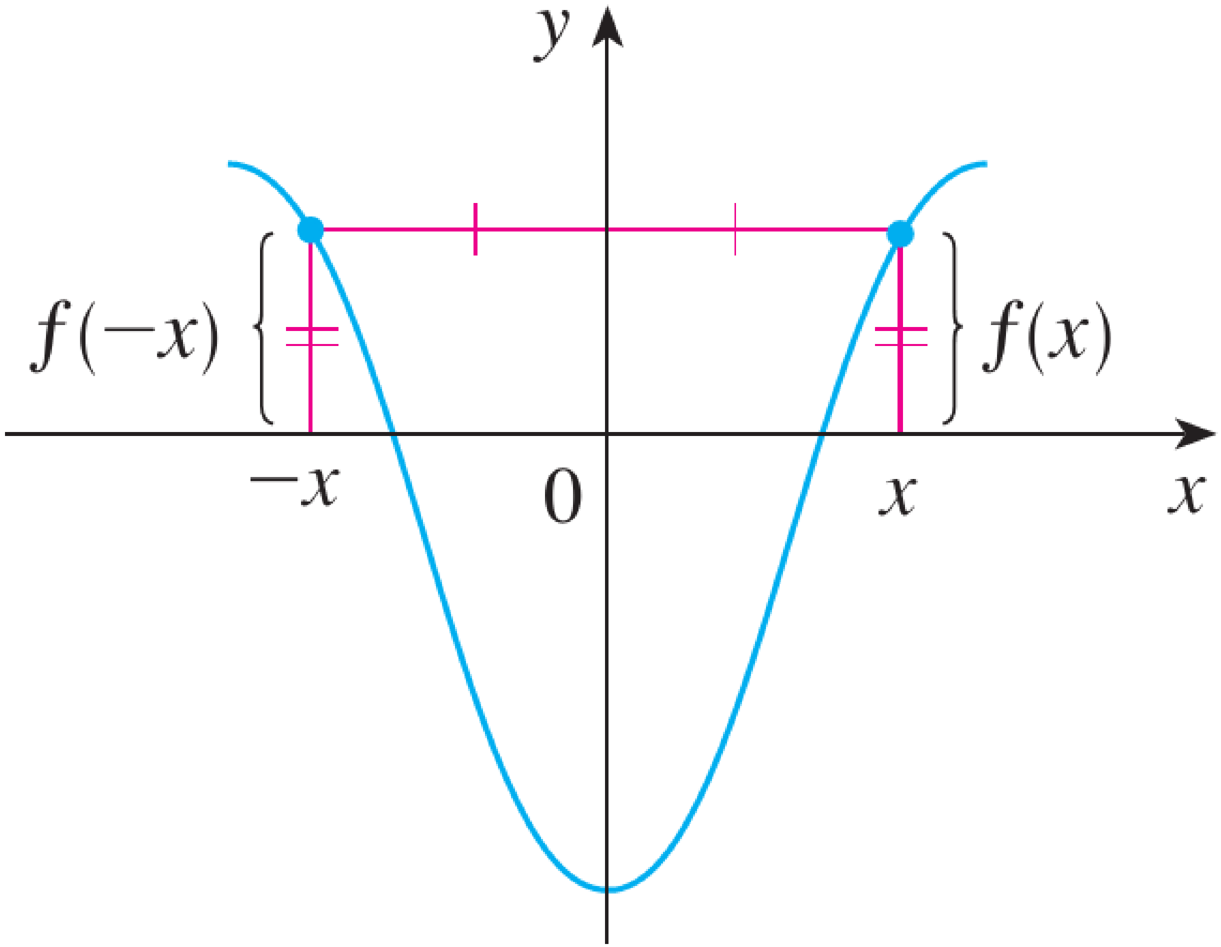


Symmetry

If a function f satisfies $f(-x) = f(x)$ for every number x in its domain, then f is called an even function.

For example the function $f(x) = x^2$ is even because

$$f(-x) = (-x)^2 = x^2 = f(x)$$



An even function

The geometric significance of an even function is that its graph is symmetric with respect to the y-axis. This means that if we have plotted the graph of f for $x > 0$, we can obtain the entire graph simply by reflecting this portion about the y-axis.

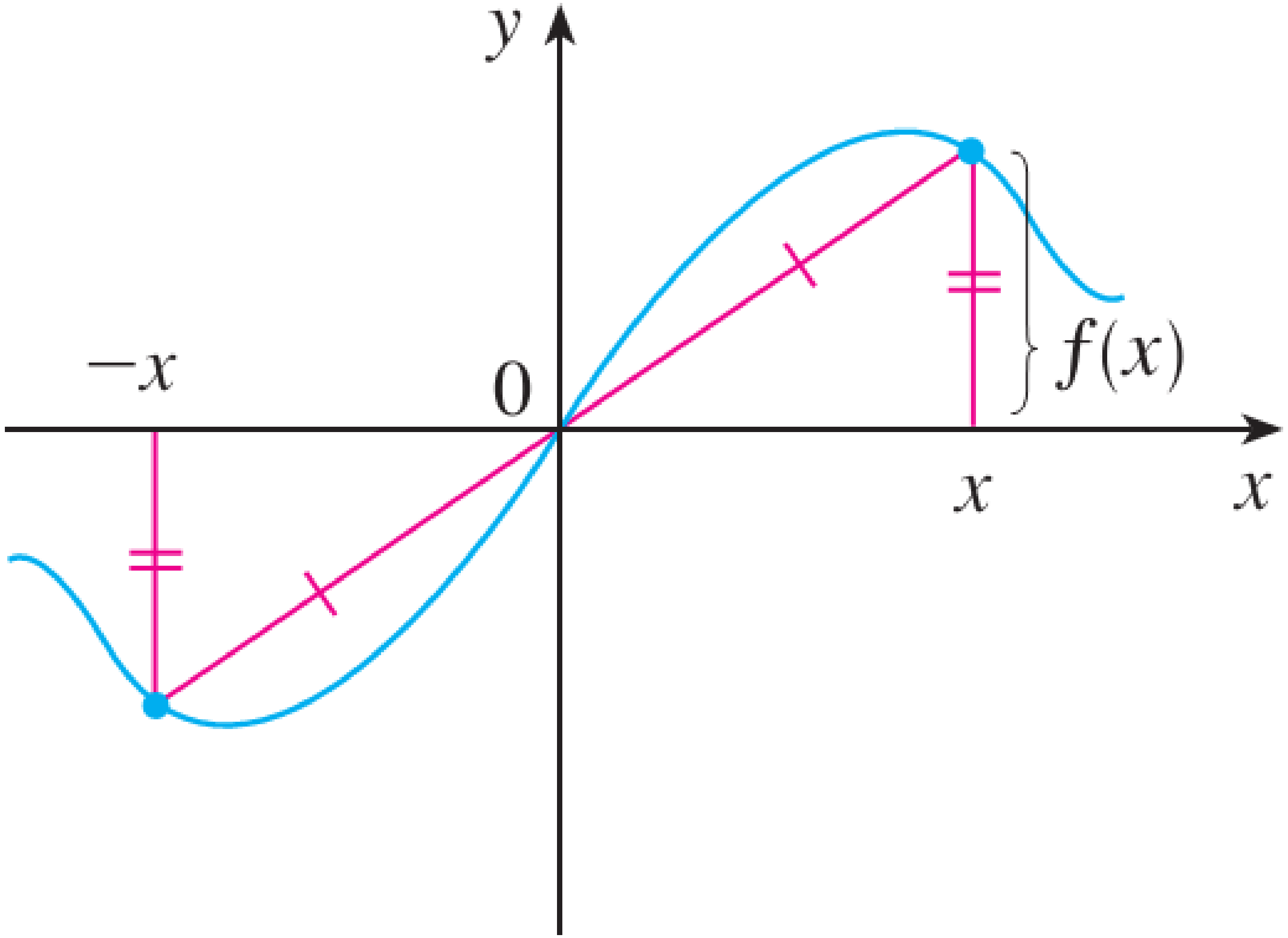
If a function f satisfies $f(-x) = -f(x)$ for every number x in its domain, then f is called an odd function.

Example

The function $f(x) = x^3$ is an odd function because

$$f(-x) = (-x)^3 = -x^3 = -f(x)$$

The graph of an odd function is symmetric about the origin. If we already have the graph of f for $x > 0$, we can obtain the entire graph by rotating this portion through 180^0 about the origin.



An odd function

Problem

Determine whether each of the following functions is even, odd, or neither even nor odd.

(a) $f(x) = x^5 + x$ (b) $g(x) = 1 - x^4$ (c) $h(x) = 2x - x^2$

SOLUTION

$$f(-x) = (-x)^5 + (-x) = (-1)^5 x^5 + (-x)$$

$$= -x^5 - x = -(x^5 + x)$$

$$= -f(x)$$

Therefore f is an odd function.

(b) $g(-x) = 1 - (-x)^4 = 1 - x^4 = g(x)$

So g is even.

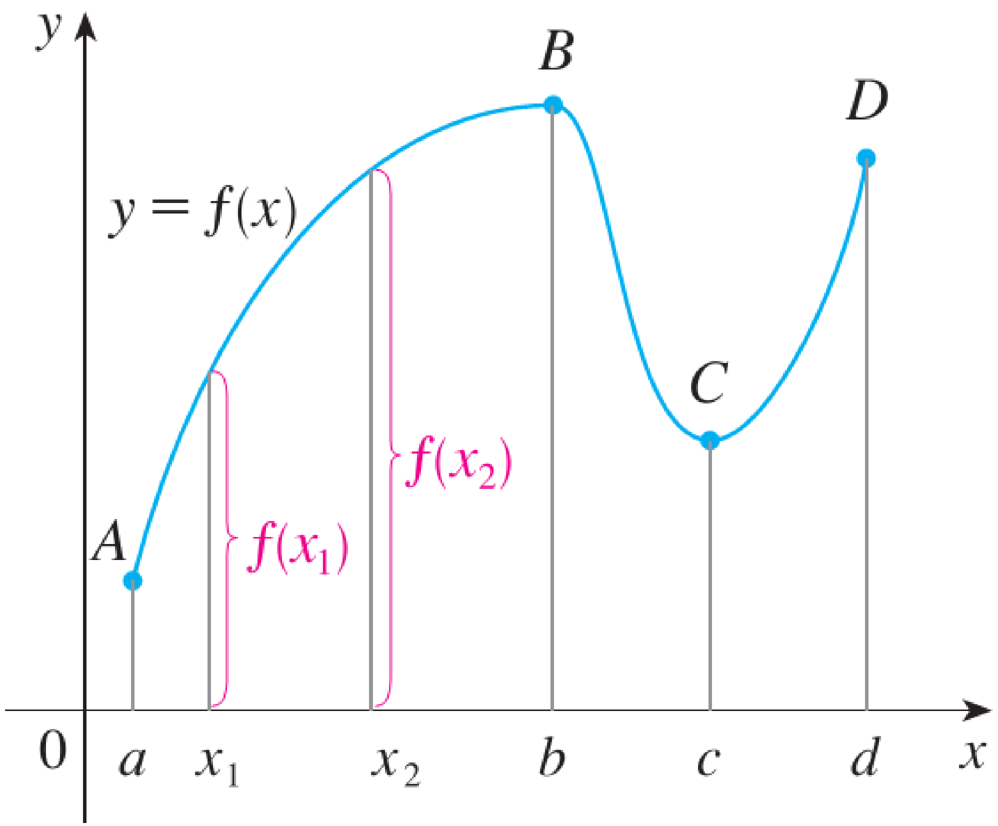
(c) $h(-x) = 2(-x) - (-x)^2 = -2x - x^2$

Since $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$,

we conclude that h is neither even nor odd.

Increasing and Decreasing Functions

The graph shown rises from A to B, falls from B to C, and rises again from C to D. The function f is said to be increasing on the interval [a, b], decreasing on [b, c], and increasing again on [c, d].



A function f is called **increasing** on an interval I if

$$f(x_1) < f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in } I$$

It is called **decreasing** on I if

$$f(x_1) > f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in } I$$

In the definition of an increasing function it is important to realize that the inequality $f(x_1) < f(x_2)$ must be satisfied for every pair of numbers x_1 and x_2 in I with $x_1 < x_2$.

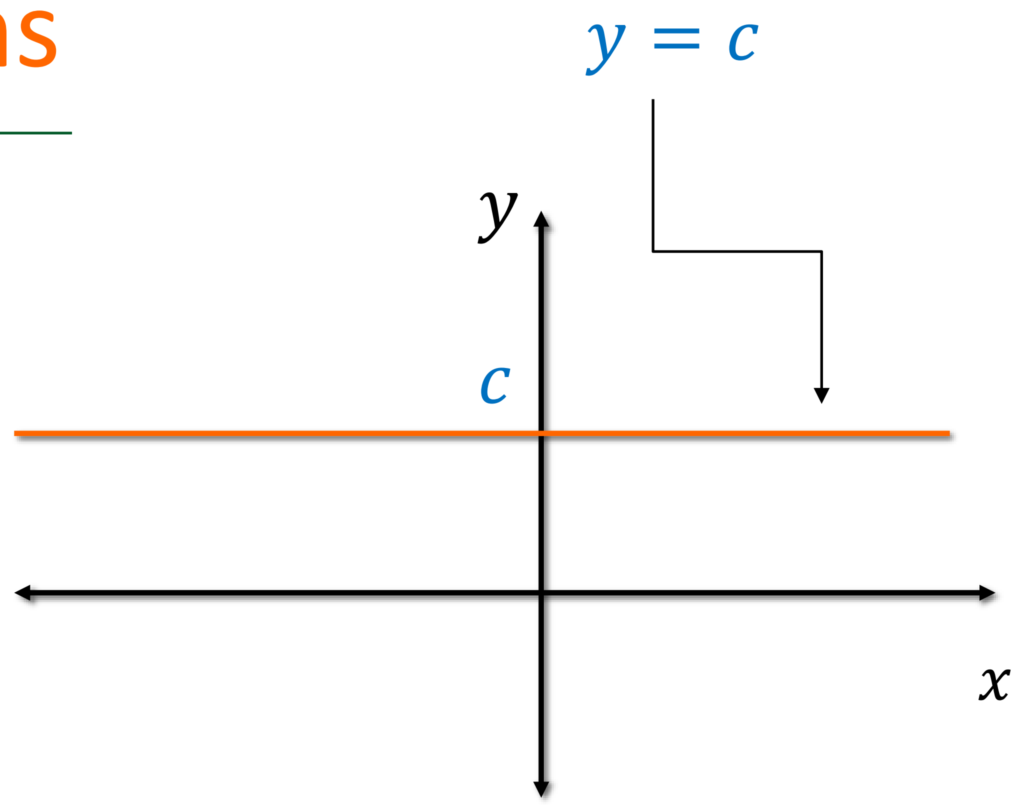
Example

The function $f(x) = x^2$ is decreasing on the interval $(-\infty, 0]$ and increasing on the interval $[0, \infty)$.

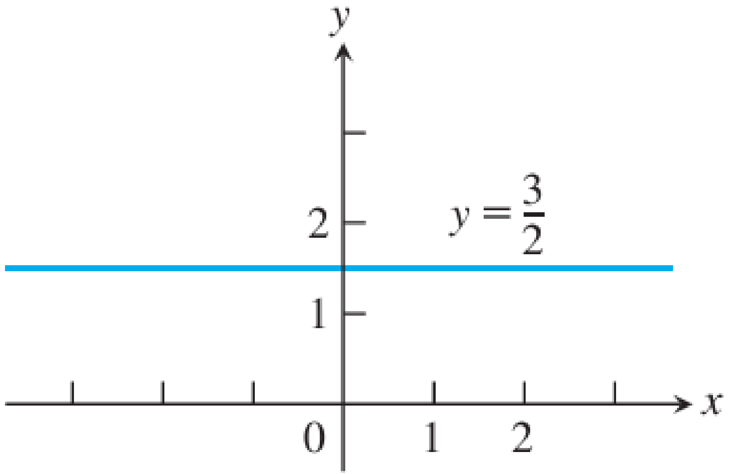
Graph of some basic Functions

Constant Function

$$f(x) = c$$



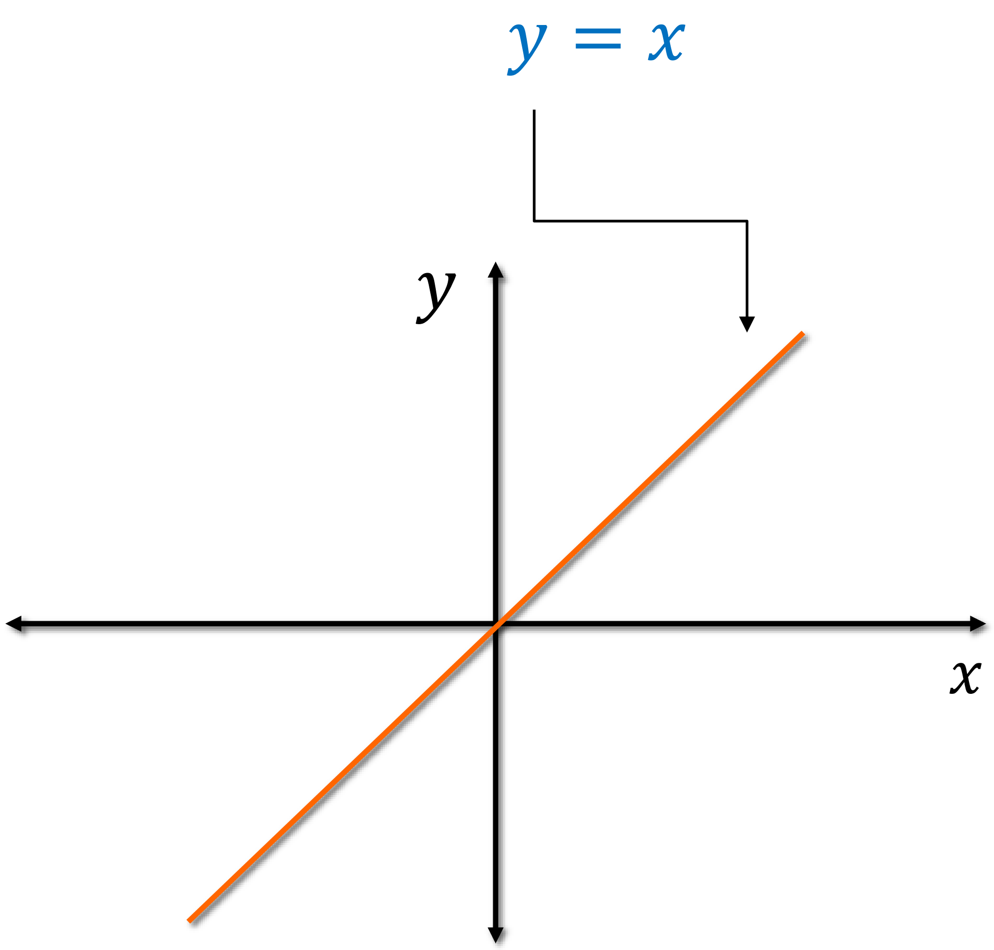
Example



The graph of a constant function is a horizontal line.

Identity Function

$$f(x) = x$$

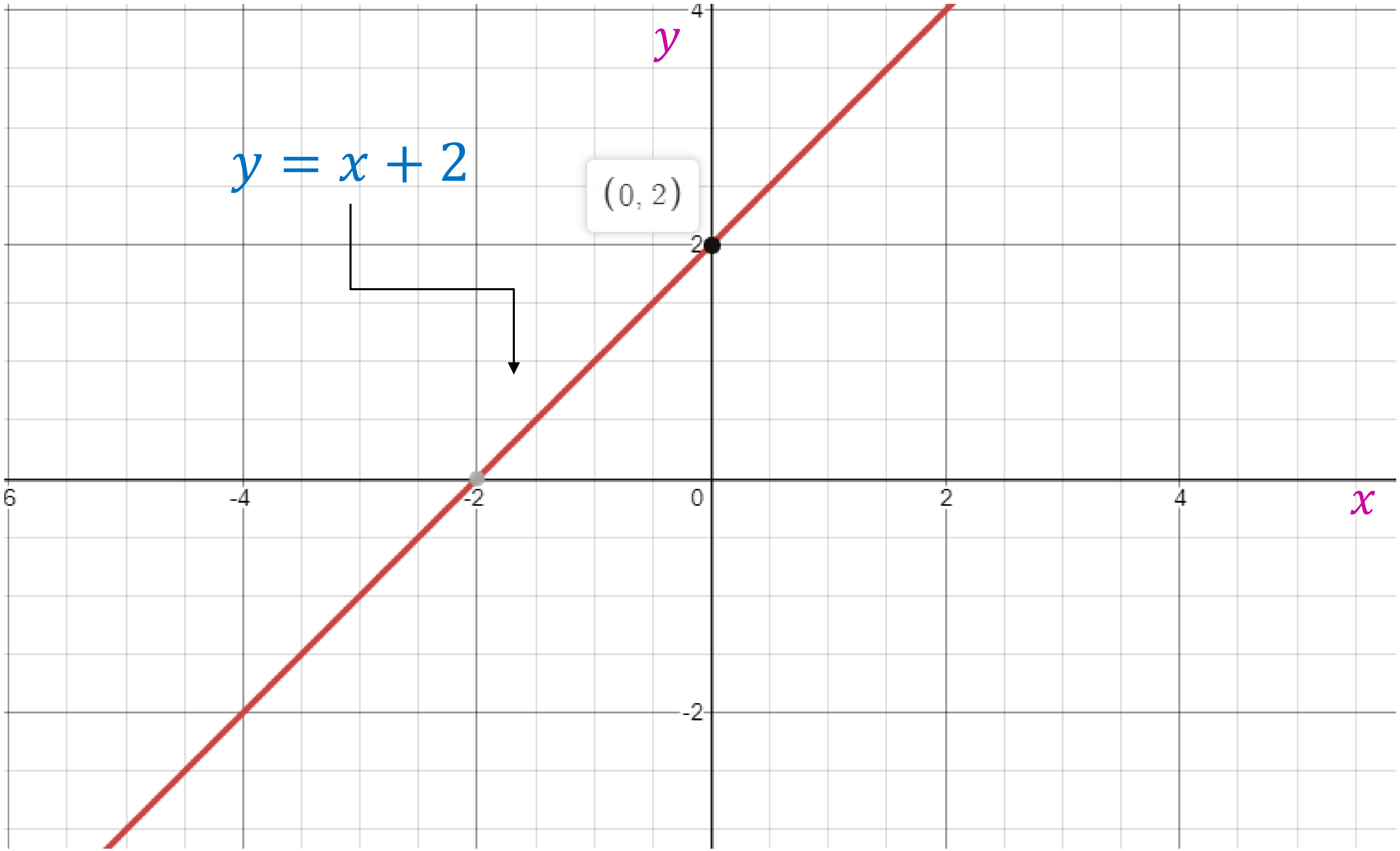


Linear Function

$$f(x) = mx + c$$

The graph of a linear function is a non vertical line. The number m is the slope of the line and c is the y-intercept.

Example: $y = x + 2$ is an equation of line that has slope 1 and y-intercept 2.



$y = x + 2$
$(0, 2)$
y
x
4
2
0
-2
-4
6
2
4
-2

Practice Problem

Climate change : Recent studies indicate that the average surface temperature of the earth has been rising steadily. Some scientists have modeled the temperature by the linear function $T = 0.02t + 8.50$, where T is temperature in °C and t represents years since 1900.

(a) What do the slope and T-intercept represent?

(b) Use the equation to predict the average global surface temperature in 2100.

Problem

Demand function for CDs : After studying sales for several months, the owner of a large CD retail outlet knows that the number of new CDs sold in a day (called the demand) decreases as the retail price increases. Specifically, his data indicate that at a price of $\$14$ per CD an average of 400 CDs are sold per day, while at a price of $\$17$ per CD an average of 250 CDs are sold per day. Assume that the demand d is a linear function of the price p.

a. Find and graph the demand function $d = f(p) = mp + b$.

b. According to this model, how many CDs (on average) are sold at a price of $\$20$?

SOLUTION

a. Two points on the graph of the demand function are given: $(p, d) = (14, 400)$ and $(17, 250)$. Therefore, the slope of the demand line is

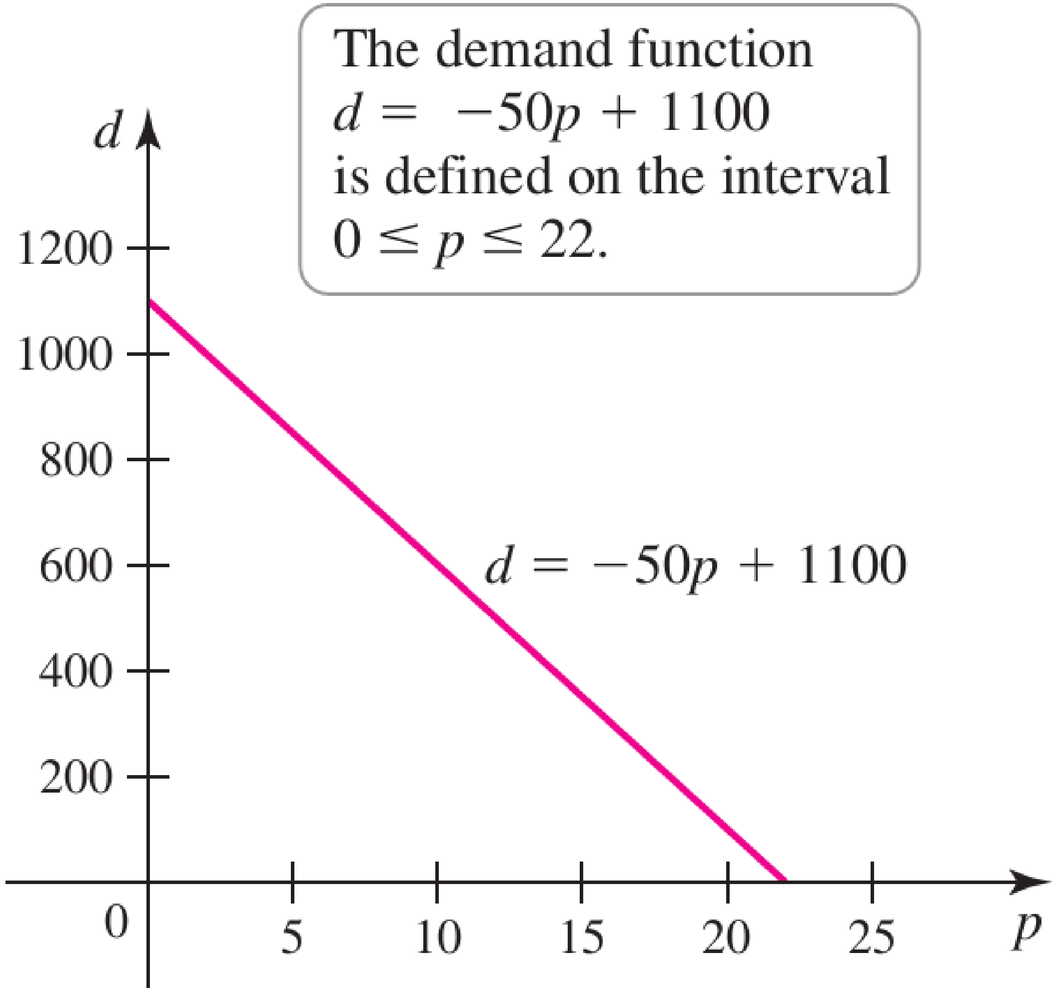
$$m = \frac{400 - 250}{14 - 17} = -50 \text{ CDs per dollar.}$$

It follows that the equation of the linear demand function is

$$d - 250 = -50(p - 17).$$

Expressing d as a function of p, we have $d = f(p) = -50p + 1100$

b. Using the demand function with a price of \$20, the average number of CDs that could be sold per day is $f(20) = 100$.



Practice Problem

The period T of a pendulum is measured for pendulums of several different lengths L. Based on the following data, does T appear to be a linear function of L?

L (cm)	20	30	40	50
T (s)	0.9	1.1	1.27	1.42

Practice Problem

A city's population was $30{,}700$ in the year 2010 and is growing by 850 people a year.

(a) Give a formula for the city's population , P , as a function of the number of years , t , since 2010.

(b) What is the population predicted to be in 2021?

(c) When is the population expected to reach $45{,}000$?

Polynomials

A function P is called a polynomial if

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + a_2 x^2 + a_1 x + a_0$$

where n is a nonnegative integer and the numbers $a_0, a_1, a_2, \ldots a_n$ are constants called the coefficients of the polynomial. If the leading coefficient $a_n \neq 0$, then the degree of the polynomial is n.

Examples:

$P(x) = 2x^6 + 5x^5 + 7x^3 - 8x^2 + x - 90$ is a polynomial of degree 6.

$P(x) = -8x^4 + x + 1$ is a polynomial of degree 4.

$f(x) = x^{\frac{1}{2}} + x + 1$ is not a polynomial because the power of first term is not a positive integer.

A polynomial of degree 1 is of the form $P(x) = a_1x + a_0$ and so is a linear function.

A polynomial of degree 2 is of the form $P(x) = a_2x^2 + a_1x + a_0$ and is called a quadratic function.

Its graph is always a parabola obtained by shifting the parabola $y = ax^2$.

The parabola opens upward if $a_2 > 0$ and downward if $a_2 < 0$.

Power Functions

A function of the form $f(x) = x^a$, where a is a constant, is called a power function.

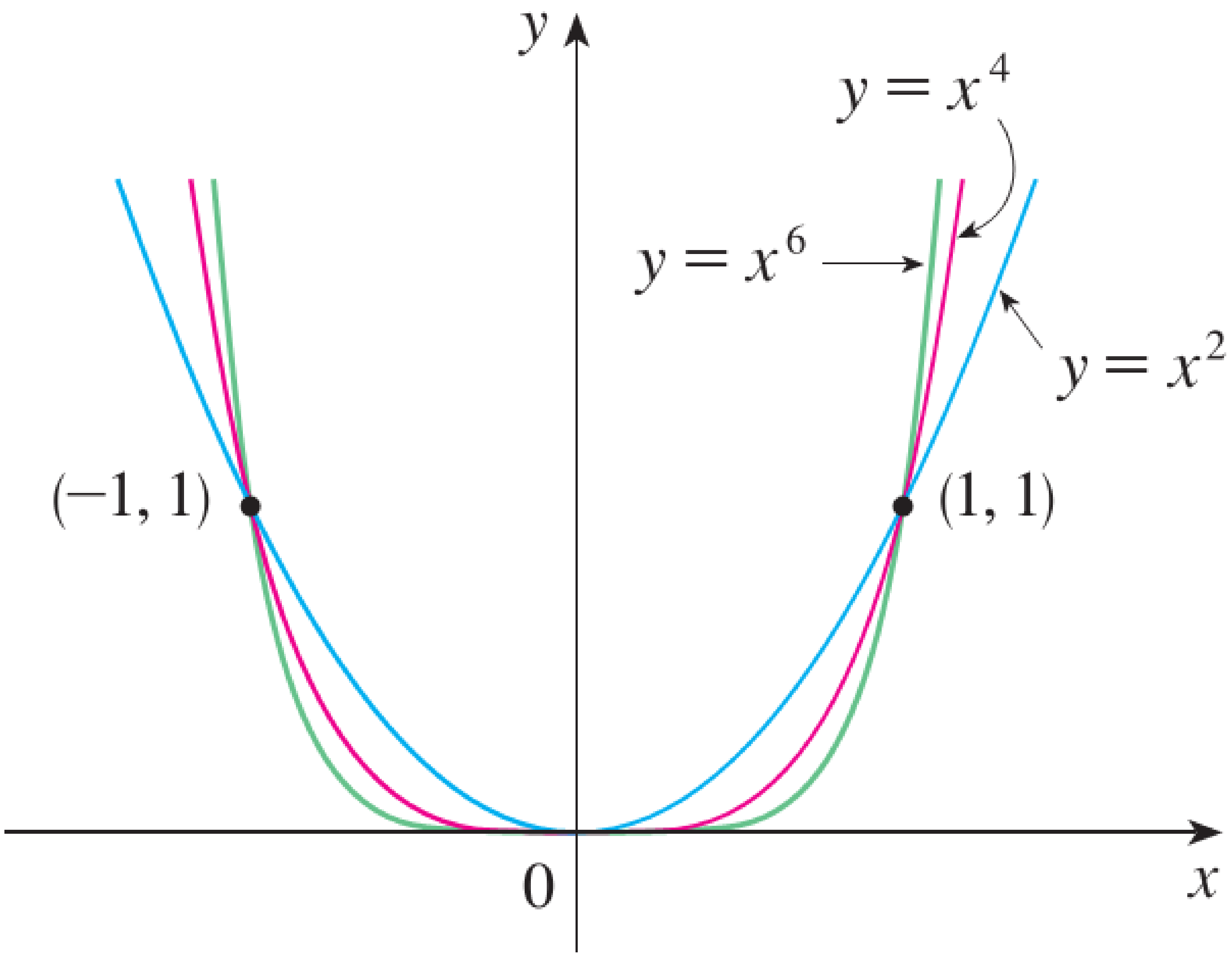
Examples

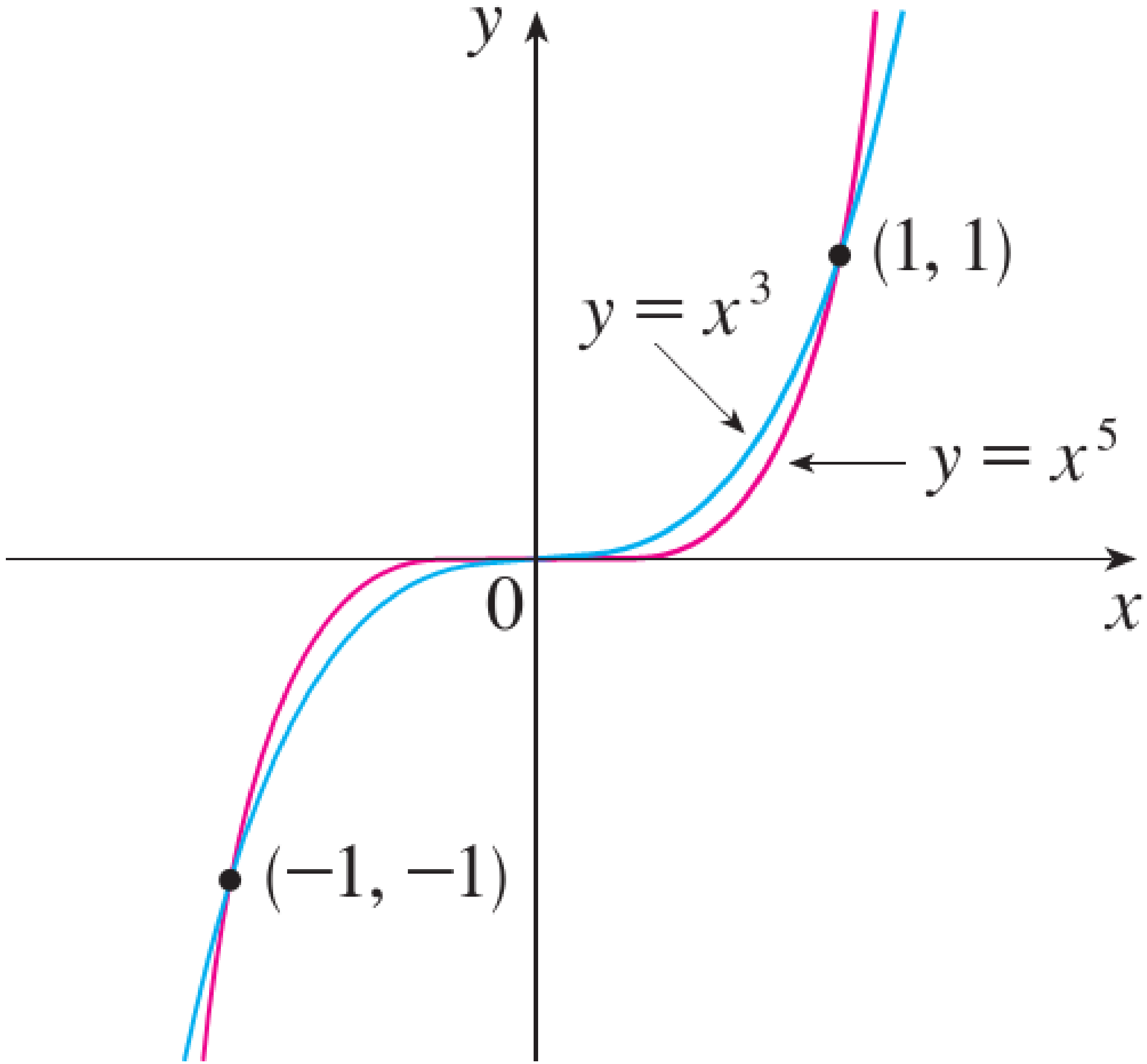
$$f(x) = x^2, \;\; g(x) = x^4 \text{ etc.}$$

(i) $a = n$, where n is a positive integer

- The general shape of the graph of $f(x) = x^n$ depends on whether n is even or odd.
- If n is even, then $f(x) = x^n$ is an even function and its graph is similar to the parabola $y = x^2$.

- If n is odd, then $f(x) = x^n$ is an odd function and its graph is similar to that of $y = x^3$. As n increases, the graph of $y = x^n$ becomes flatter near 0 and steeper when $|x| > 1$. (If x is small, then x^2 is smaller, x^3 is more smaller than x^2 and so on.)


y
y = x^4
y = x^6
y = x^2
(−1, 1)
(1, 1)
0
x

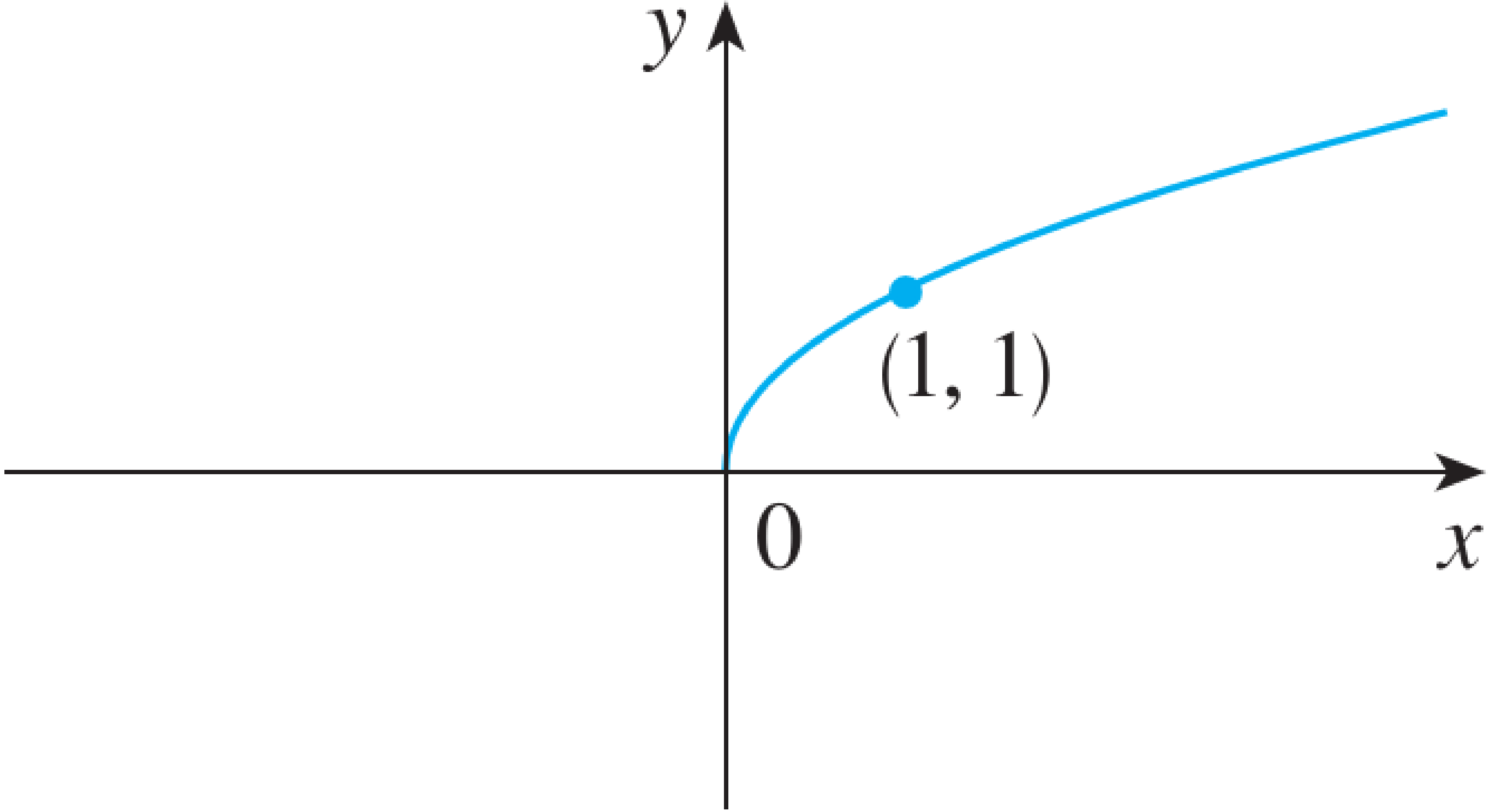

y
x
0
(1, 1)
(−1, −1)
y = x^3
y = x^5

(ii) $a = \frac{1}{n}$, where n is a positive integer

- The function $f(x) = x^{\frac{1}{n}} = \sqrt[n]{x}$ is a root function.

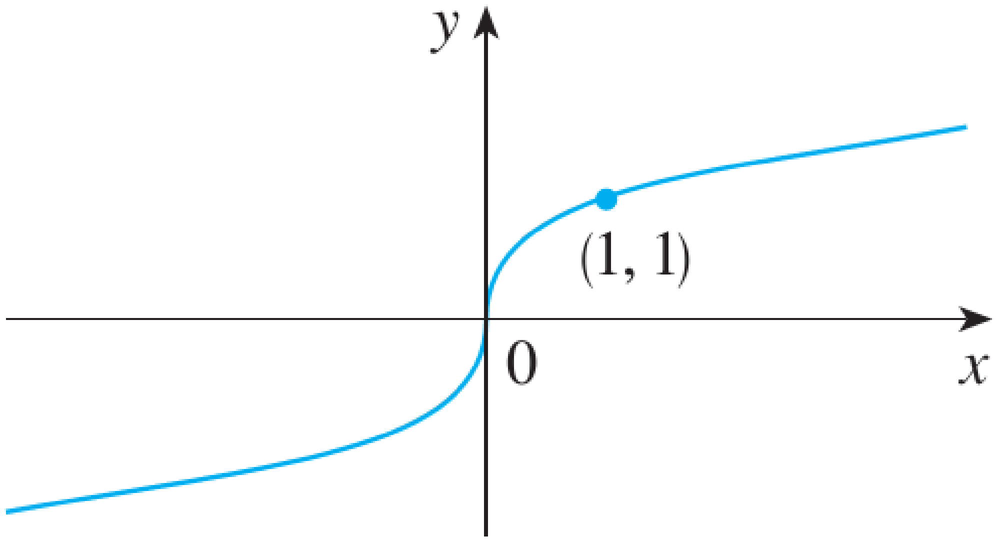
- For $n = 2$ it is the square root function $f(x) = \sqrt{x}$ whose domain is $[0, \infty)$ and whose graph is the upper half of the parabola $x = y^2$.

- For other even values of n, the graph of $y = \sqrt[n]{x}$ is similar to that of $y = \sqrt{x}$.



$f(x) = \sqrt{x}$

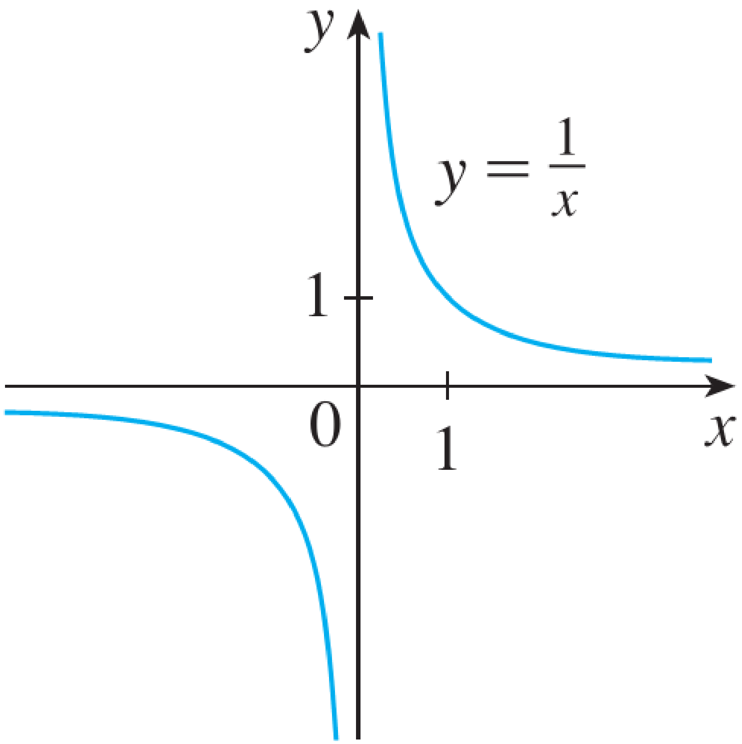
For $n = 3$ we have the cube root function $f(x) = \sqrt[3]{x}$ whose domain is R (recall that every real number has a cube root) and whose graph is shown in Figure.



$f(x) = \sqrt[3]{x}$

(ii) $a = -1$

The graph of the reciprocal function $f(x) = x^{-1} = \frac{1}{x}$ is shown in the figure.

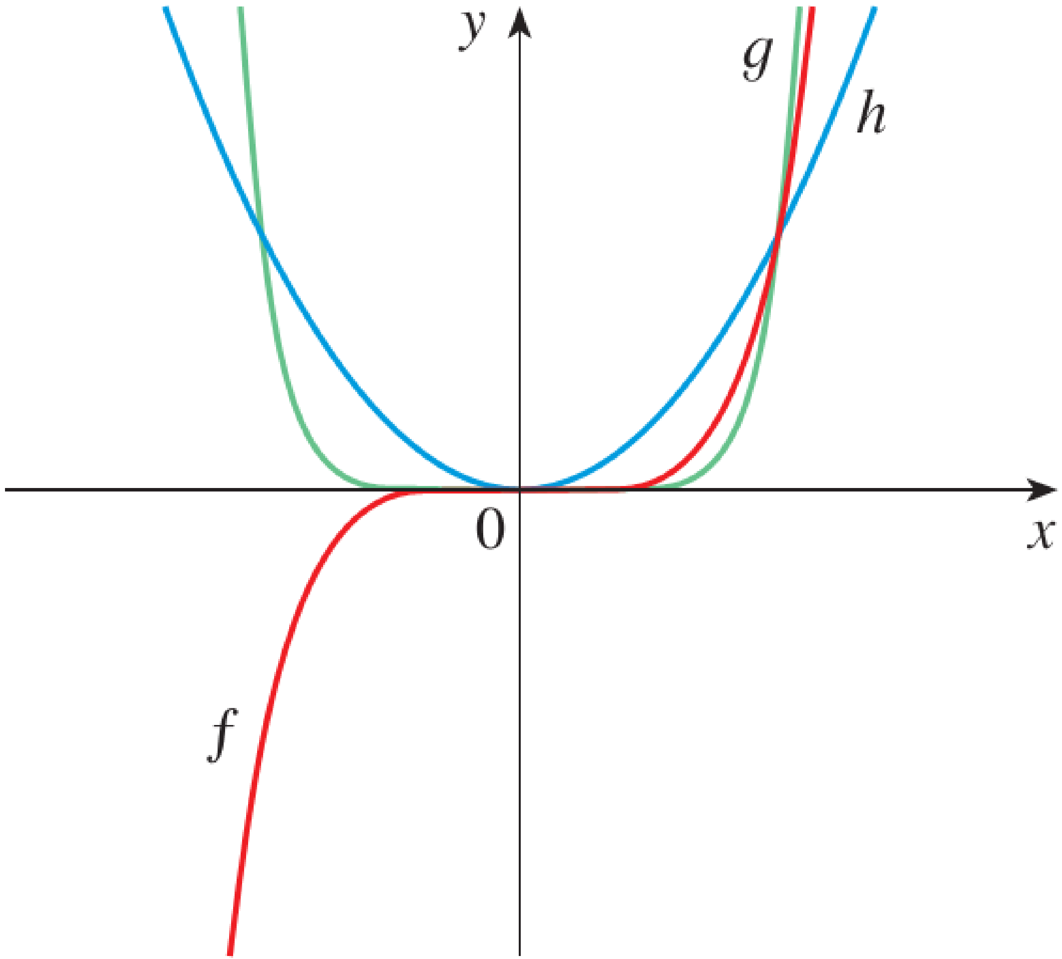


Practice Problem

Match each equation with its graph.

(Don’t use a computer or graphing calculator.)

(a) $y = x^2$ (b) $y = x^5$ (c) $y = x^8$



Rational Functions

A **rational function** f is a ratio of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials.

The domain consists of all values of x such that $Q(x) \neq 0$.

Example

$$f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 4}$$

is a rational function with domain $\{x \mid x \neq \pm 2\}$.

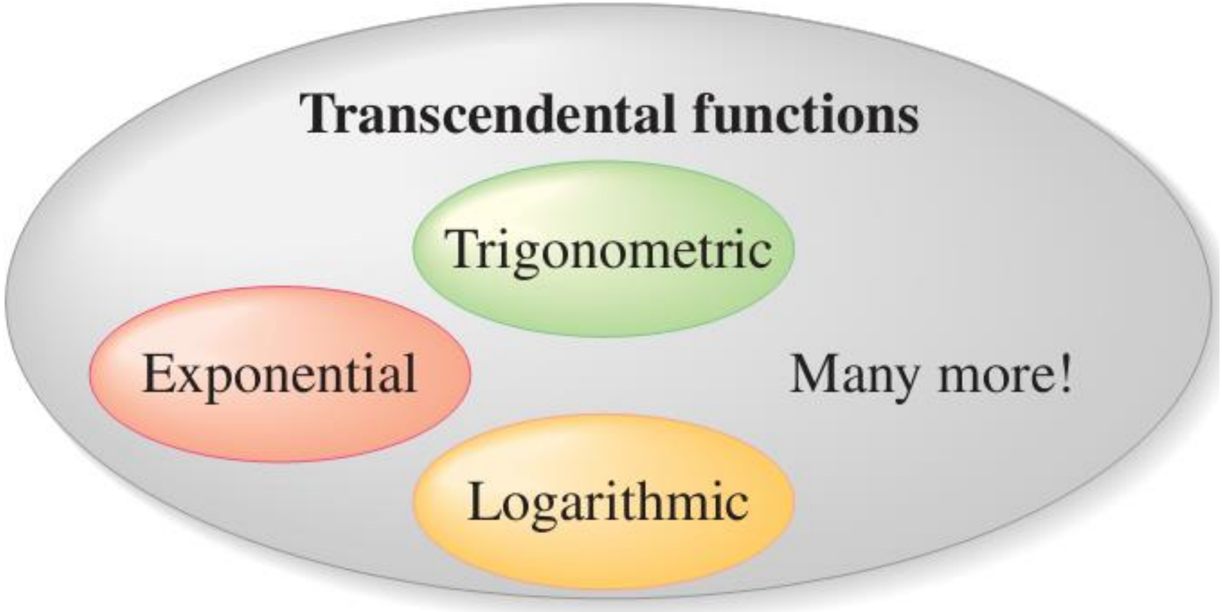
Algebraic Function

A function f is called an **algebraic function** if it can be constructed using algebraic operations (such as addition, subtraction, multiplication, division, and taking roots) starting with polynomials. Any rational function is automatically an algebraic function. Here are two more examples:

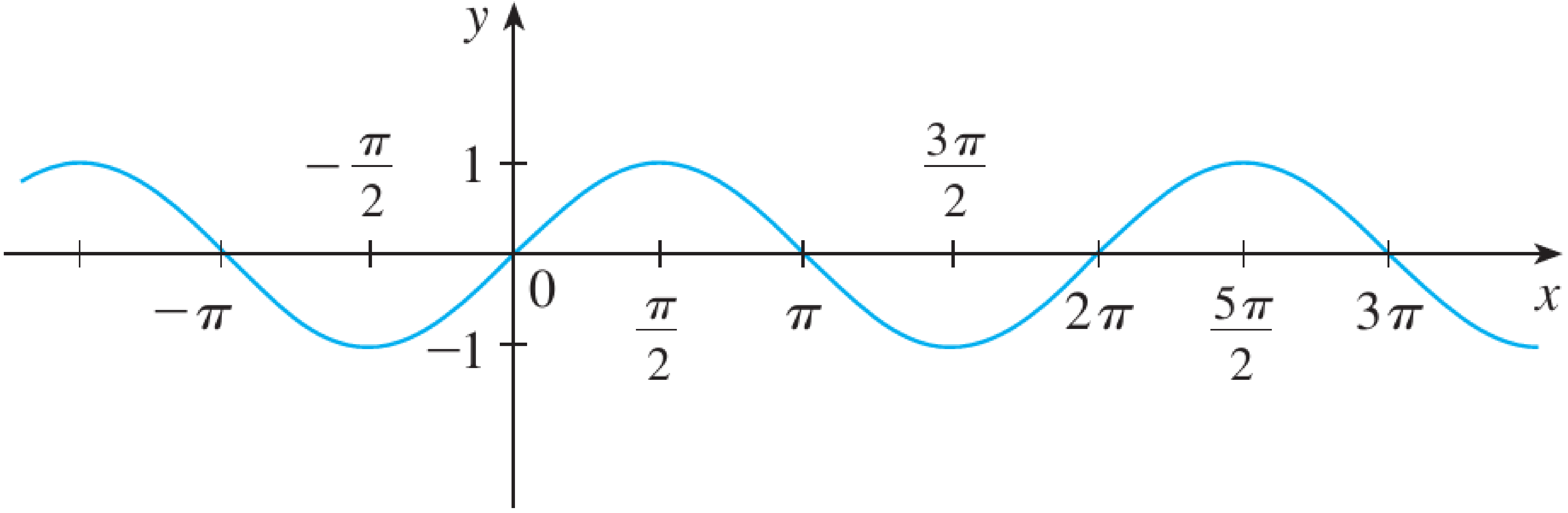
$$f(x) = \sqrt{x^2 + 1} \qquad\qquad g(x) = \frac{x^4 - 16x^2}{x + \sqrt{x}} + (x - 2)\sqrt[3]{x + 1}$$

Transcendental Functions

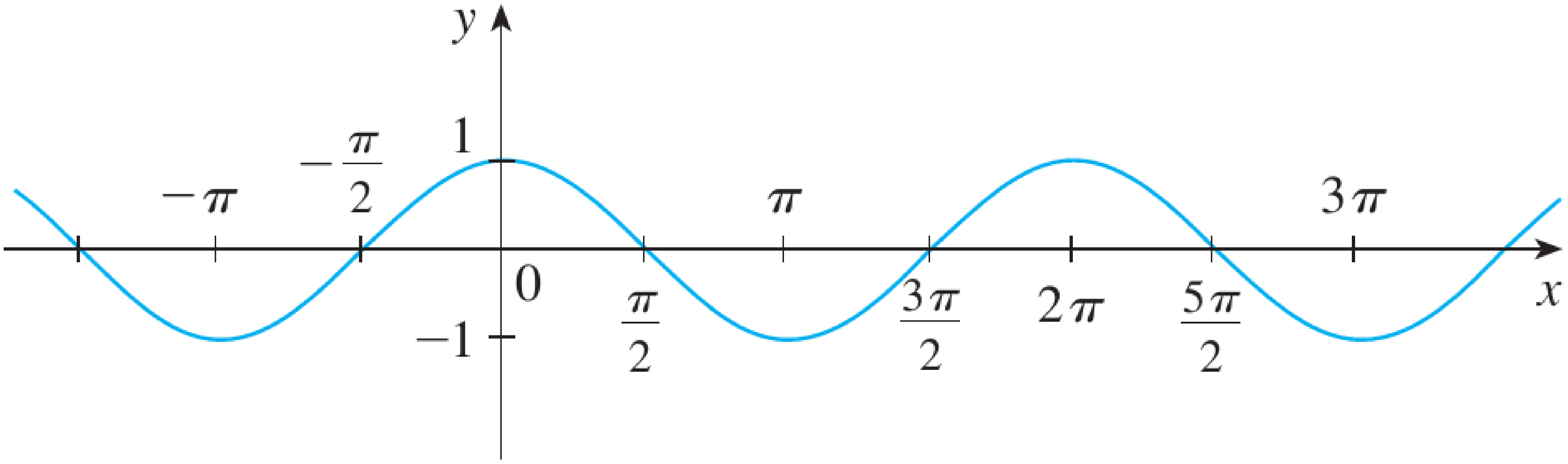
These are functions that are not algebraic. They include the trigonometric, inverse trigonometric, exponential, and logarithmic functions, and many other functions as well.



Trigonometric Functions



$$f(x) = \sin x$$



$$g(x) = \cos x$$

Home Work

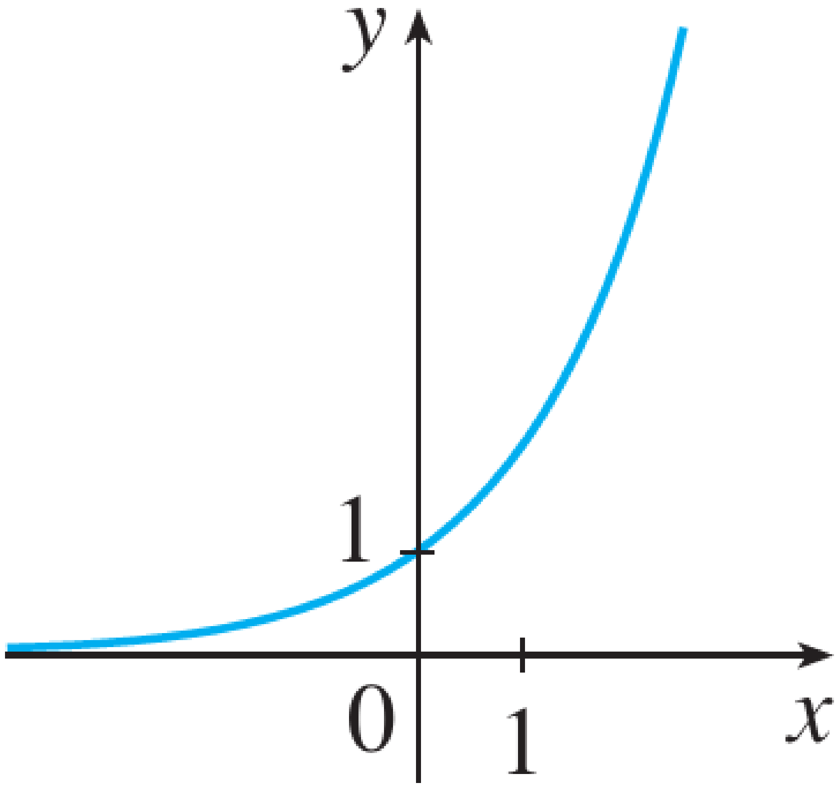
Sketch the graph of all trigonometric function.

Exponential Functions

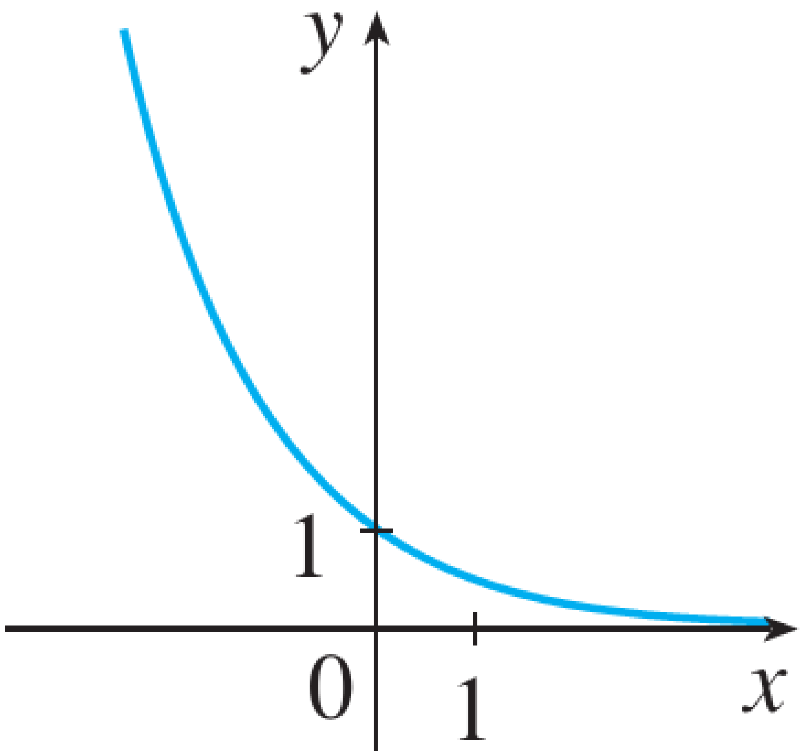
The **exponential functions** are the functions of the form $f(x) = b^x$, where the base b is a positive constant.

The graphs of $y = 2^x$ and $y = (0.5)^x$ are shown in Figure **on the next slide**.

In both cases the domain is $(-\infty, \infty)$ and the range is $(0, \infty)$.

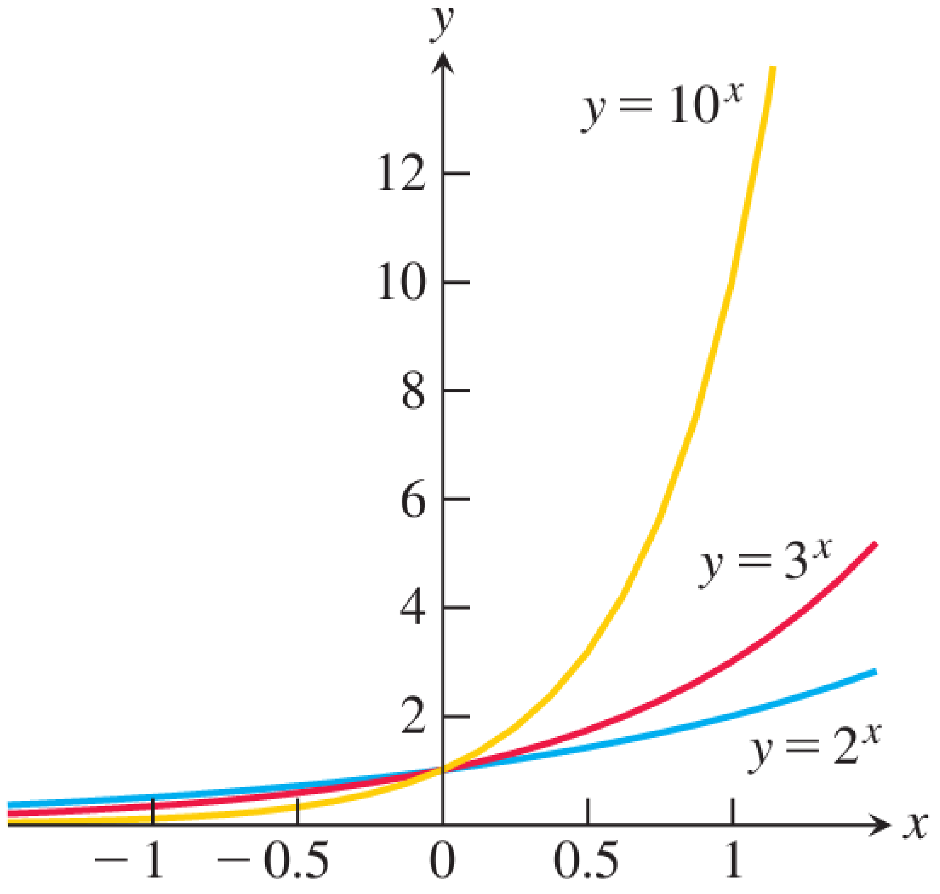


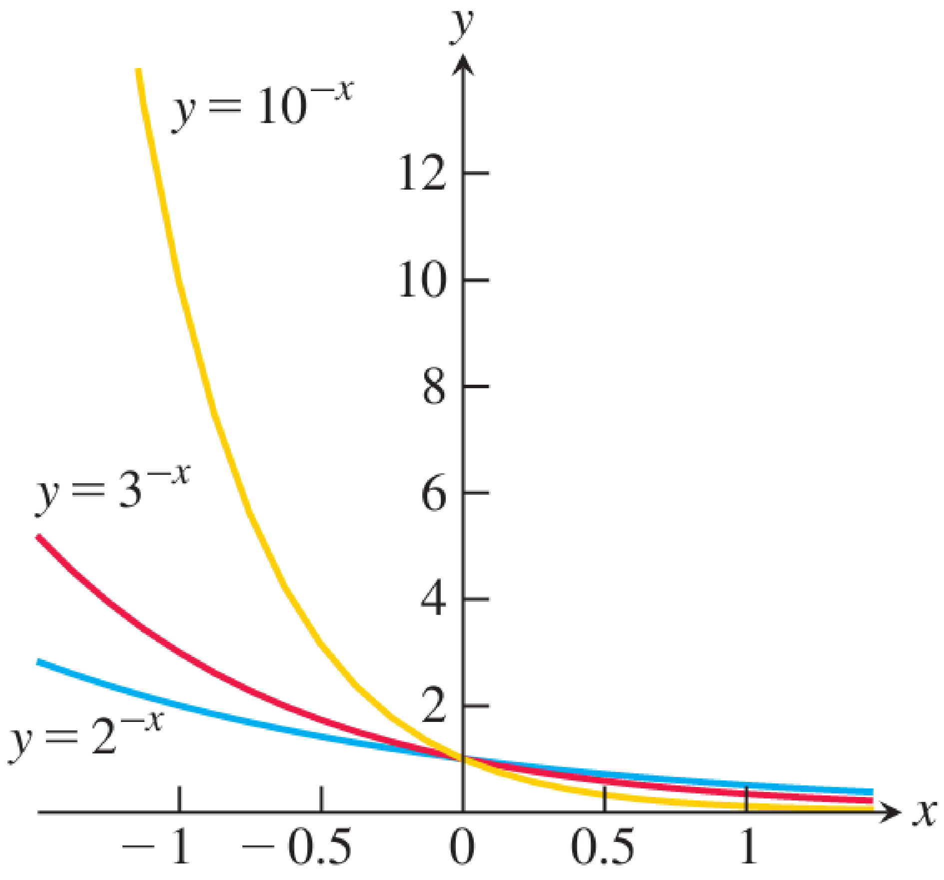
$y = 2^x$



$y = (0.5)^x$

Graph of some Exponential Functions



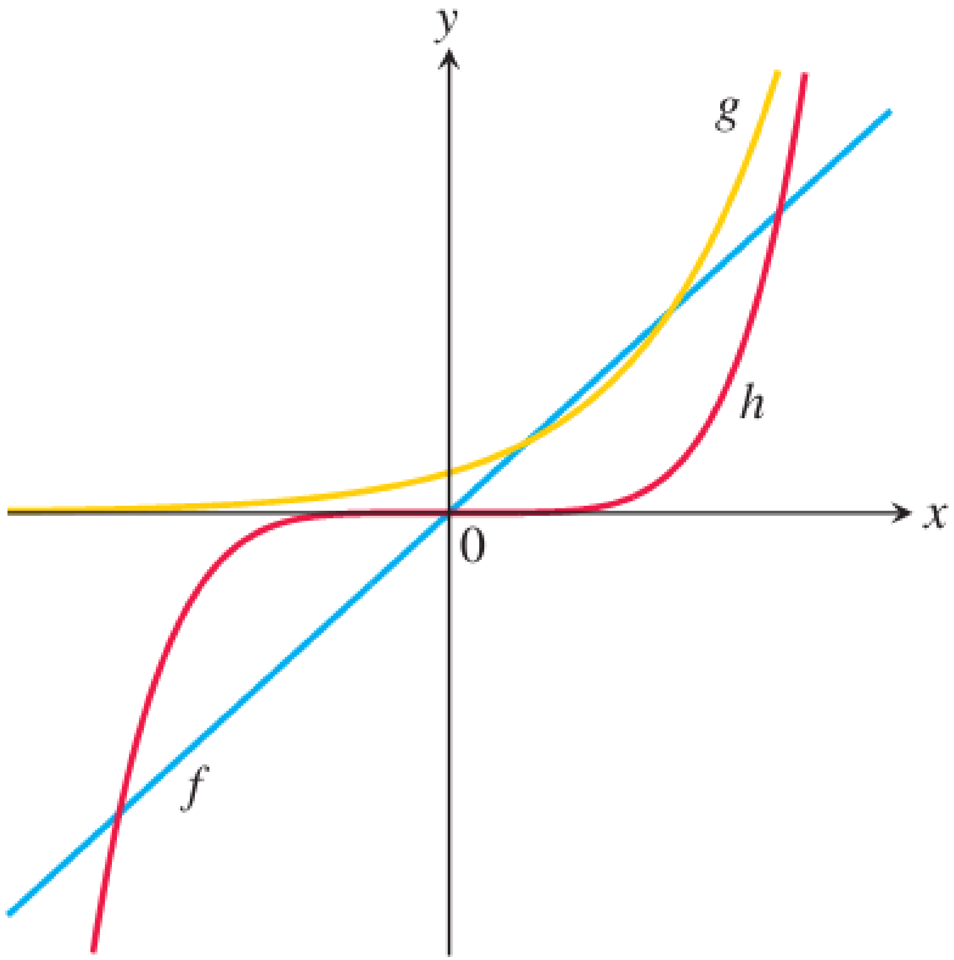


Practice Problem

Match each equation with its graph.

(Don't use a computer or graphing calculator.)

a. $y = 5x$ **b.** $y = 5^x$ **c.** $y = x^5$



Practice Problem

Wingspan and weight : The weight W (in pounds) of a bird (that can fly) has been related to the wingspan L (in inches) of the bird by the power function $L = 30.6\, W^{0.3952}$.

(i) An eagle has a wingspan of about 90 inches. Use the model to estimate the weight of the eagle.

(ii) An ostrich weighs about 300 pounds. Use the model to estimate what the wingspan of an ostrich should be in order for it to fly.

(iii) The wingspan of an ostrich is about 72 inches. Use your answer to part (b) to explain why ostriches can't fly.

Hints :

$$L = 30.6\,W^{.3952}$$

$$90 = 30.6\,w^{.3952}$$

$$\frac{90}{30.6} = w^{.3952}$$

$$w = \left(\frac{90}{30.6}\right)^{\frac{1}{.3952}} \approx 15.3$$

$$L = 30.6(300)^{.3952}$$

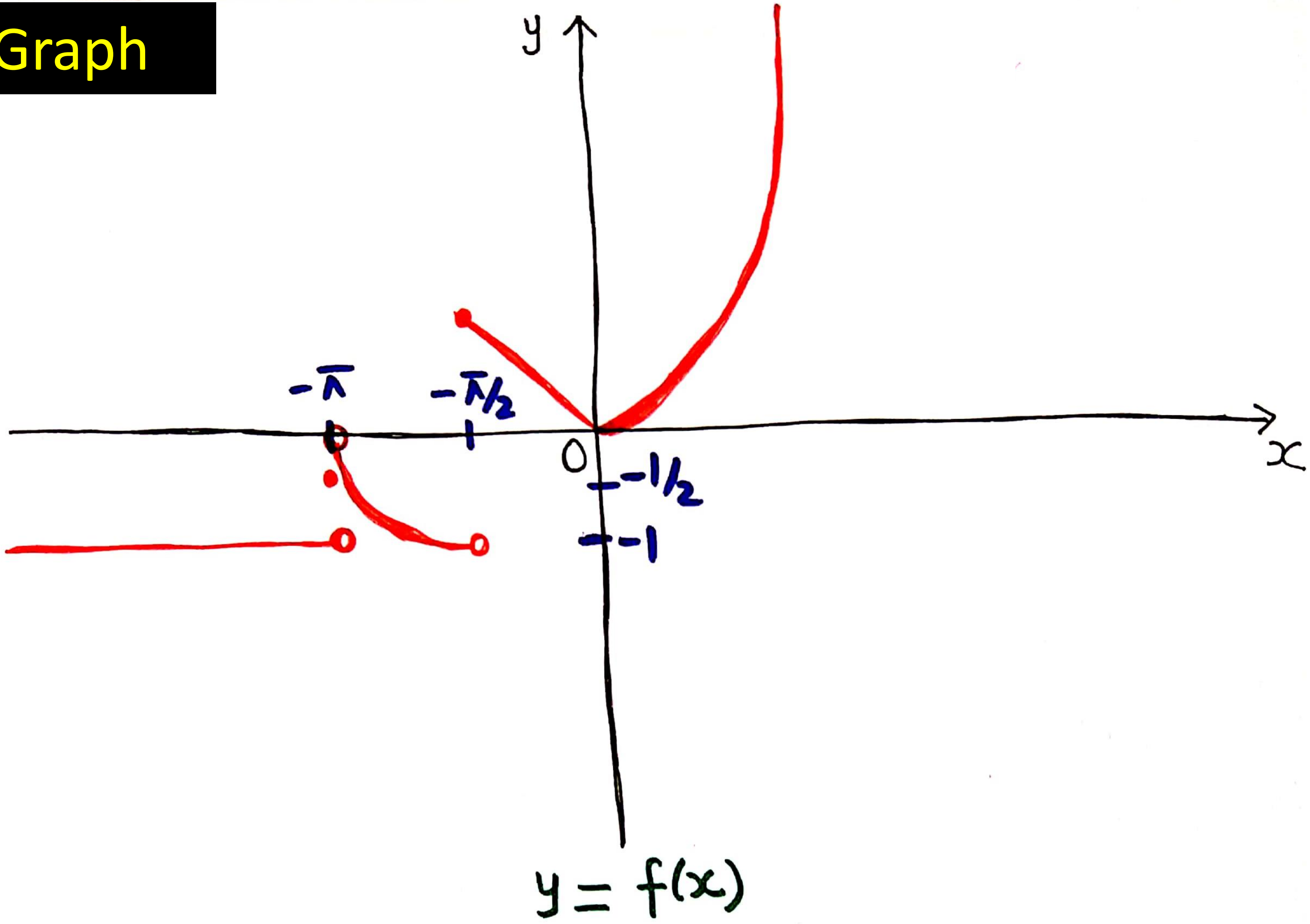
$$\approx 291.5$$

Problem

Sketch the graph of the following function.

$$f(x)=\begin{cases} -1 & \text{if } x<-\pi \\ -\dfrac{1}{2} & \text{if } x=-\pi \\ \sin x & \text{if } -\pi<x<-\dfrac{\pi}{2} \\ |x| & \text{if } -\dfrac{\pi}{2}\le x<0 \\ x^2 & \text{if } x\ge 0 \end{cases}$$

Graph

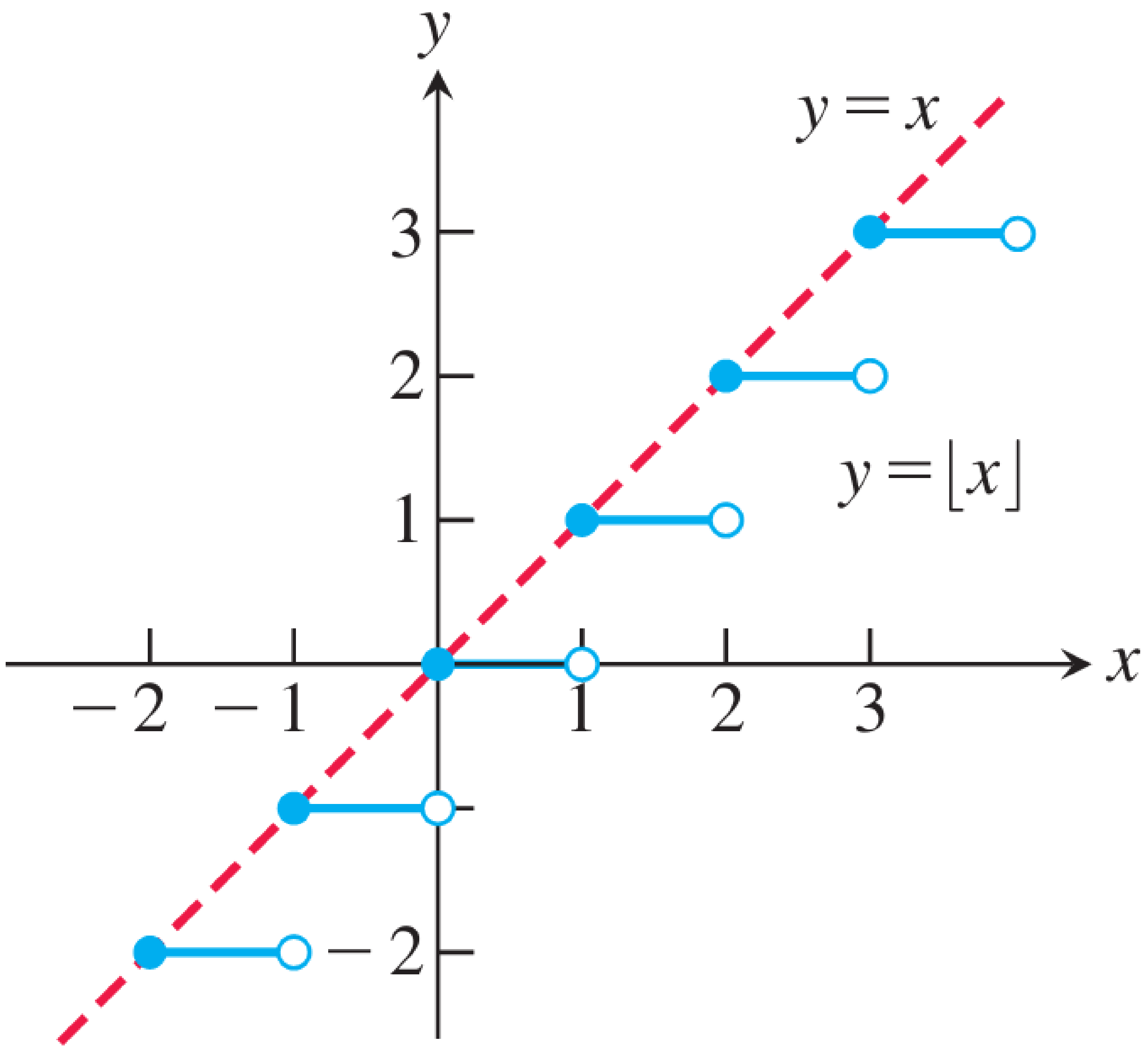


Greatest Integer Function or Integer Floor function

The function whose value at any number x is the greatest integer less than or equal to x is called the greatest integer function or the integer floor function. It is denoted by $y = \lfloor x \rfloor$.

$$\lfloor 2.4 \rfloor = 2, \quad \lfloor 1.9 \rfloor = 1, \quad \lfloor 0 \rfloor = 0, \quad \lfloor -1.2 \rfloor = -2,$$
$$\lfloor 2 \rfloor = 2, \quad \lfloor 0.2 \rfloor = 0, \quad \lfloor -0.3 \rfloor = -1, \quad \lfloor -2 \rfloor = -2.$$

$$y = \lfloor x \rfloor$$

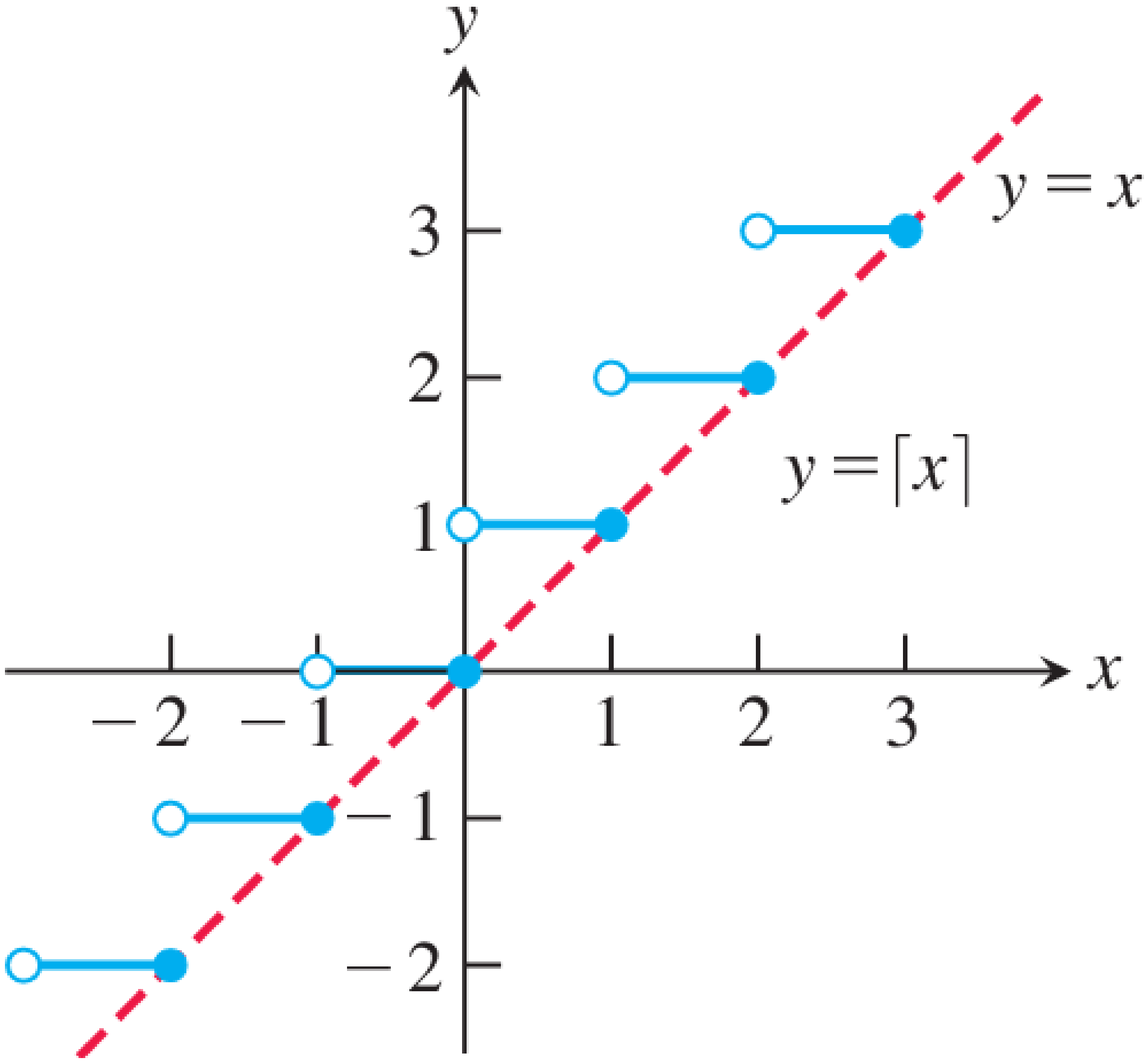


Least integer function or integer ceiling function

The function whose value at any number x is the smallest integer greater than or equal to x is called the least integer function or the integer ceiling function. It is denoted by $y = \lceil x \rceil$.

$\lceil 1 \rceil = 1$, $\lceil 1.1 \rceil = 2$, $\lceil 2.8 \rceil = 3$

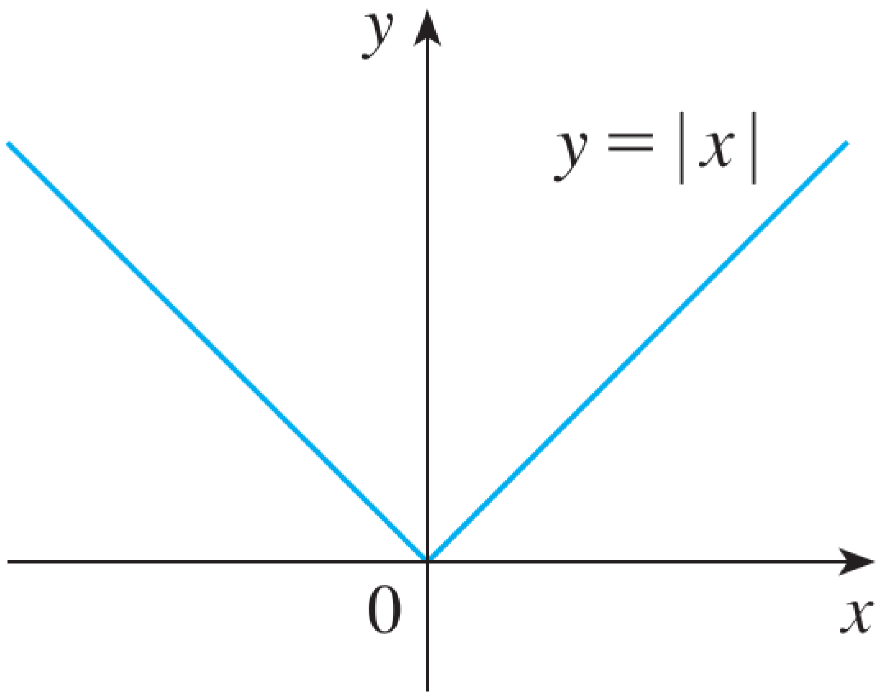
$\lceil -0.1 \rceil = 0$, $\lceil -2.9 \rceil = -2$ etc.


y
x
y = x
y = ⌈x⌉
3
2
1
−1
−2
−2
−1
1
2
3

Absolute Function

$$|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$$

Graph



Practice Problems

Q1. Sketch the graph of the following functions.

a.

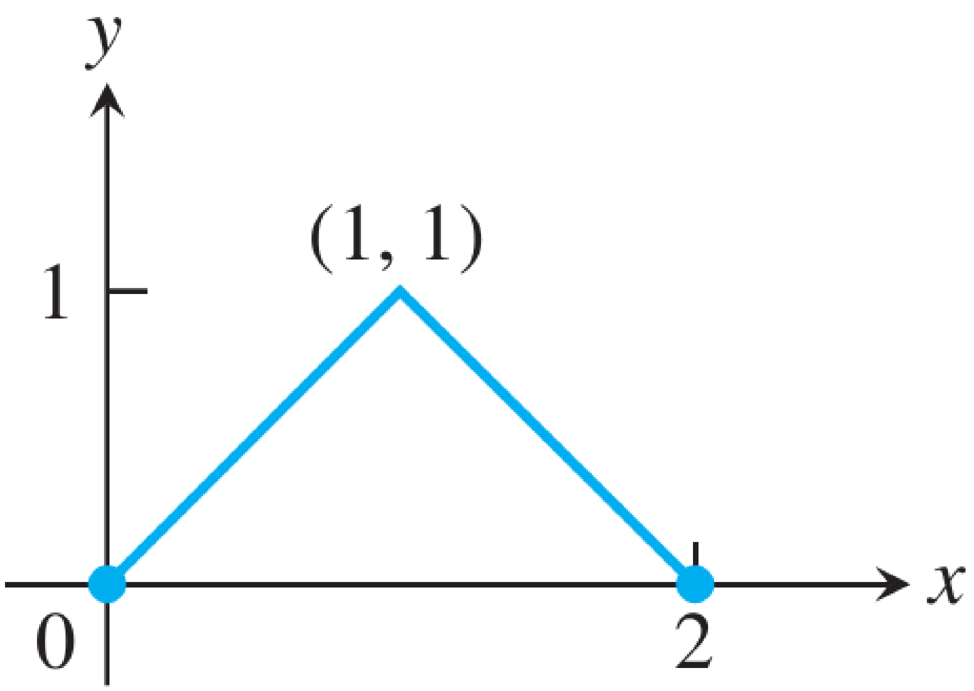
$$f(x) = \begin{cases} -1 & \text{if } x < -1 \\ 0 & \text{if } x = -1 \\ -x & \text{if } -1 < x < 1 \\ x^2 & \text{if } 1 \le x < 2 \\ -2 & \text{if } x \geq 2 \end{cases}$$

b.

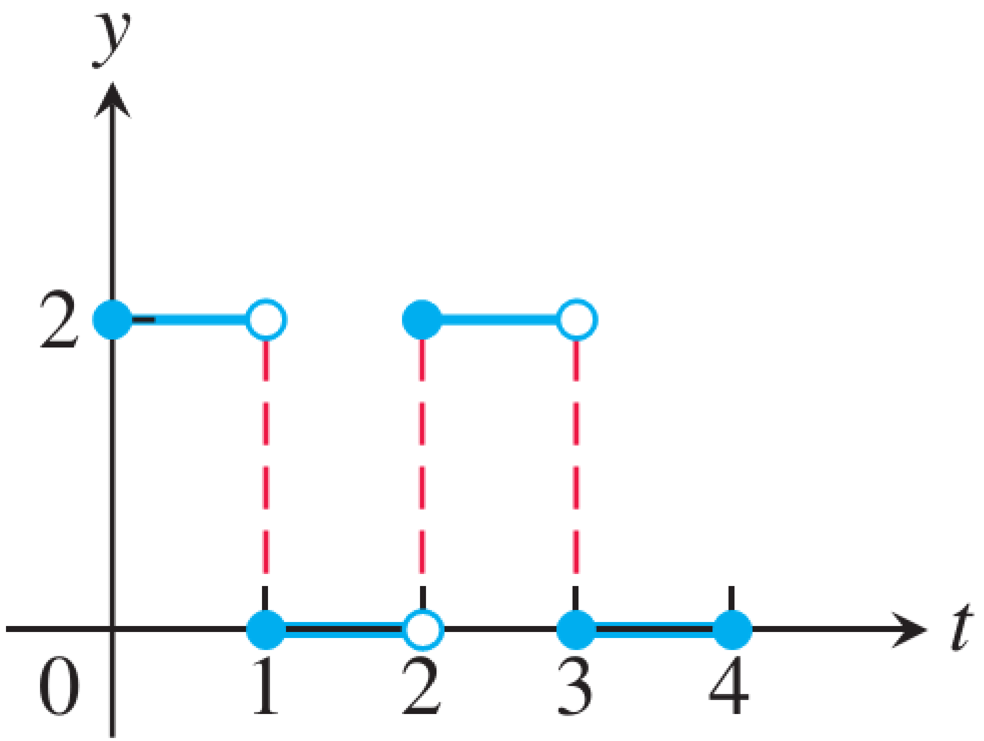
$$G(x) = \begin{cases} 1/x, & x < 0 \\ x, & 0 \le x \end{cases}$$

Q2. Find a formula for each function graphed.

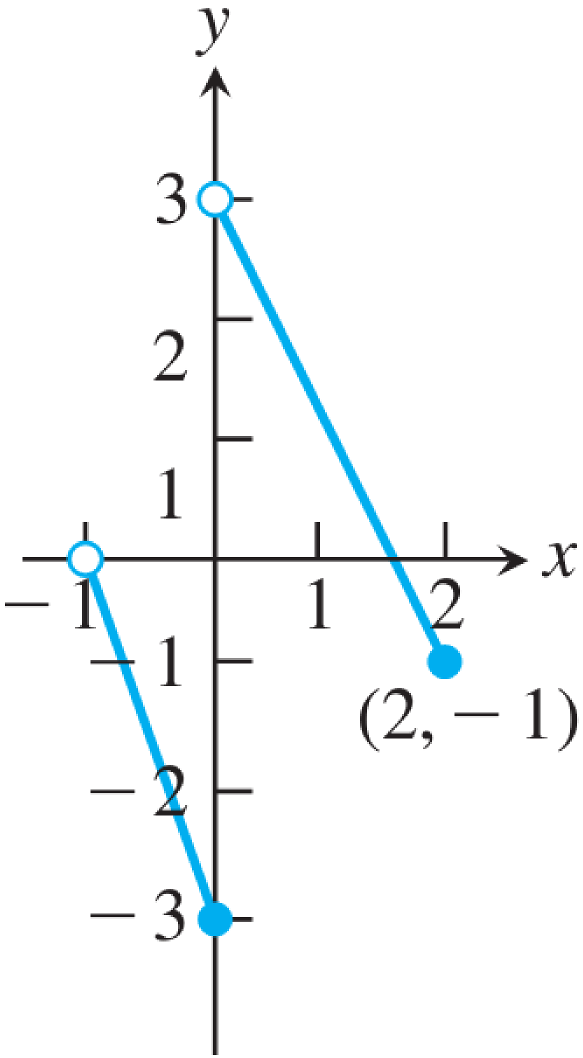
a.



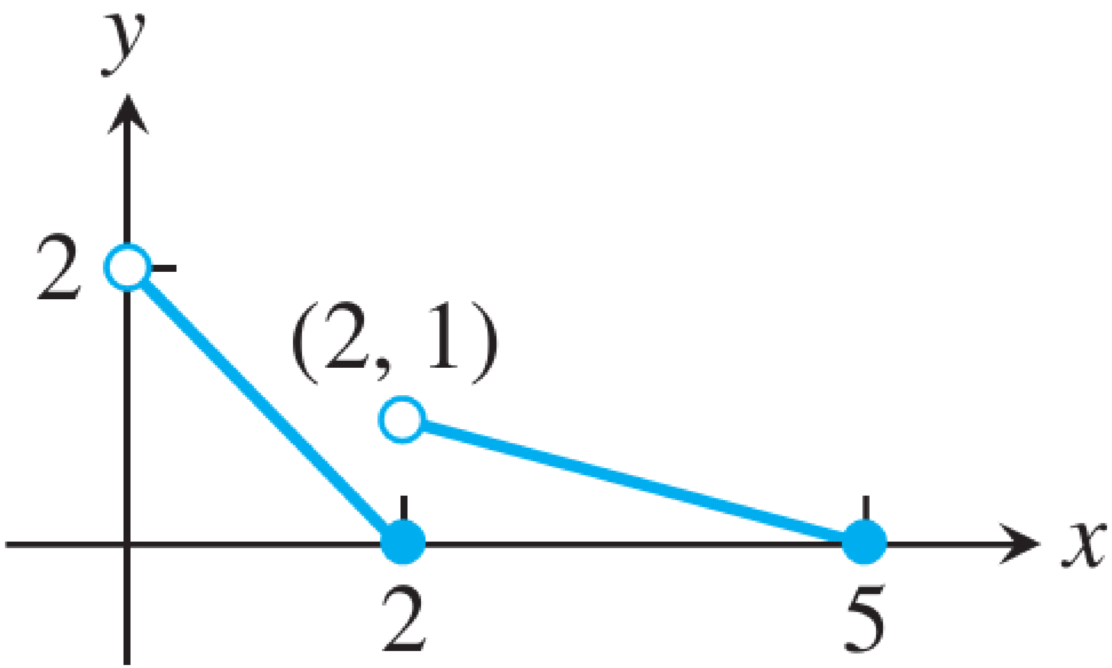
b.



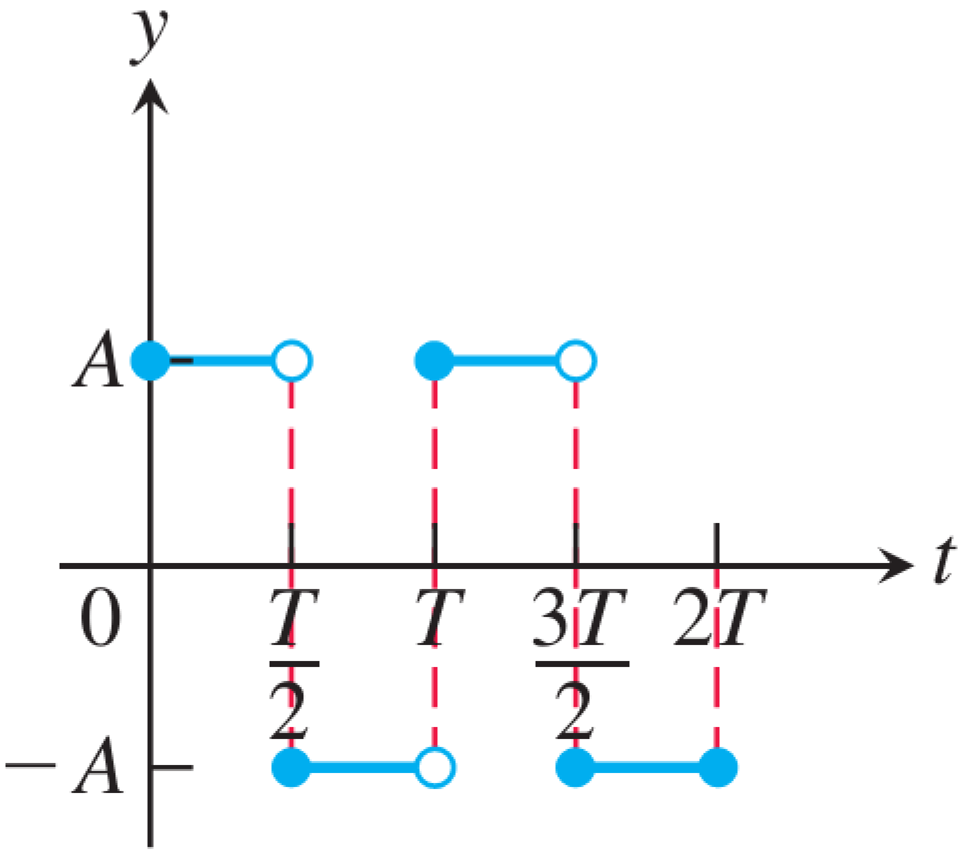
c.



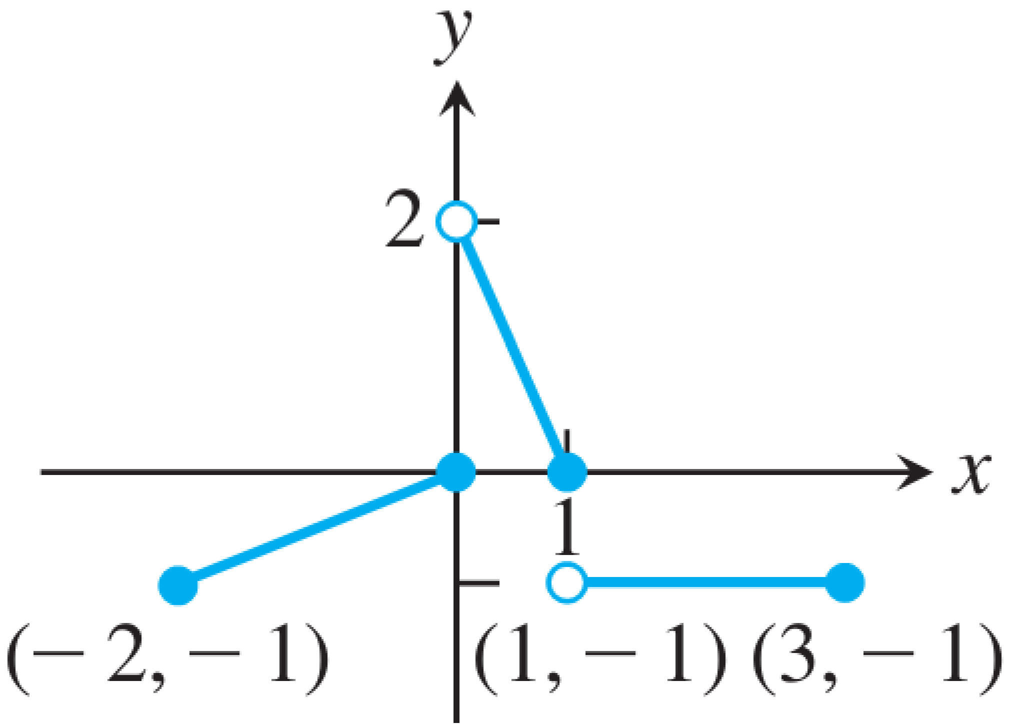
d.



e.



f.



New Functions from Old Functions

Translation

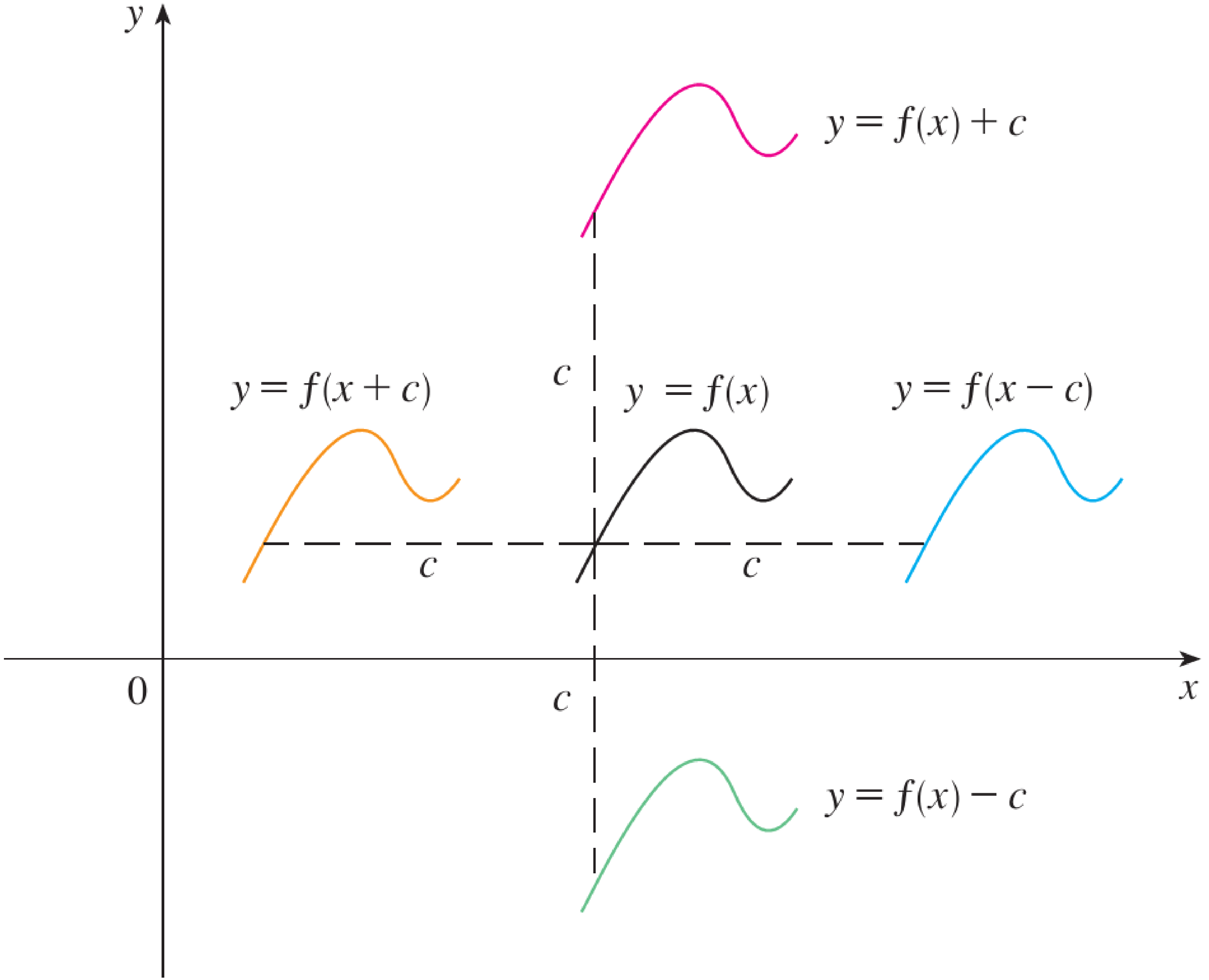
Vertical and Horizontal Shifts Suppose $c > 0$. To obtain the graph of

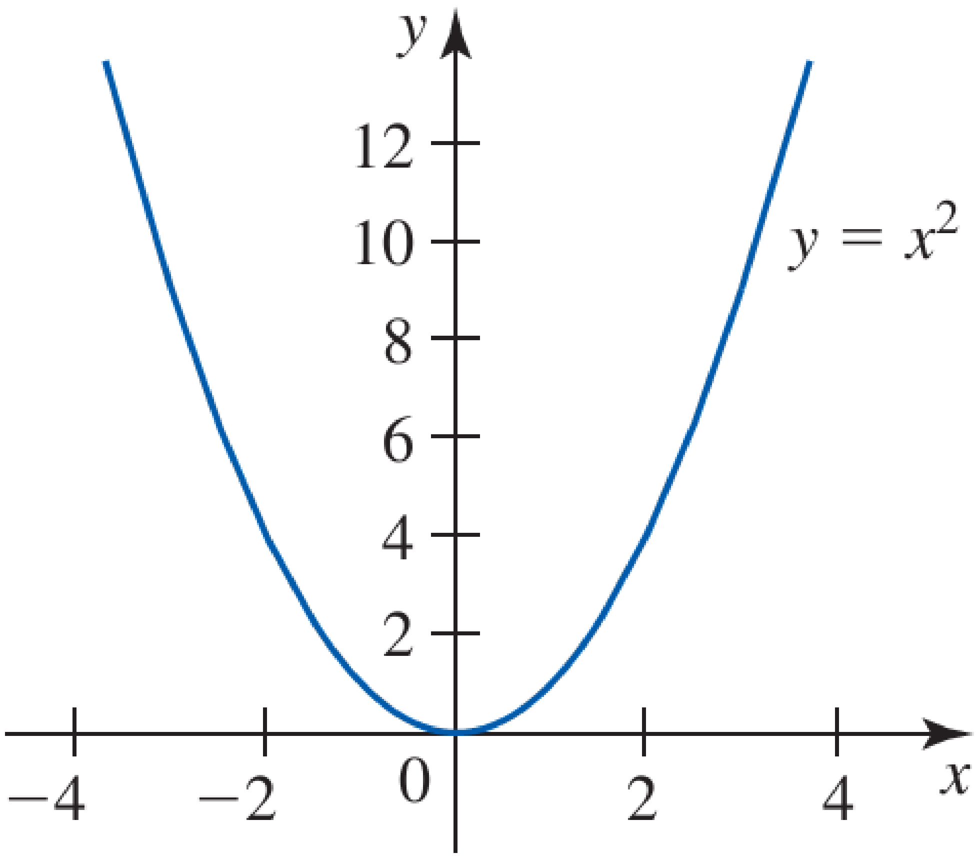
$y = f(x) + c$, shift the graph of $y = f(x)$ a distance c units upward

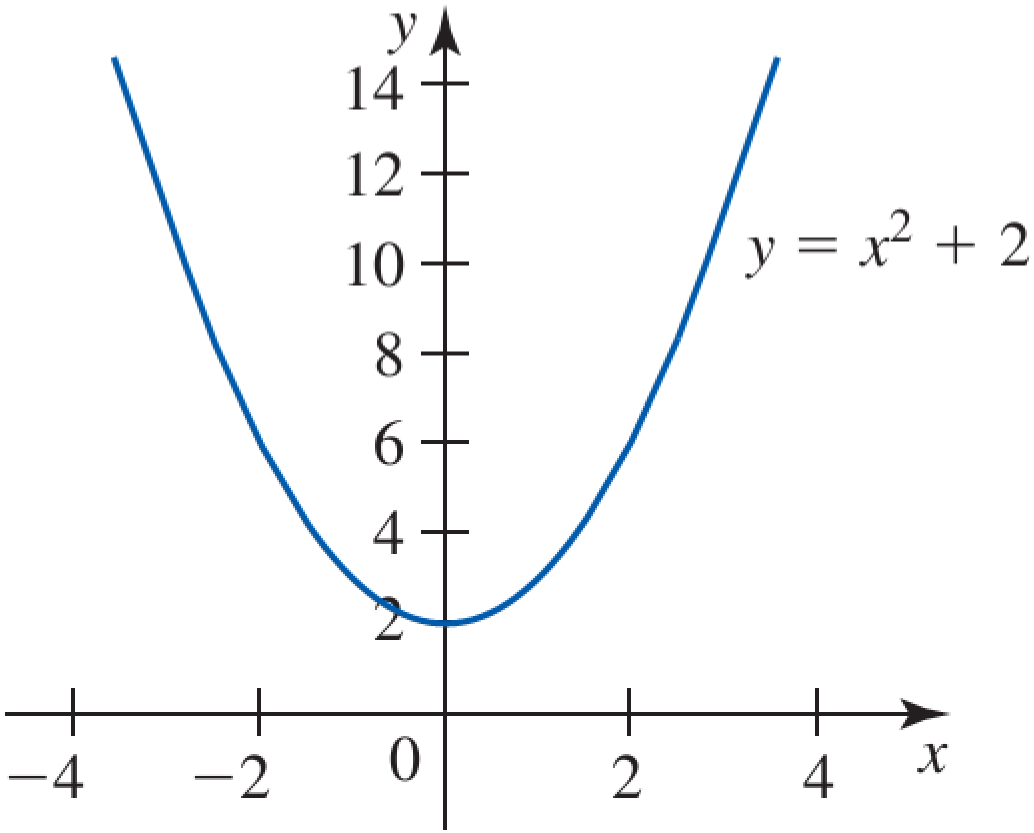
$y = f(x) - c$, shift the graph of $y = f(x)$ a distance c units downward

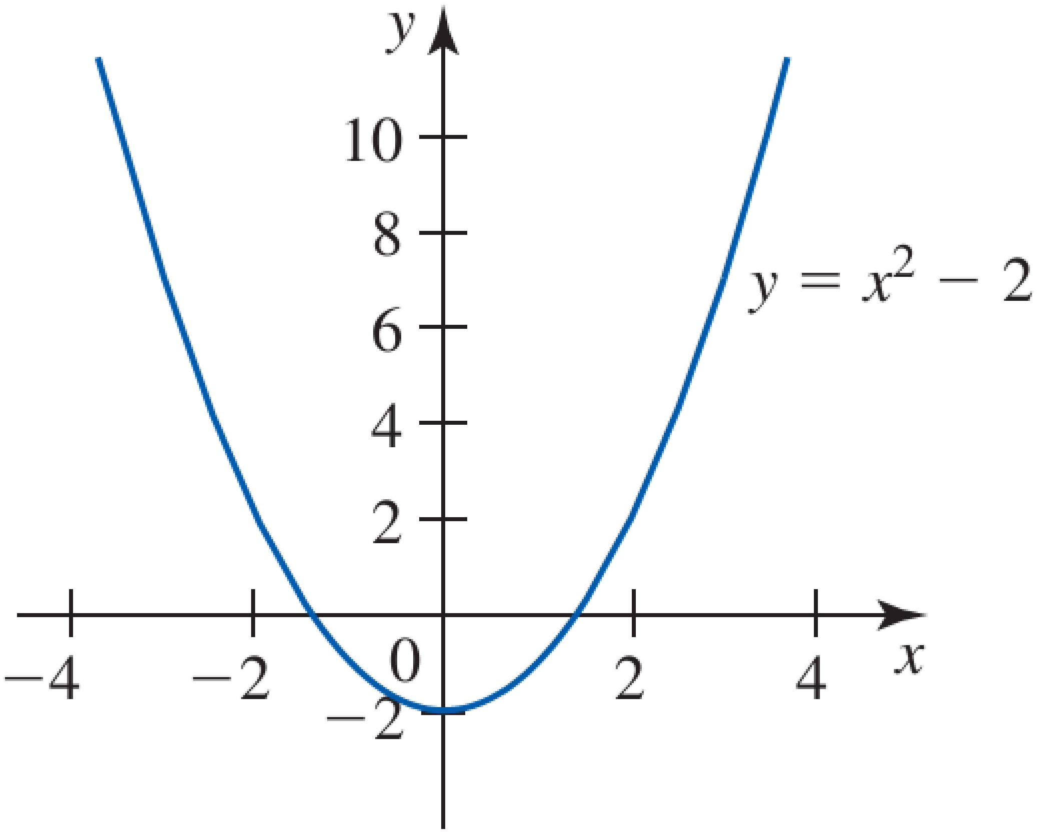
$y = f(x - c)$, shift the graph of $y = f(x)$ a distance c units to the right

$y = f(x + c)$, shift the graph of $y = f(x)$ a distance c units to the left

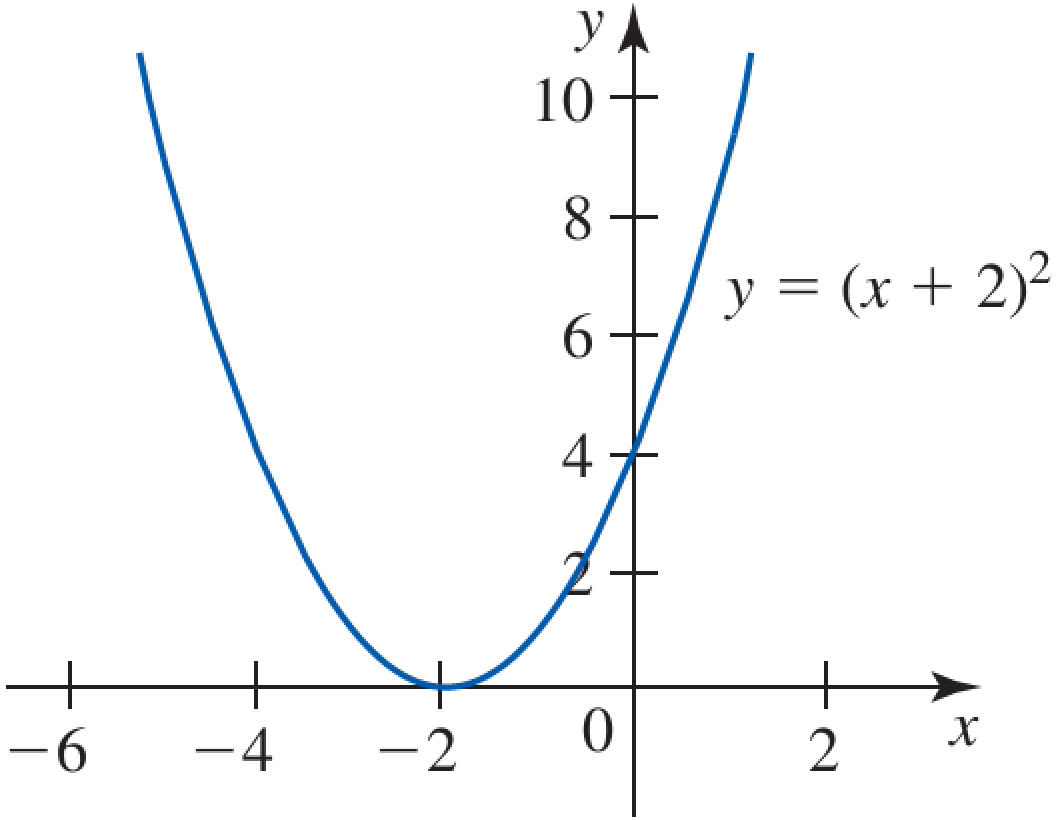

y
y = f(x) + c
c
y = f(x + c)
y = f(x)
y = f(x − c)
c
c
0
c
x
y = f(x) − c


y
12
10
8
6
4
2
$y = x^2$
−4
−2
0
2
4
x

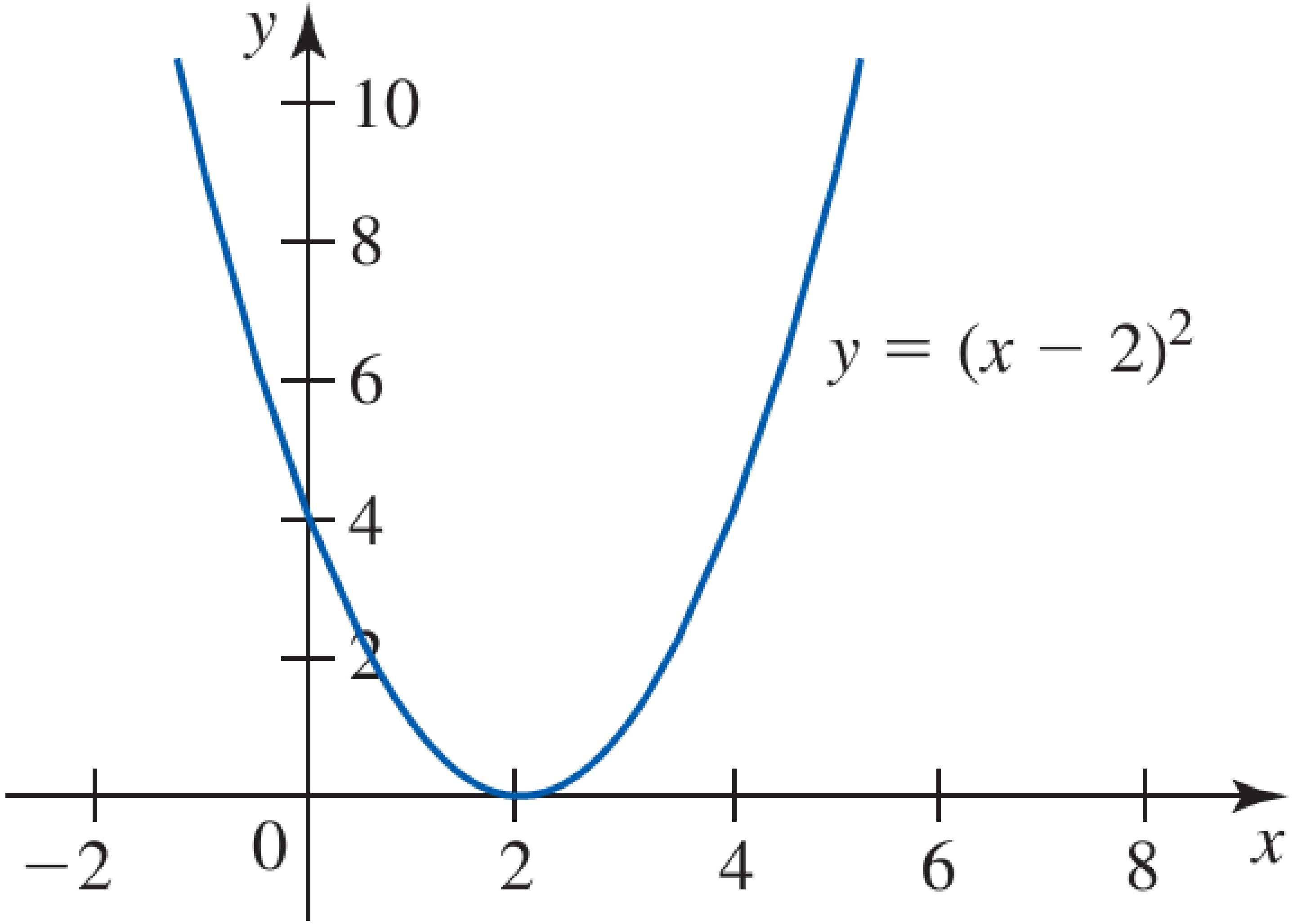

y
14
12
10
8
6
4
2
$y = x^2 + 2$
−4
−2
0
2
4
x



y
10
8
6
4
2
−4
−2
0
2
4
x
−2
y = x² − 2



y
10
8
6
4
2
−6
−4
−2
0
2
x
y = (x + 2)²


y
10
8
6
4
2
0
−2
2
4
6
8
x
$y = (x - 2)^2$

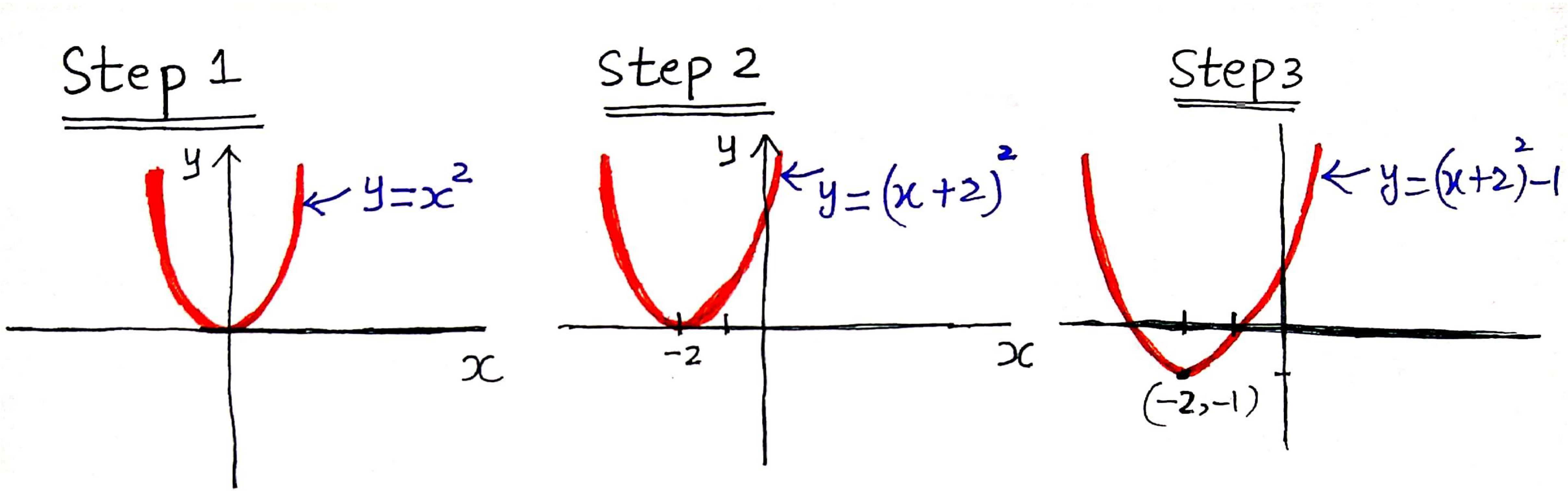
Example (Horizontal & Vertical Shifting both in one function)

We can get the graph of $y = (x + 2)^2 - 1$ in the following way:

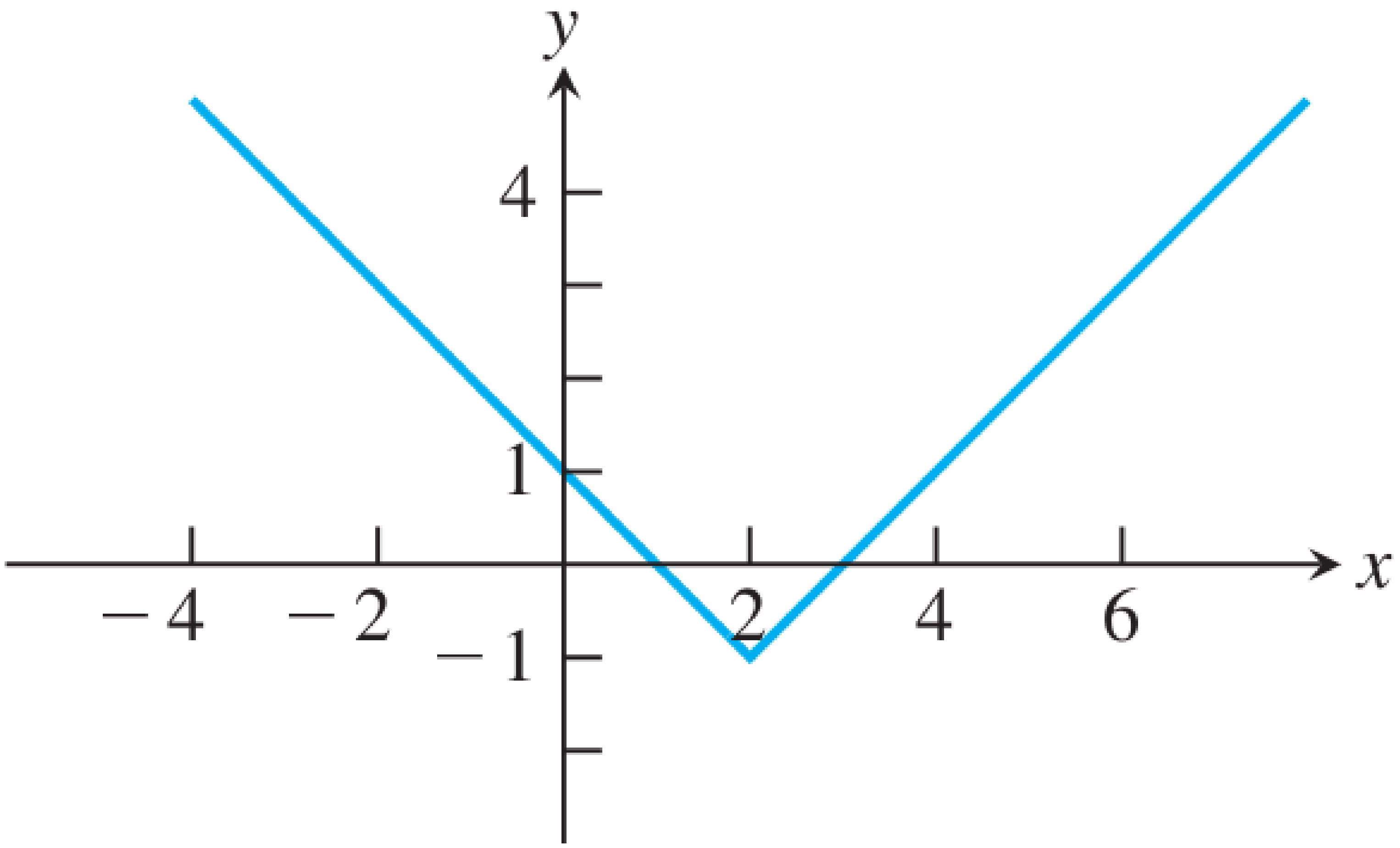
(i) We will first sketch the graph of the parabola $y = x^2$.

(ii) Then we will move the parabola 2 unit to the left to get the graph of $y = (x + 2)^2$.

(iii) Finally we will move the graph of $y = (x + 2)^2$ 1 unit down to get the graph of $y = (x + 2)^2 - 1$.


Step 1
y
$y = x^2$
x
Step 2
y
$y = (x+2)^2$
-2
x
Step 3
$y = (x+2)^2 - 1$
(-2, -1)

Problem: Find a formula for the function graphed.



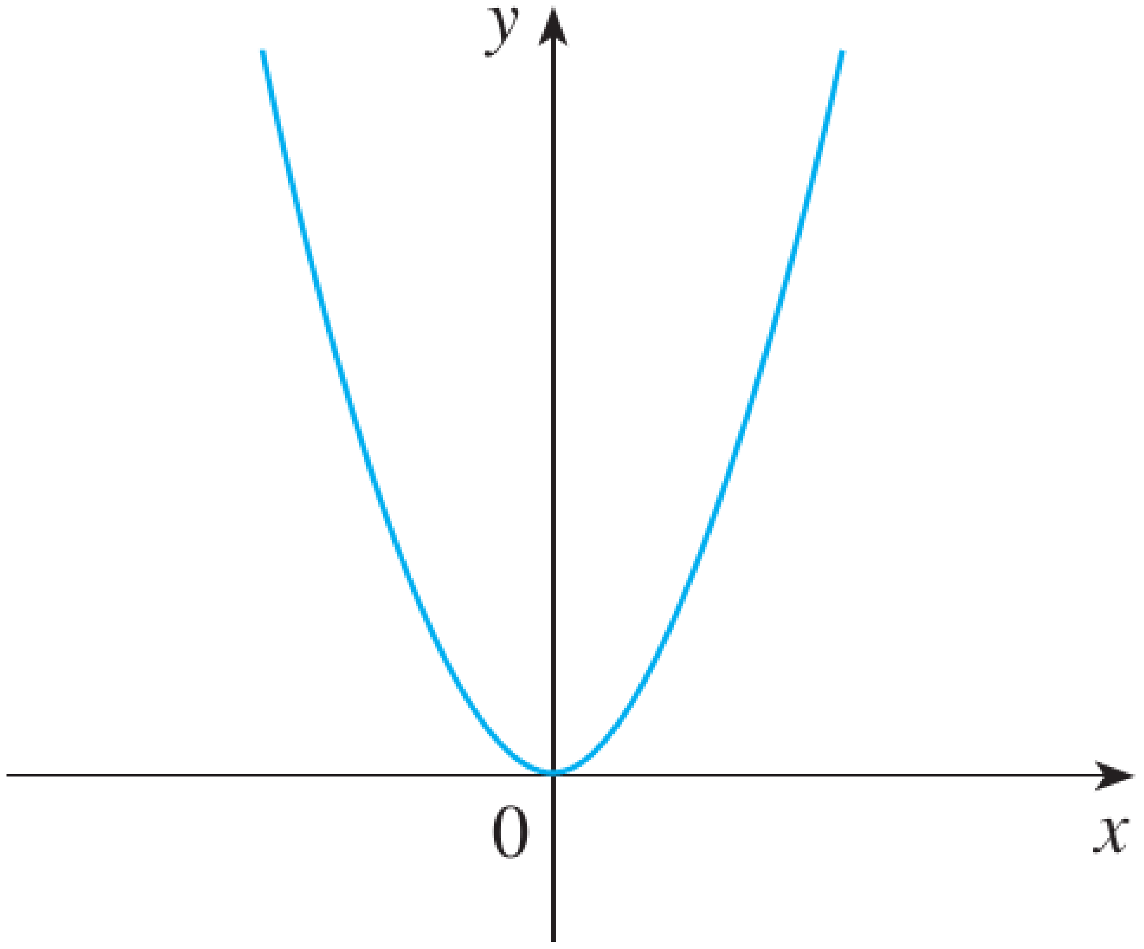
Problem

Sketch the graph of the function $f(x) = x^2 + 6x + 10$.

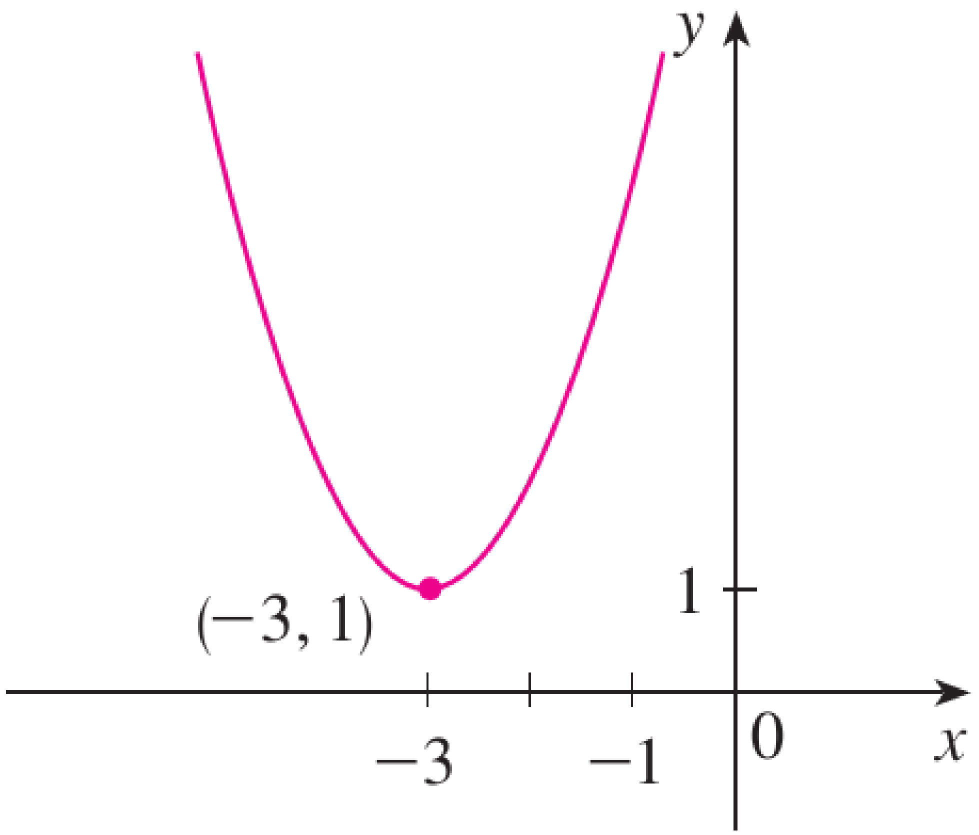
SOLUTION Completing the square, we write the equation of the graph as

$$y = x^2 + 6x + 10 = (x + 3)^2 + 1$$

This means we obtain the desired graph by starting with the parabola $y = x^2$ and shifting 3 units to the left and then 1 unit upward



(a) $y = x^2$



(b) $y = (x + 3)^2 + 1$

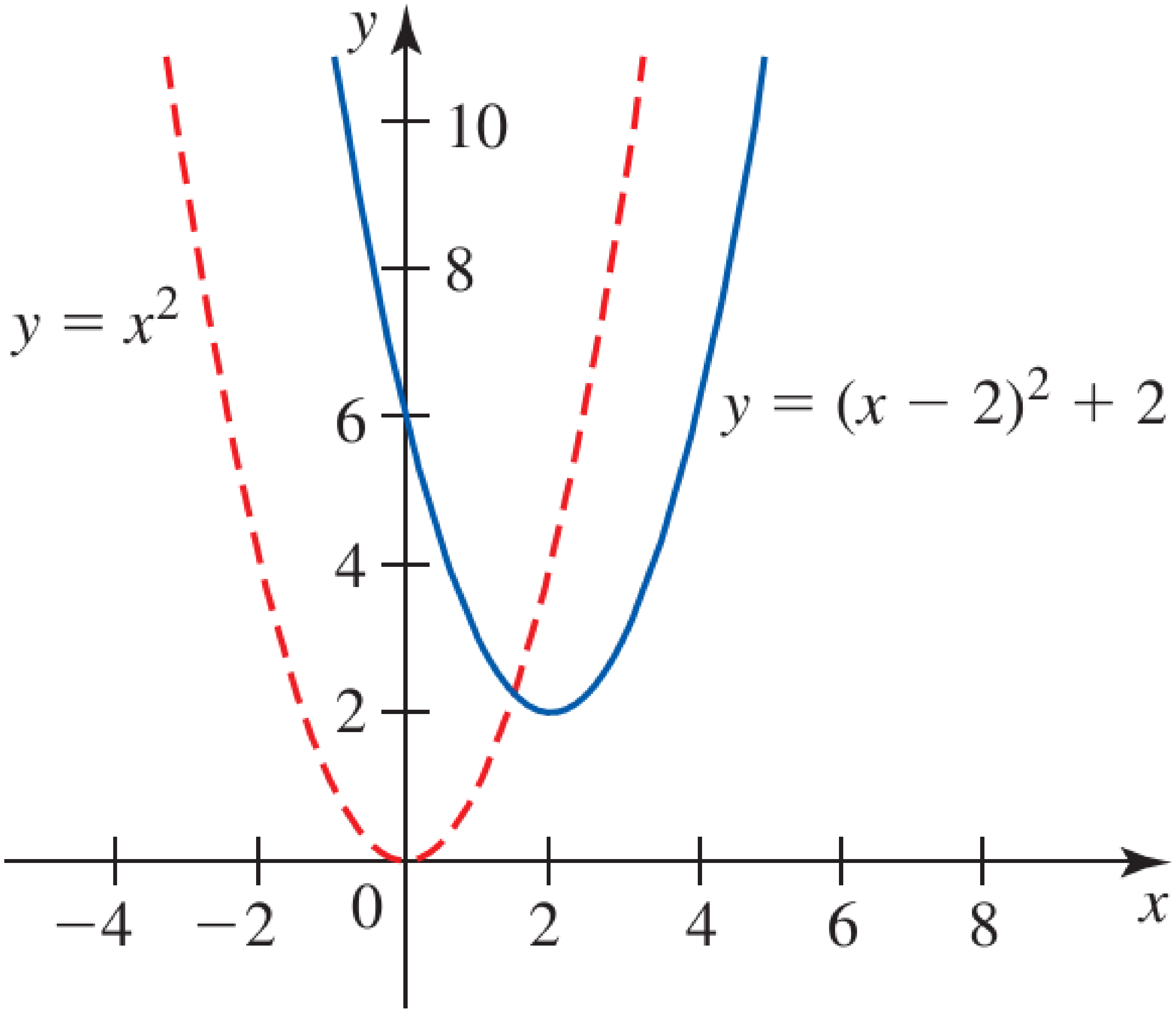
Problem

Sketch the graph of the function f defined by $f(x) = x^2 - 4x + 6$.

Solution By completing the square, we can rewrite the given equation in the form

$$\begin{aligned} y &= [x^2 - 4x + (-2)^2] + 6 - (-2)^2 \\ &= (x - 2)^2 + 2 \end{aligned}$$

We see that the required graph can be obtained from the graph of $y = x^2$ by shifting it 2 units to the right and 2 units upward

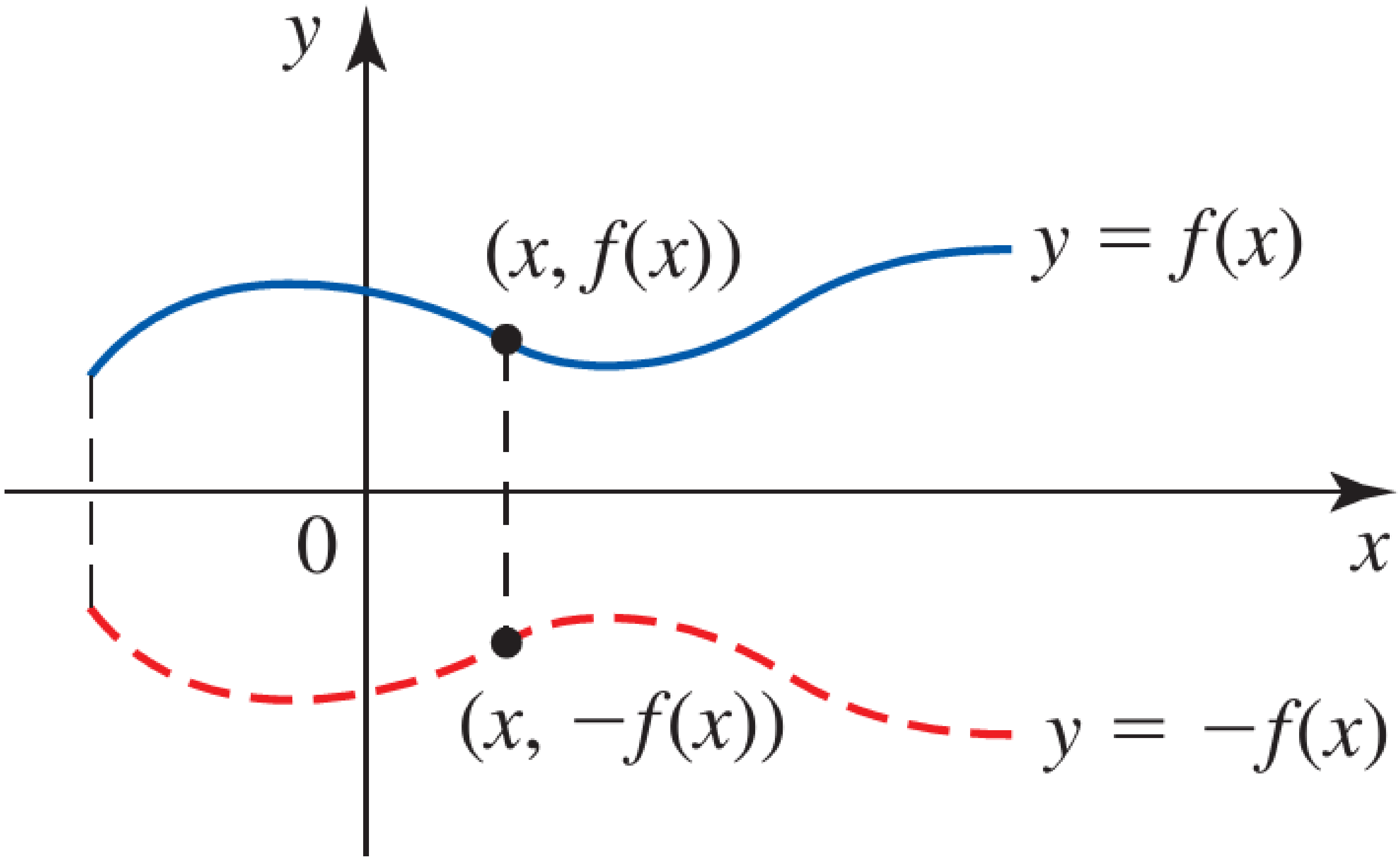

y
10
8
6
4
2
y = x²
y = (x − 2)² + 2
−4
−2
0
2
4
6
8
x

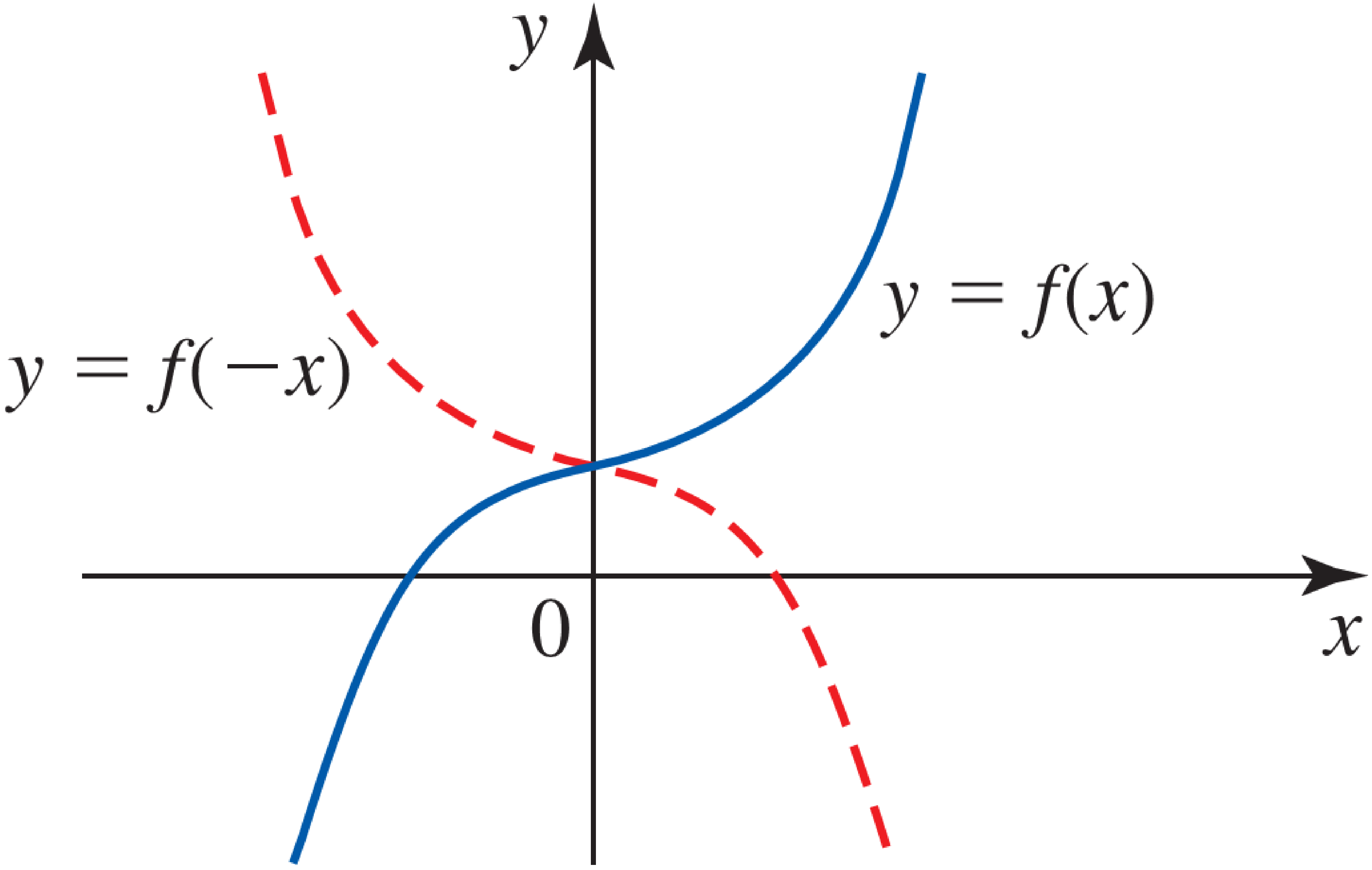
Reflection

To obtain the graph of

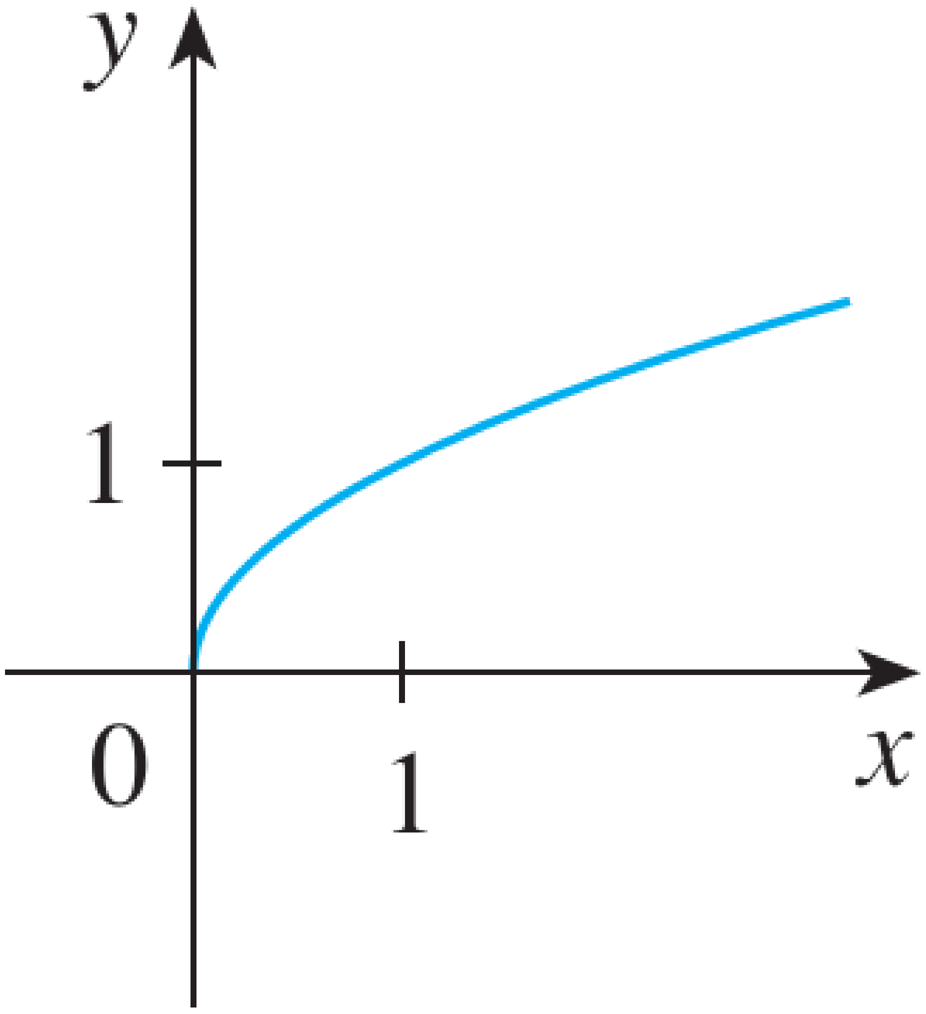
(i) $y = -f(x)$, reflect the graph of $y = f(x)$ about the x-axis.

(ii) $y = f(-x)$, reflect the graph of $y = f(x)$ about the y-axis.

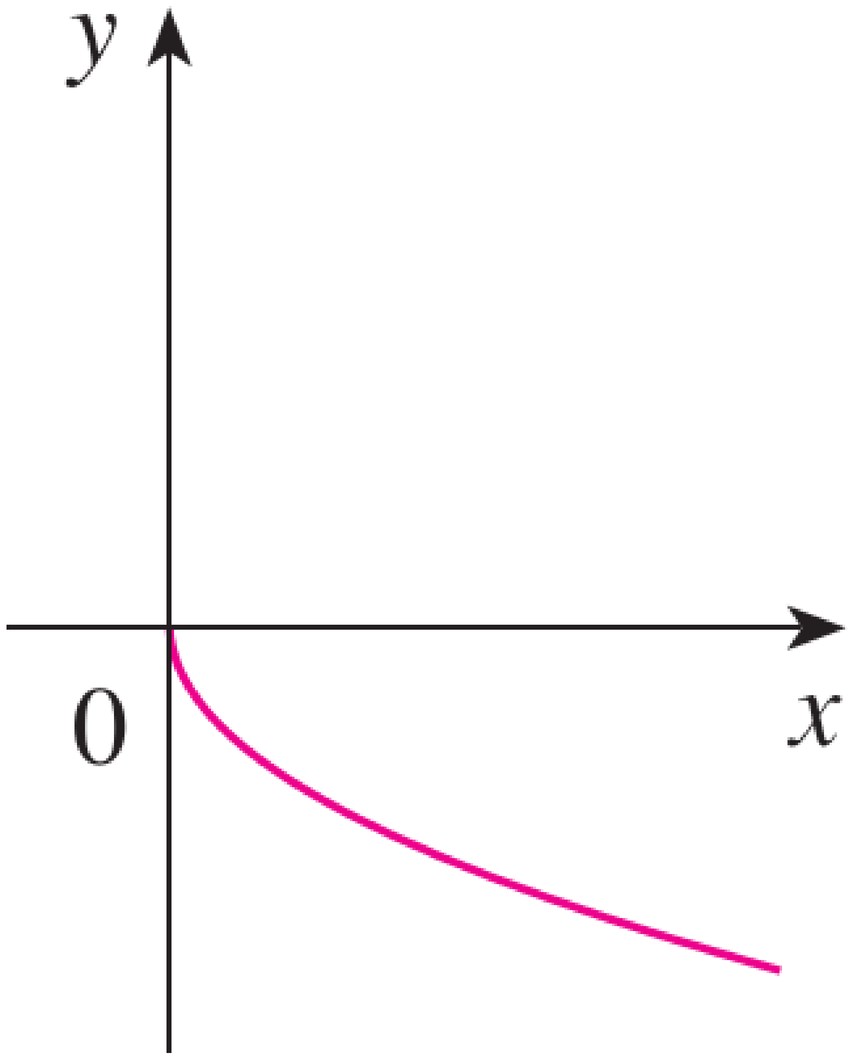

y
(x, f(x))
y = f(x)
0
x
(x, −f(x))
y = −f(x)



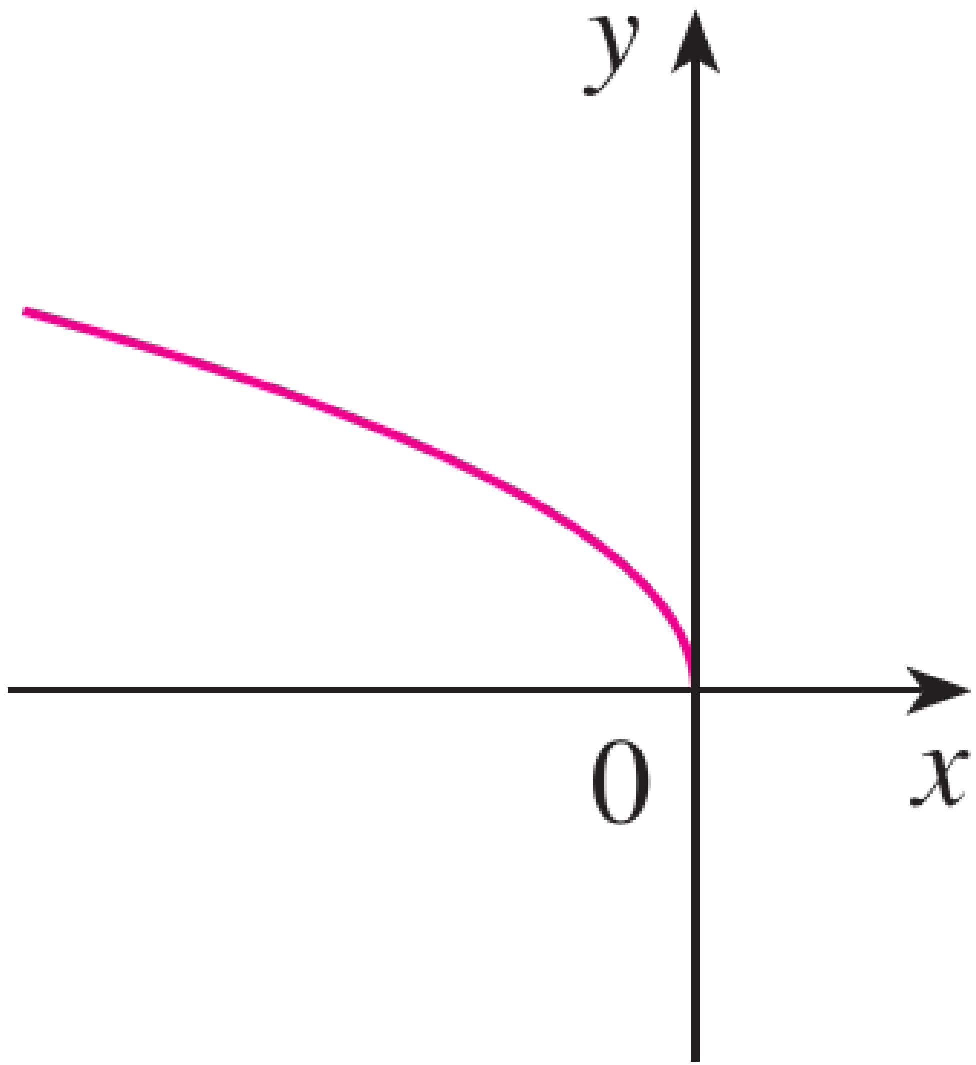
y
y = f(x)
y = f(−x)
0
x



$y = \sqrt{x}$



$y = -\sqrt{x}$



$y = \sqrt{-x}$

Stretching & Shrinking

Suppose $c > 1$. To obtain the graph of

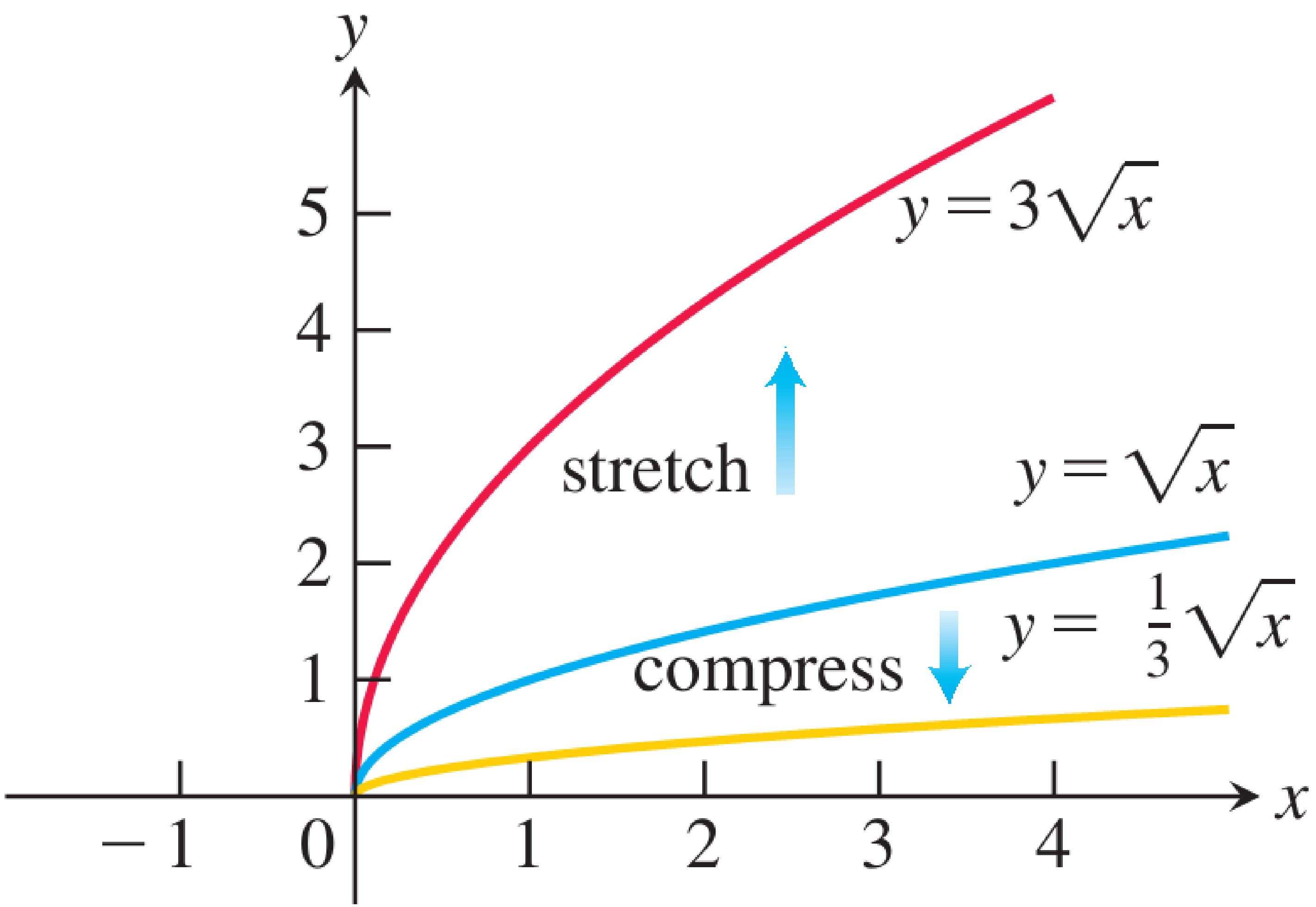
(i) $y = cf(x)$, stretch the graph of $y = f(x)$ vertically by a factor of c.

(ii) $y = \left(\frac{1}{c}\right) f(x)$, shrink the graph of $y = f(x)$ vertically by a factor of c.

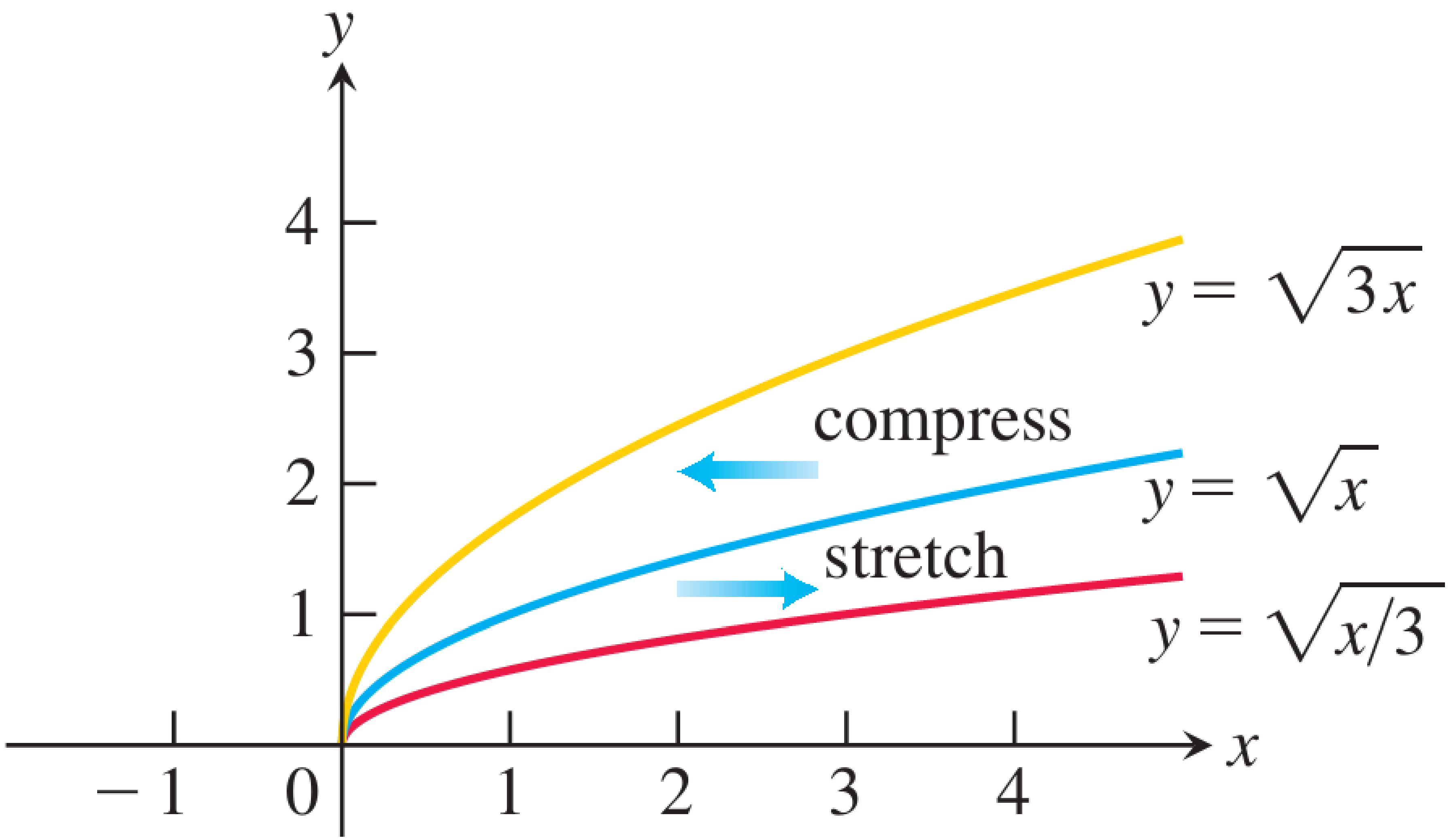
(iii) $y = f(cx)$, shrink the graph of $y = f(x)$ horizontally by a factor of c.

(iv) $y = f\left(\frac{x}{c}\right)$, stretch the graph of $y = f(x)$ horizontally by a factor of c.

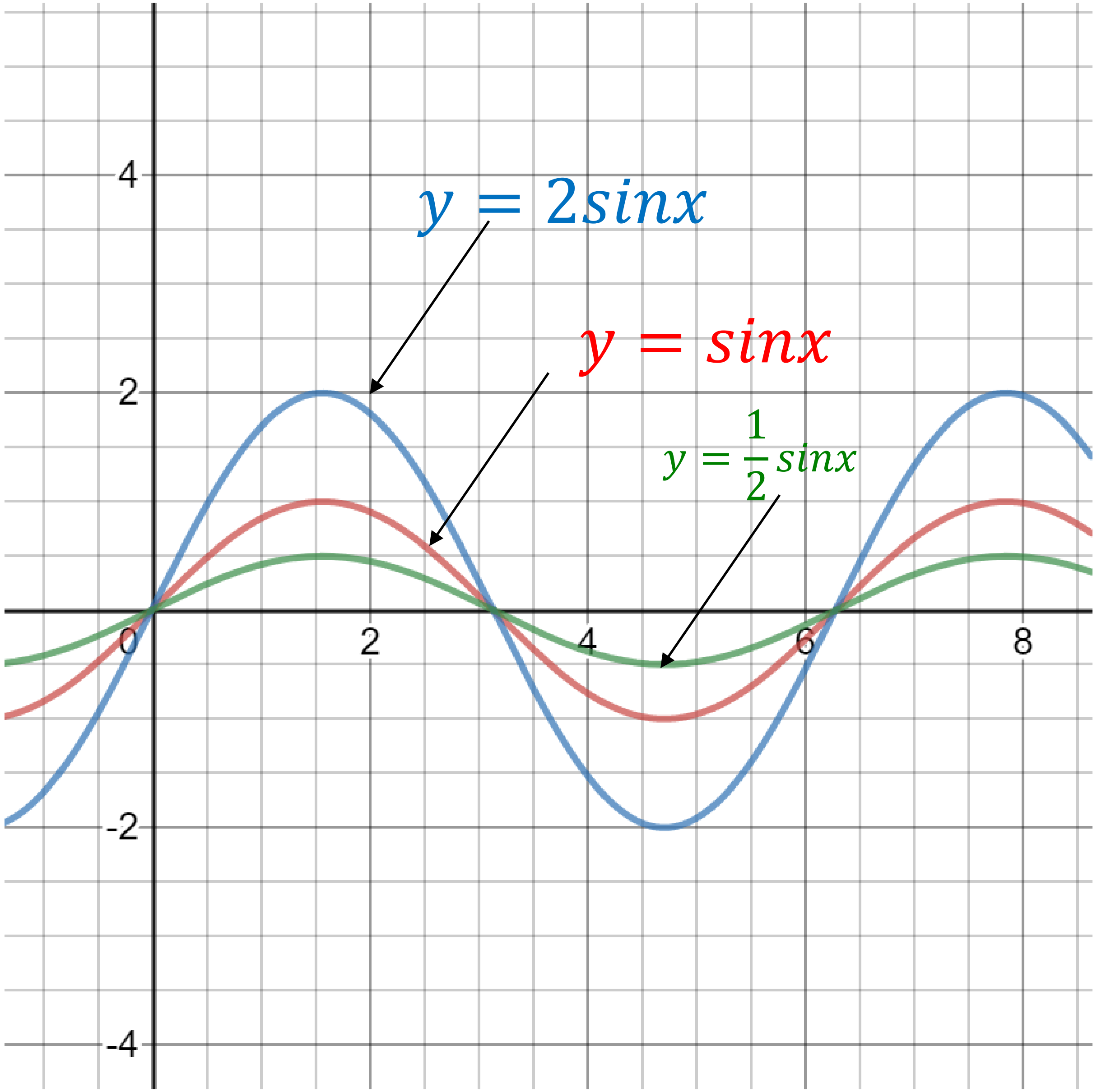
Example: We can get the graph of the function $y = 3\sqrt{x}$ by vertically stretching the graph of $y = \sqrt{x}$ by a factor of 3.



Vertically stretching and shrinking of the graph $y = \sqrt{x}$ by a factor of 3.

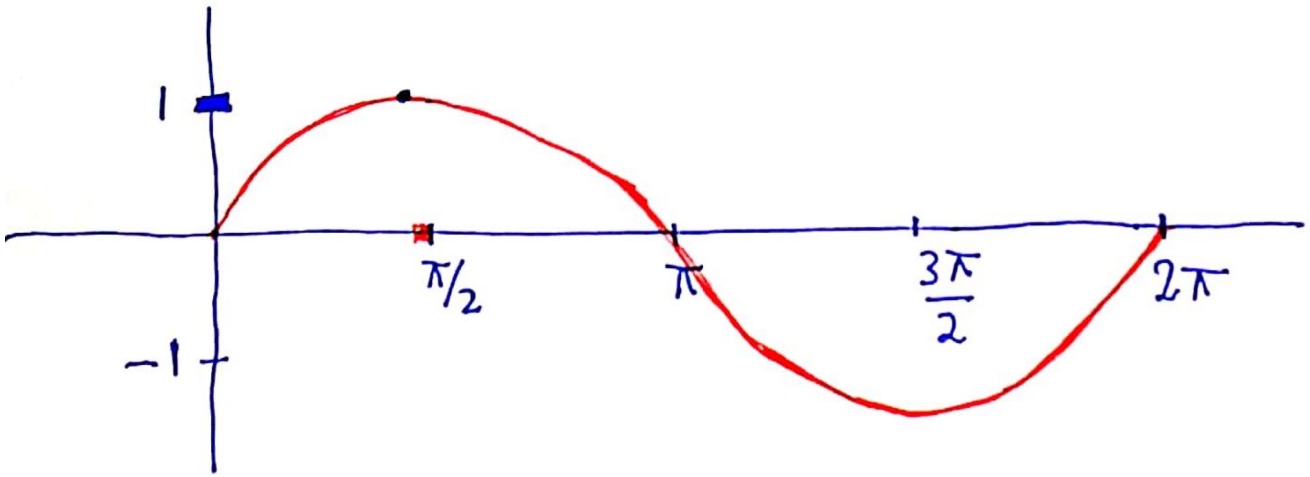


Horizontal stretching and shrinking of the graph $y = \sqrt{x}$ by a factor of 3.

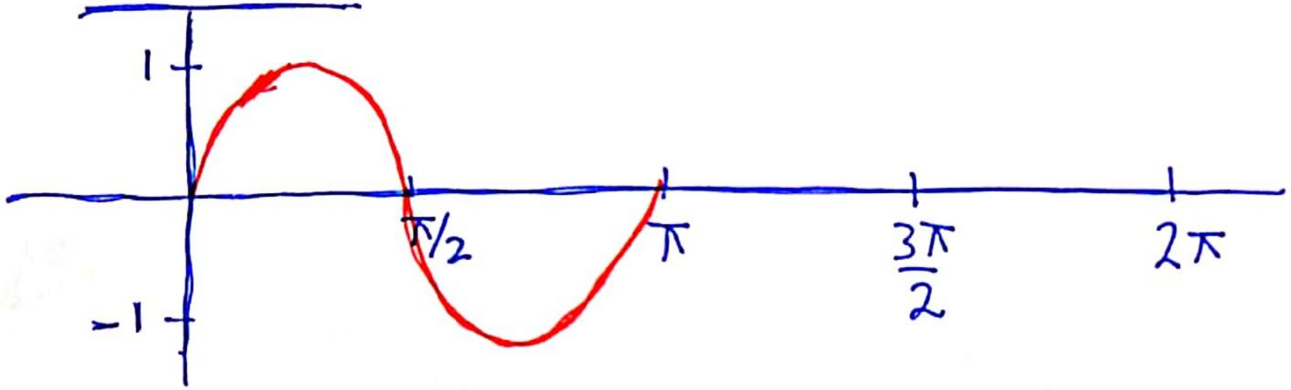


$y = 2sinx$
$y = sinx$
$y = \frac{1}{2}sinx$
4
2
-2
-4
0
2
4
6
8

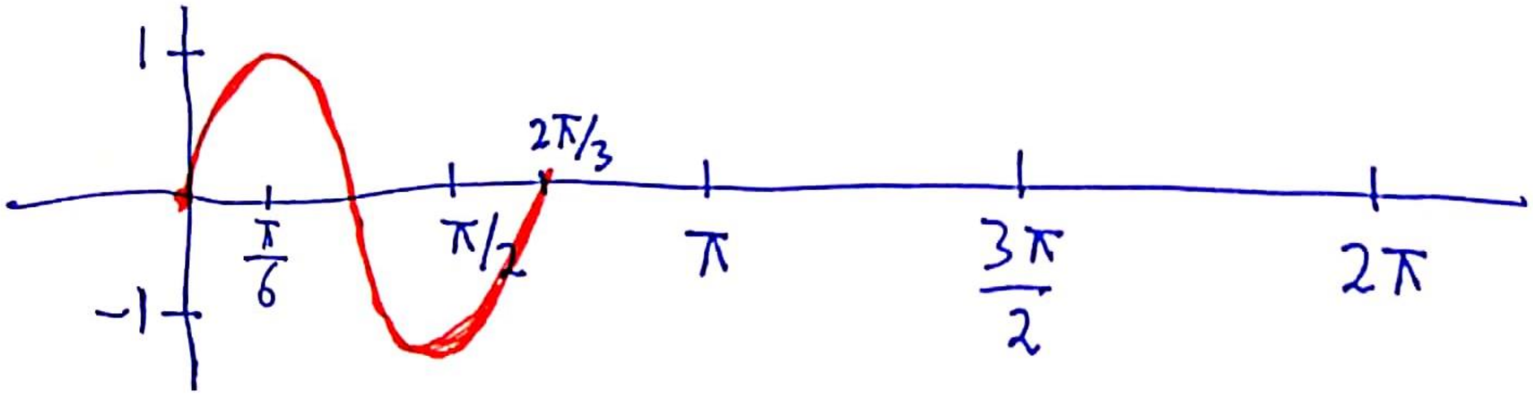
$y = \sin x$



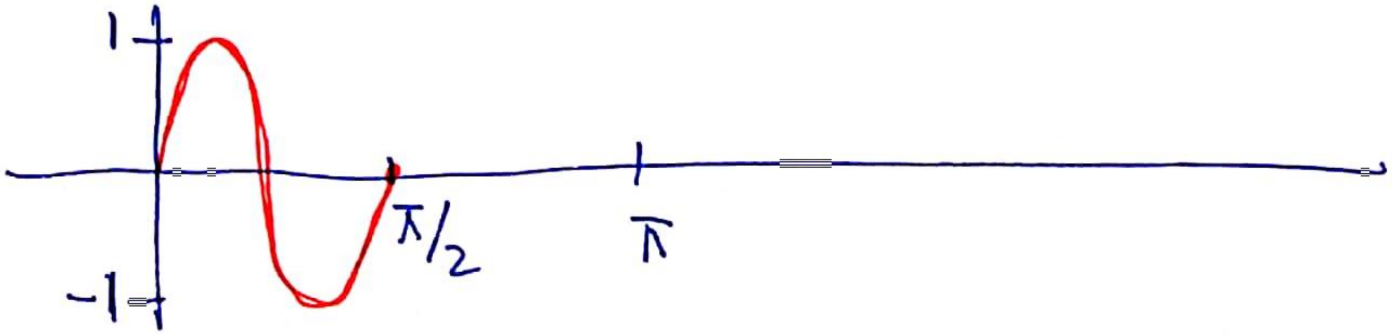
$y = \sin 2x$

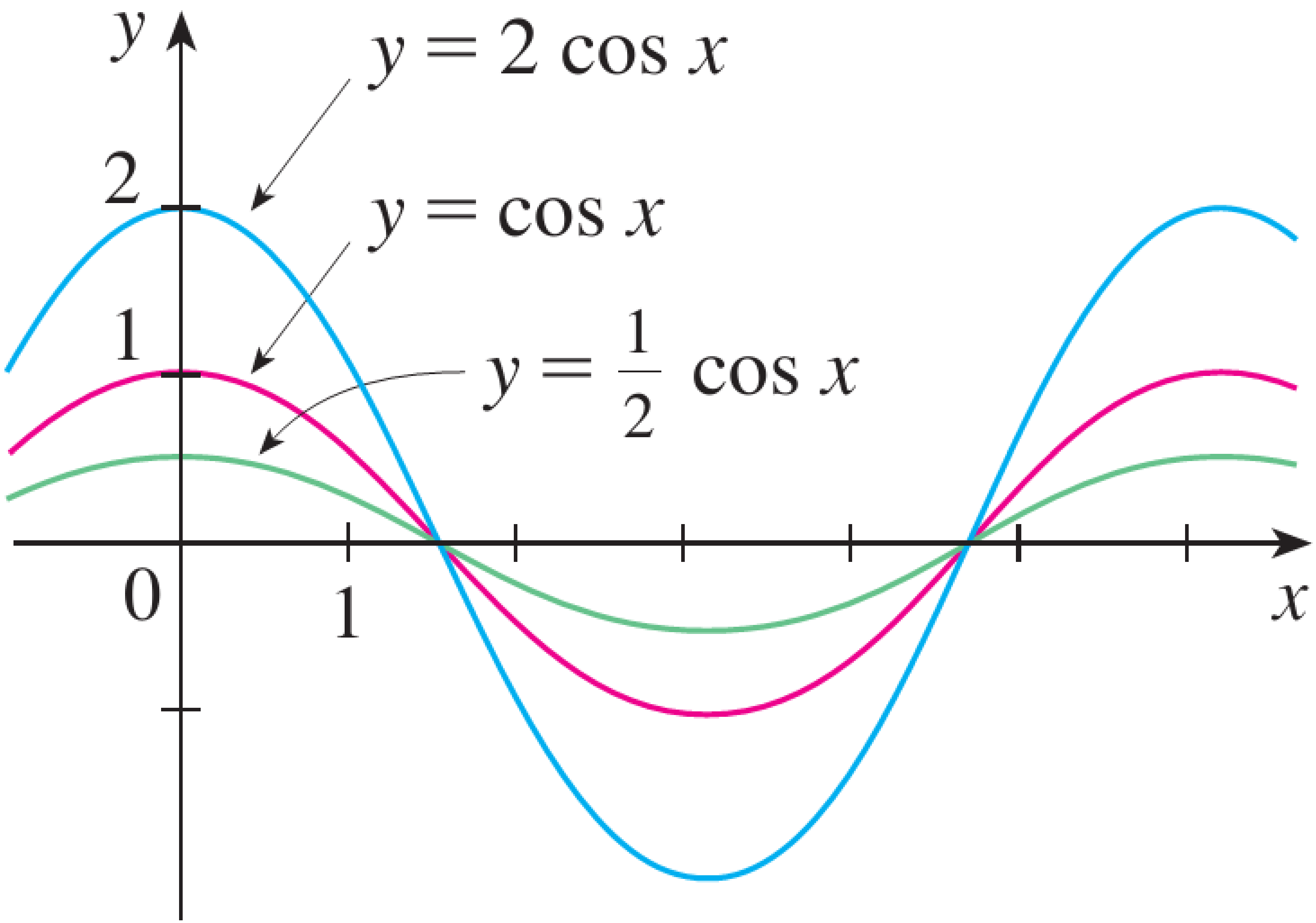


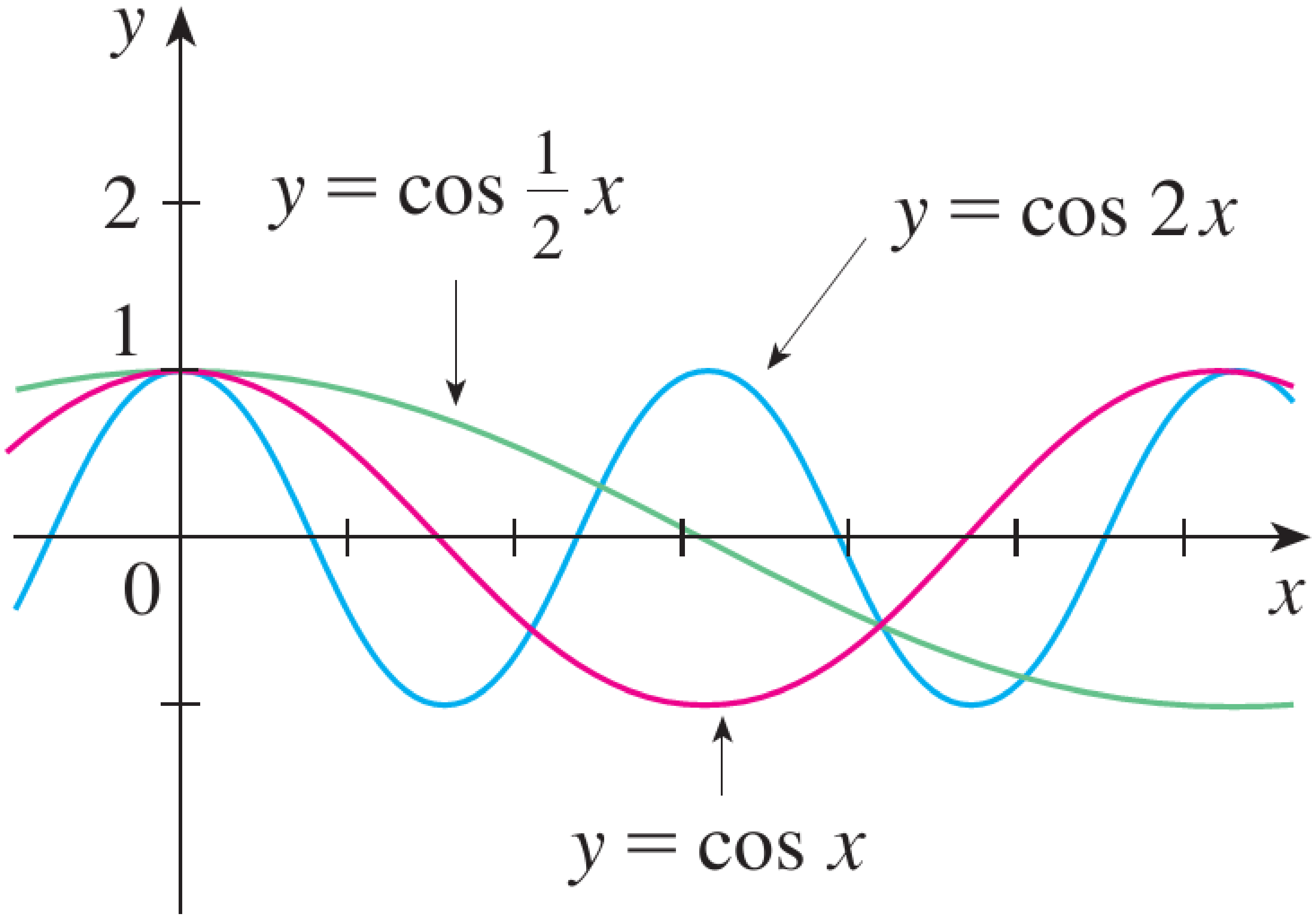
$y = \sin 3x$



$y = \sin 4x$




y
y = 2 cos x
2
y = cos x
1
y = 1/2 cos x
0
1
x



y
2
1
0
x
$y = \cos \frac{1}{2}x$
$y = \cos 2x$
$y = \cos x$

Practice Problems

Graph each function, not by plotting points, but by starting with the graph of one of the standard function and apply an appropriate transformation:

(a) $y = |2x| + 1$

(b) $y = 5\sin x$

(c) $y = 2x^2 + 4x + 2$

(d) $y = \cos(3x)$

Graph of an absolute of a function

To obtain the graph of $y = |f(x)|$ from the graph of $y = f(x)$ follow the following procedure:

(1) The part of the graph that lies above the x-axis will remain the same.

(2) The part that lies below the x-axis is reflected about the x-axis.

Problem

Sketch the graph of the function $y = |x^2 - 1|$.

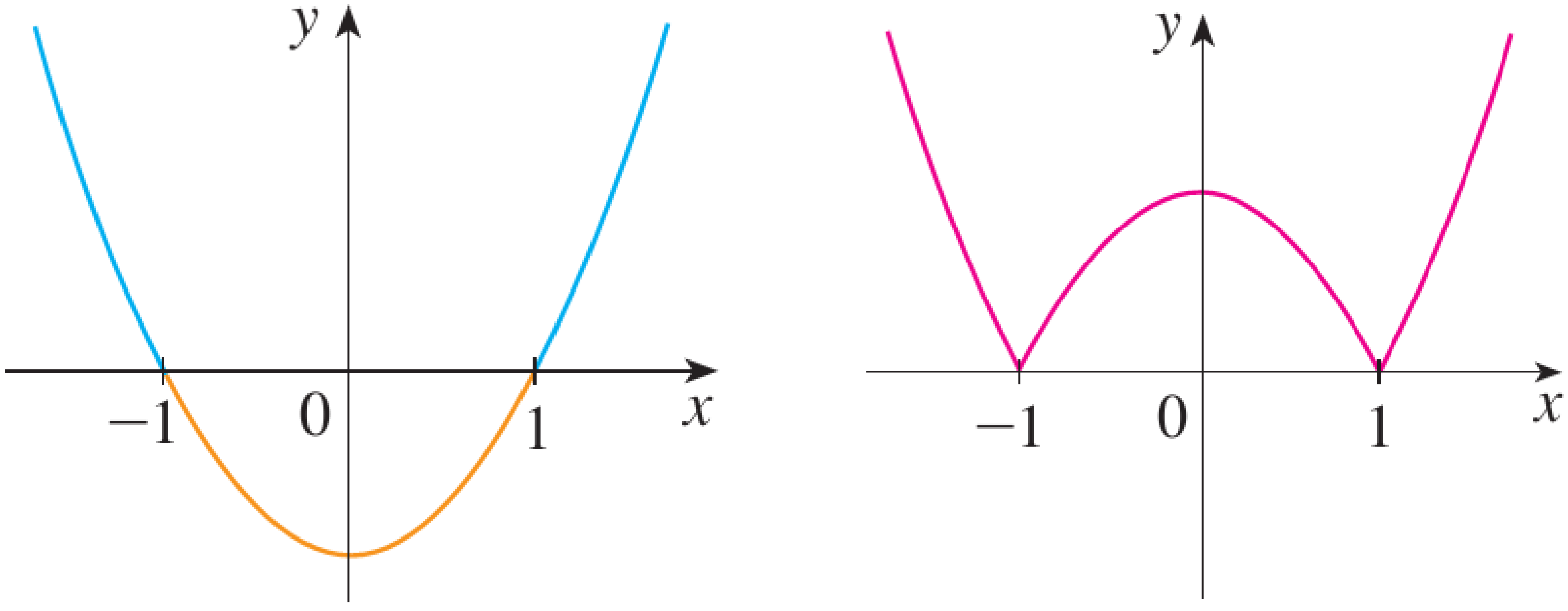
SOLUTION We first graph the parabola $y = x^2 - 1$ by shifting the parabola $y = x^2$ downward 1 unit.

We see that the graph lies below the x-axis when $-1 < x < 1$,

so we reflect that part of the graph about the x-axis to obtain the graph of

$$y = |x^2 - 1|$$

Graph is on next slide…



(a) $y = x^2 - 1$ (b) $y = |x^2 - 1|$

Practice Problems

Graph each function, not by plotting points, but by starting with the graph of one of the standard function and apply an appropriate transformation:

(i) $f(x) = -\sqrt{x-1} + 2$ (ii) $g(x) = x^2 + 2x + 1$ (iii) $h(x) = \left|-x^2 + 5\right|$

(iv) $p(x) = |x+1| - 5$ (v) $q(x) = \lceil x \rceil - 1$ (vi) $r(x) = \dfrac{1}{1-x}$

(vii) $s(x) = \lfloor x \rfloor + 1$

(viii) $t(x) = \dfrac{1}{x^2} + 1$

(ix) $u(x) = |\cos x|$

(x) $v(x) = -x^2 + 1$

(xi) $t(x) = \ln(x+1)$

(xii) $w(x) = e^x - 1$

Combination of Functions

DEFINITION

Let f and g be functions with domains A and B, respectively. Then their sum $f + g$, difference $f - g$, product fg, and quotient f/g are defined as follows:

$$(f + g)(x) = f(x) + g(x) \qquad \text{with domain } A \cap B \tag{1a}$$

$$(f - g)(x) = f(x) - g(x) \qquad \text{with domain } A \cap B \tag{1b}$$

$$(fg)(x) = f(x)g(x) \qquad \text{with domain } A \cap B \tag{1c}$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad \text{with domain } \{x \mid x \in A \cap B \text{ and } g(x) \neq 0\} \tag{1d}$$

Problem

Let f and g be functions defined by $f(x) = \sqrt{x}$ and $g(x) = \sqrt{3 - x}$. Find the domain and the rule for each of the functions $f + g$, $f - g$, fg, and f/g.

Solution

The domain of f is $[0, \infty)$, and the domain of g is $(-\infty, 3]$. Therefore, the domain of $f + g$, $f - g$, and fg is

$$[0, \infty) \cap (-\infty, 3] = [0, 3]$$

The rules for these functions are

$$(f + g)(x) = f(x) + g(x) = \sqrt{x} + \sqrt{3 - x}$$

$$(f - g)(x) = f(x) - g(x) = \sqrt{x} - \sqrt{3 - x}$$

and

$$(fg)(x) = f(x)g(x) = \sqrt{x}\sqrt{3 - x} = \sqrt{3x - x^2}$$

For the domain of f/g we must exclude the value of x for which $g(x) = \sqrt{3 - x} = 0$ or $x = 3$.

Therefore, f/g is defined by

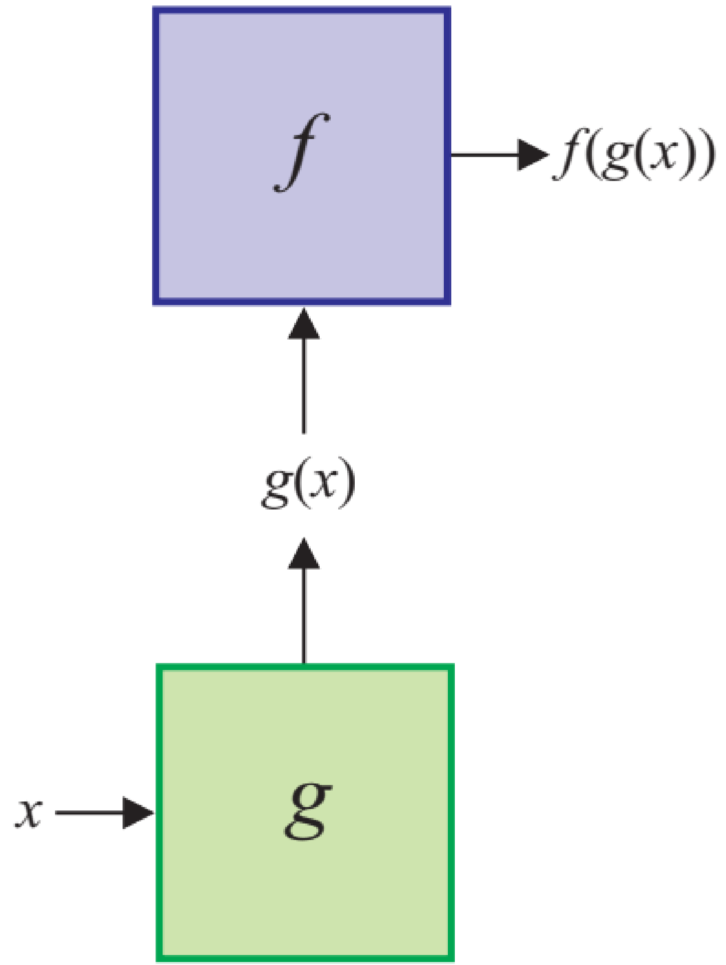
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x}}{\sqrt{3 - x}} = \sqrt{\frac{x}{3 - x}}$$

with domain $[0, 3)$.

Definition Given two functions f and g, the **composite function** $f \circ g$ (also called the **composition** of f and g) is defined by

$$(f \circ g)(x) = f(g(x))$$

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f. In other words, $(f \circ g)(x)$ is defined whenever both $g(x)$ and $f(g(x))$ are defined.



$$(f \circ g)(x) = f(g(x))$$

Problem

For $f(x) = x^2 + 1$ and $g(x) = \sqrt{x-2}$, find the compositions $f \circ g$ and $g \circ f$ and identify the domain of each.

Solution

First, we have

$$(f \circ g)(x) = f(g(x)) = f(\sqrt{x-2})$$
$$= (\sqrt{x-2})^2 + 1 = x - 2 + 1 = x - 1.$$

It's tempting to write that the domain of $f \circ g$ is the entire real line, but look more carefully. Note that for x to be in the domain of g, we must have $x \geq 2$. The domain of f is the whole real line, so this places no further restrictions on the domain of $f \circ g$. Even though the final expression $x - 1$ is defined for all x, the domain of $(f \circ g)$ is $\{x \,|\, x \geq 2\}$.

For the second composition,

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) = g(x^2 + 1) \\ &= \sqrt{(x^2 + 1) - 2} = \sqrt{x^2 - 1}.\end{aligned}$$

The resulting square root requires $x^2 - 1 \geq 0$ or $|x| \geq 1$. Since the "inside" function f is defined for all x, the domain of $g \circ f$ is $\{x \in \mathbb{R} \,\big|\, |x| \geq 1\}$, which we write in interval notation as $(-\infty, -1] \cup [1, \infty)$.

Practice Problems 1

Evaluating Composite Functions Use the graphs of f and g to evaluate each expression. If the result is undefined, explain why.

(a) $(f \circ g)(3)$

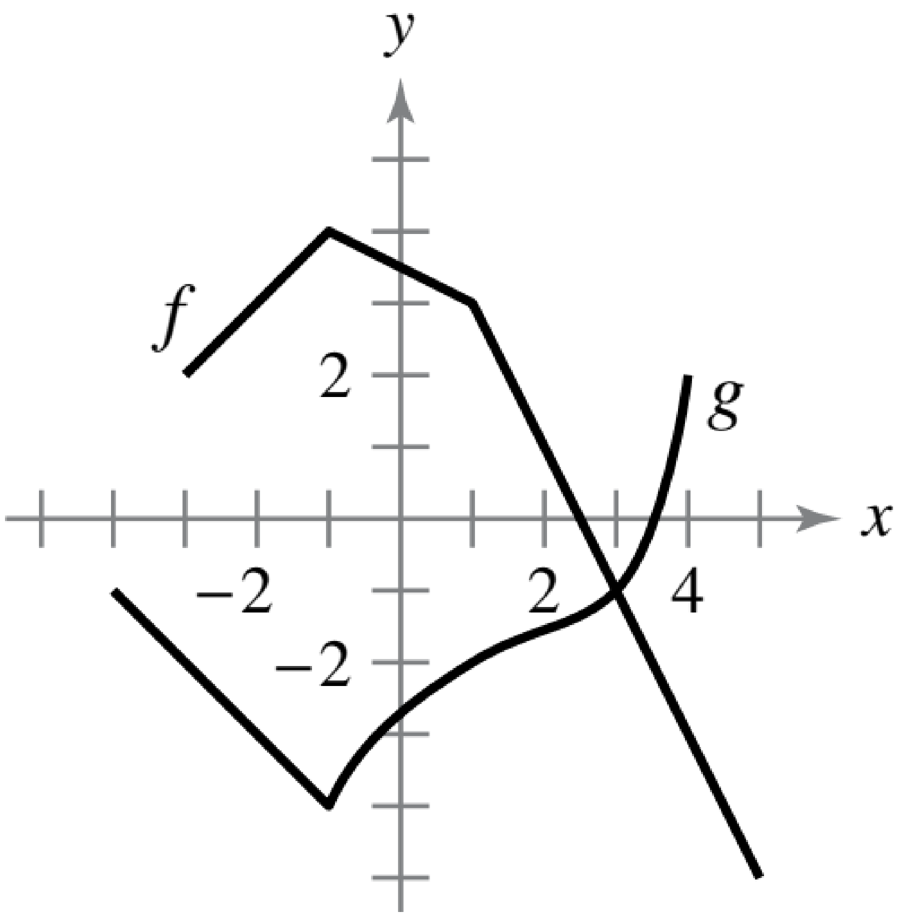
(b) $g(f(2))$

(c) $g(f(5))$

(d) $(f \circ g)(-3)$

(e) $(g \circ f)(-1)$

(f) $f(g(-1))$



Practice Problem 2

State the domain of the following functions.

(i) $f(x) = \frac{e^x + \ln(x)}{\sqrt{x+1}}$

(ii) $g(x) = \sqrt{|x|}$

(iii) $h(x) = \tan^{-1}(x^2 + x + 1)$

(iv) $R(x) = \tan^{-1}(x^2 + x + 1)$

(v) $L(x) = \sin^{-1}(x^2 + 2x)$

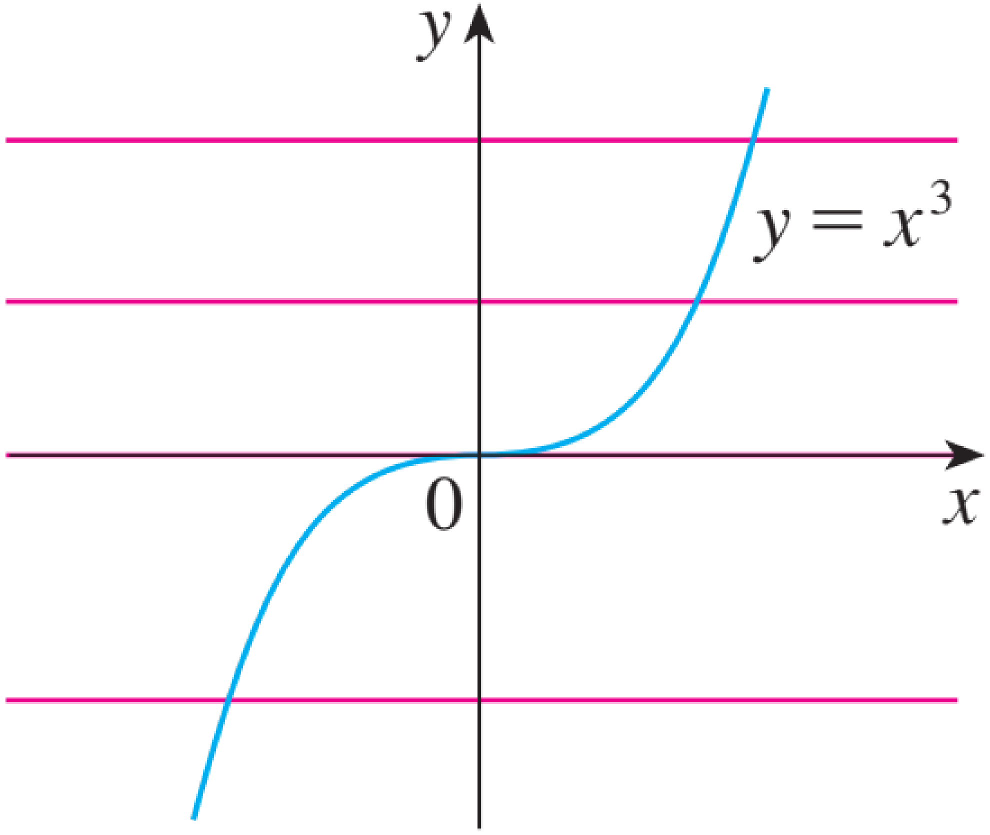
(vi) $s(x) = \ln(-(x + 1))$

(vii) $r(x) = \cos^{-1}\left(\frac{1}{x}\right)$

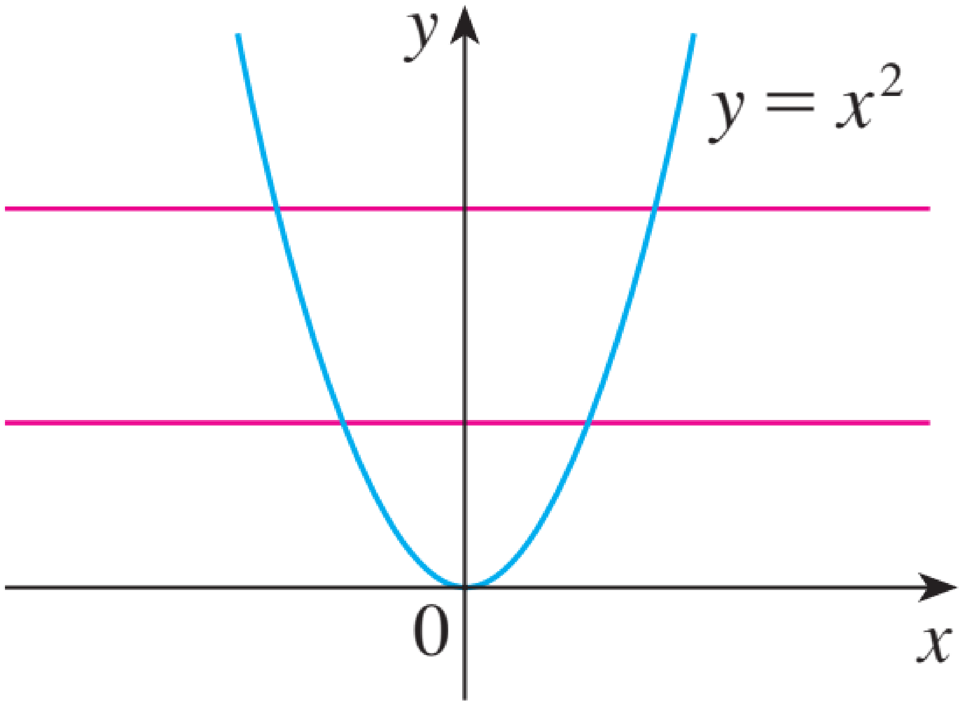
Inverse Functions

DEFINITION A function $f(x)$ is **one-to-one** on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in D.

Horizontal Line Test A function is one-to-one if and only if no horizontal line intersects its graph more than once.



$f(x) = x^3$ is one-to-one.



$g(x) = x^2$ is not one-to-one.

One-to-one functions are important because they are precisely the functions that possess inverse functions according to the following definition.

Definition Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \quad \Longleftrightarrow \quad f(x) = y$$

for any y in B.

This definition says that if f maps x into y, then f^{-1} maps y back into x.

CAUTION Do not mistake the -1 in f^{-1} for an exponent. Thus

$$f^{-1}(x) \quad \text{does } \textit{not} \text{ mean} \quad \frac{1}{f(x)}$$

$$f^{-1}(f(x)) = x \qquad \text{for every } x \text{ in } A$$

$$f(f^{-1}(x)) = x \qquad \text{for every } x \text{ in } B$$

How to Find the Inverse Function of a One-to-One Function f

STEP 1 Write $y = f(x)$.

STEP 2 Solve this equation for x in terms of y (if possible).

STEP 3 To express f^{-1} as a function of x, interchange x and y. The resulting equation is $y = f^{-1}(x)$.

Problem Find the inverse function of $f(x) = x^3 + 2$.

Solution

$$y = x^3 + 2$$

we solve this equation for x:

$$x^3 = y - 2 \quad \longrightarrow \quad x = \sqrt[3]{y - 2}$$

Finally, we interchange x and y:

$$y = \sqrt[3]{x - 2}$$

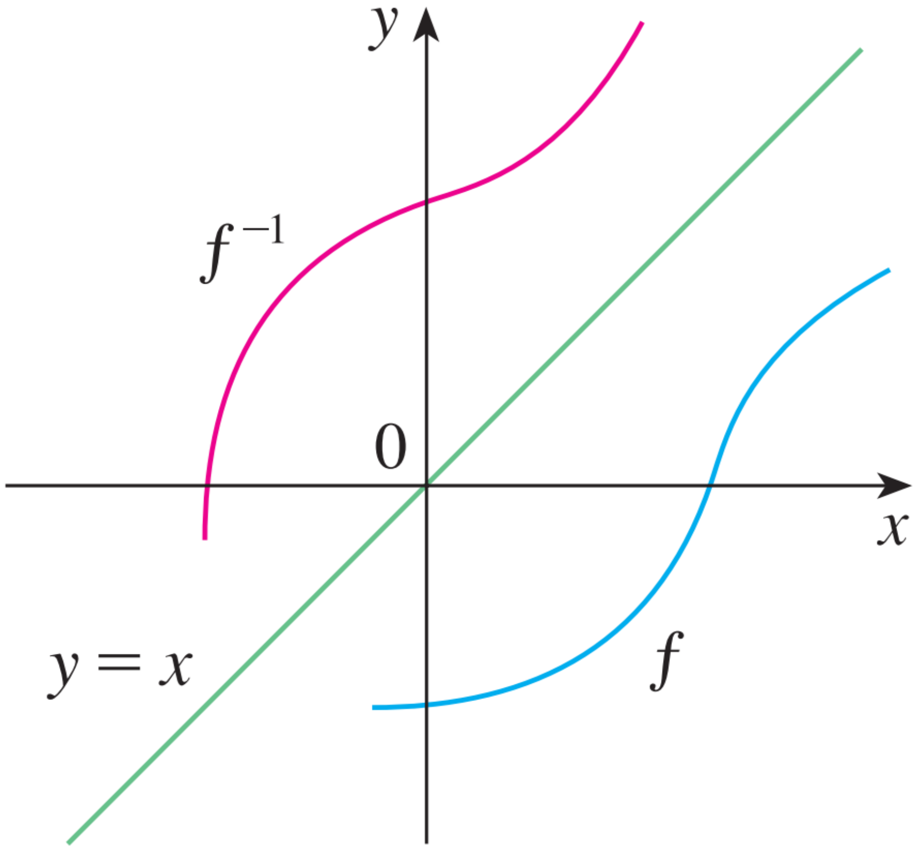
Therefore the inverse function is $f^{-1}(x) = \sqrt[3]{x - 2}$.

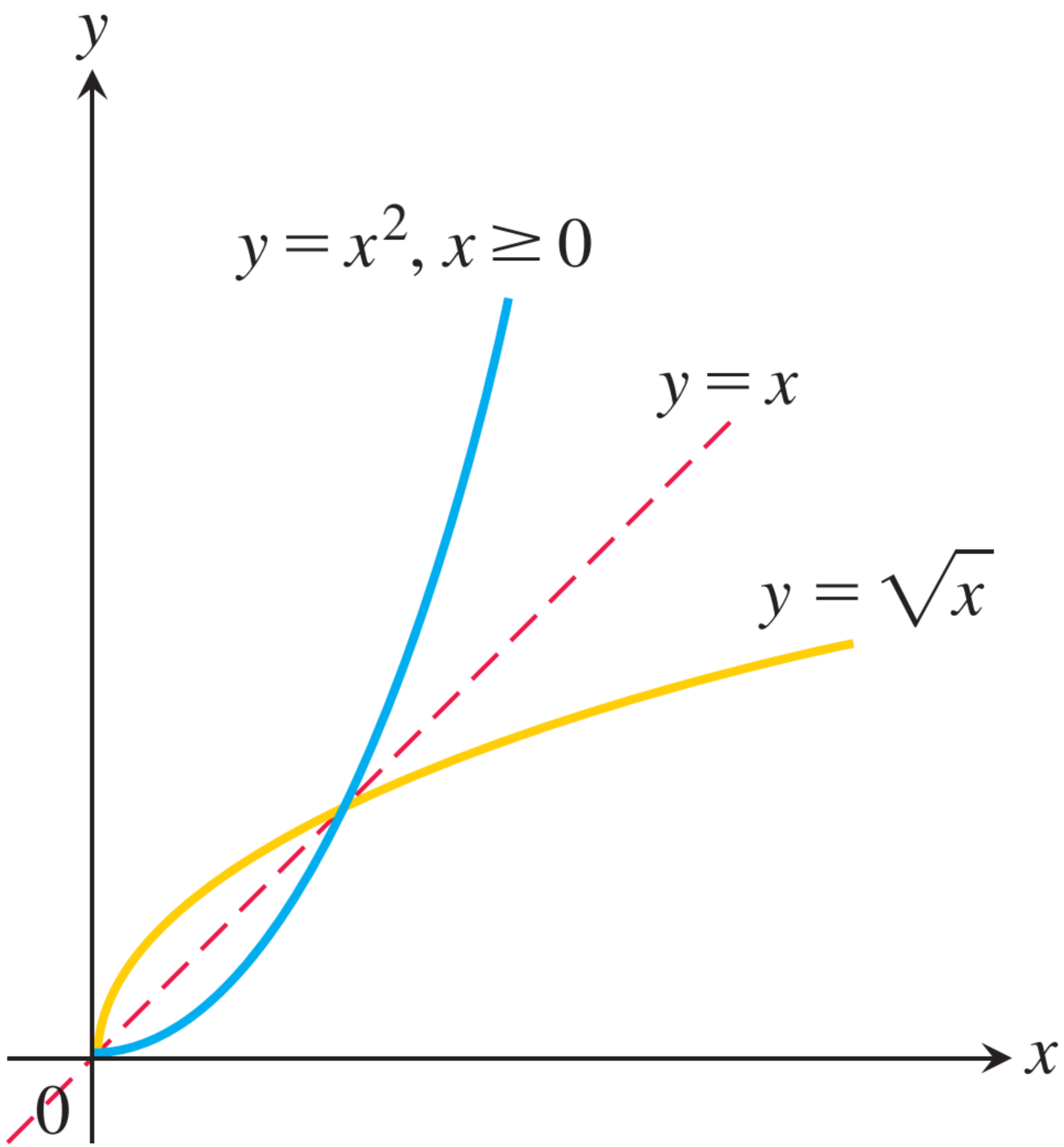
Practice Problem

Find the inverse of $y = \frac{1}{2}x + 1$.

Graph of f^{-1} from the graph of f

The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.





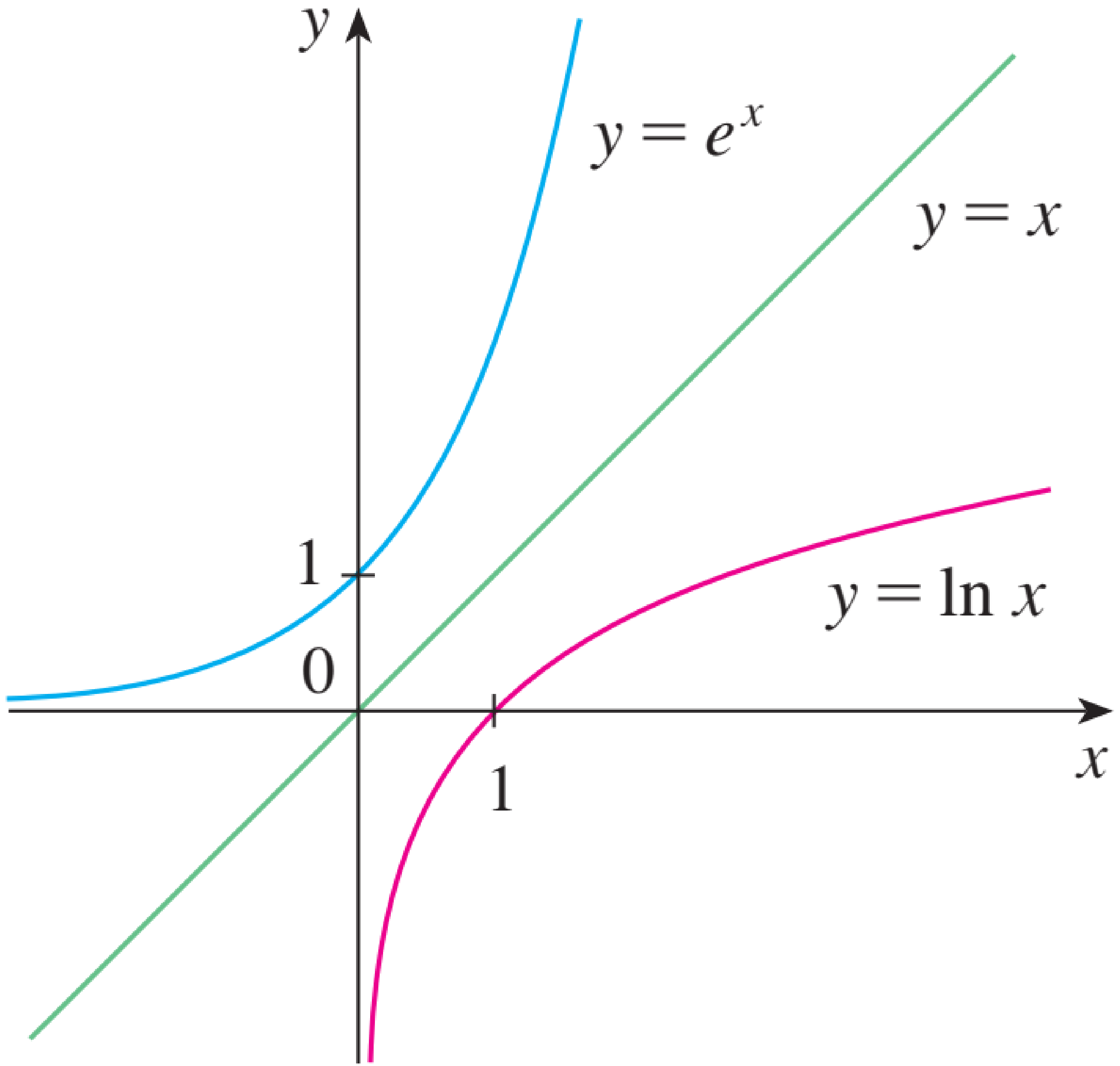
y
$y = x^2, x \geq 0$
$y = x$
$y = \sqrt{x}$
x
0

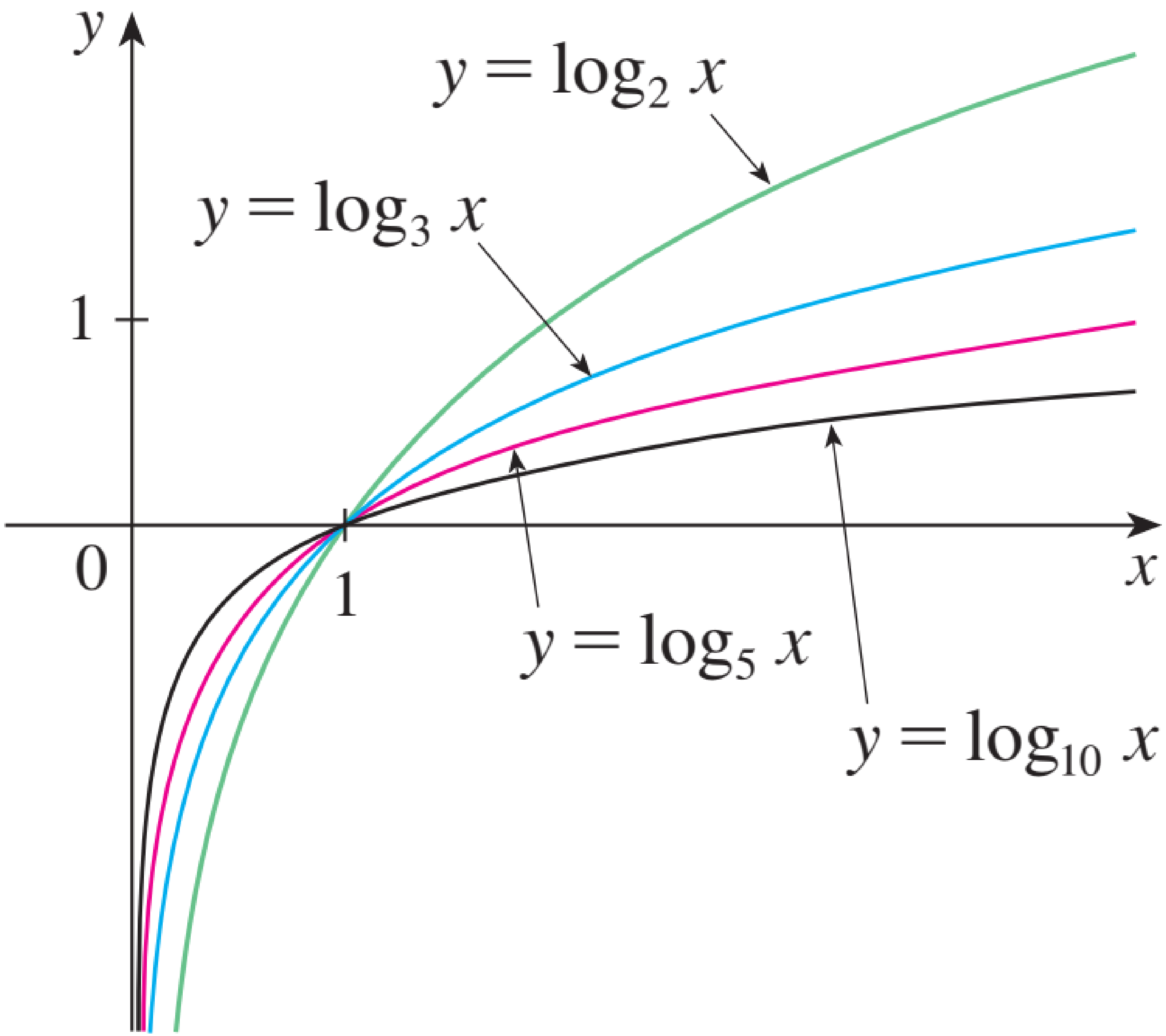
Logarithmic Functions

DEFINITION The **logarithm function with base** a, written $y = \log_a x$, is the inverse of the base a exponential function $y = a^x$ $(a > 0, a \neq 1)$.

$\log_e x$	is written as	$\ln x$.
$\log_{10} x$	is written as	$\log x$.

The function $y = \ln x$ is called the **natural logarithm function**, and $y = \log x$ is often called the **common logarithm function**.


y
y = e^x
y = x
1
y = ln x
0
1
x


y
y = log₂ x
y = log₃ x
1
0
1
x
y = log₅ x
y = log₁₀ x

The limit of a function at a number

Suppose f is a function defined on some open interval that contains a, if the value of $f(x)$ tends to get closer and closer to the number L as x gets closer and closer to the number a (from either side of a) but $x \neq a$ then we say that "the limit of $f(x)$, as x approaches a, equals L" and we write

$$\lim_{x \to a} f(x) = L$$

Let $f(x) = \frac{x^2-1}{x-1}$

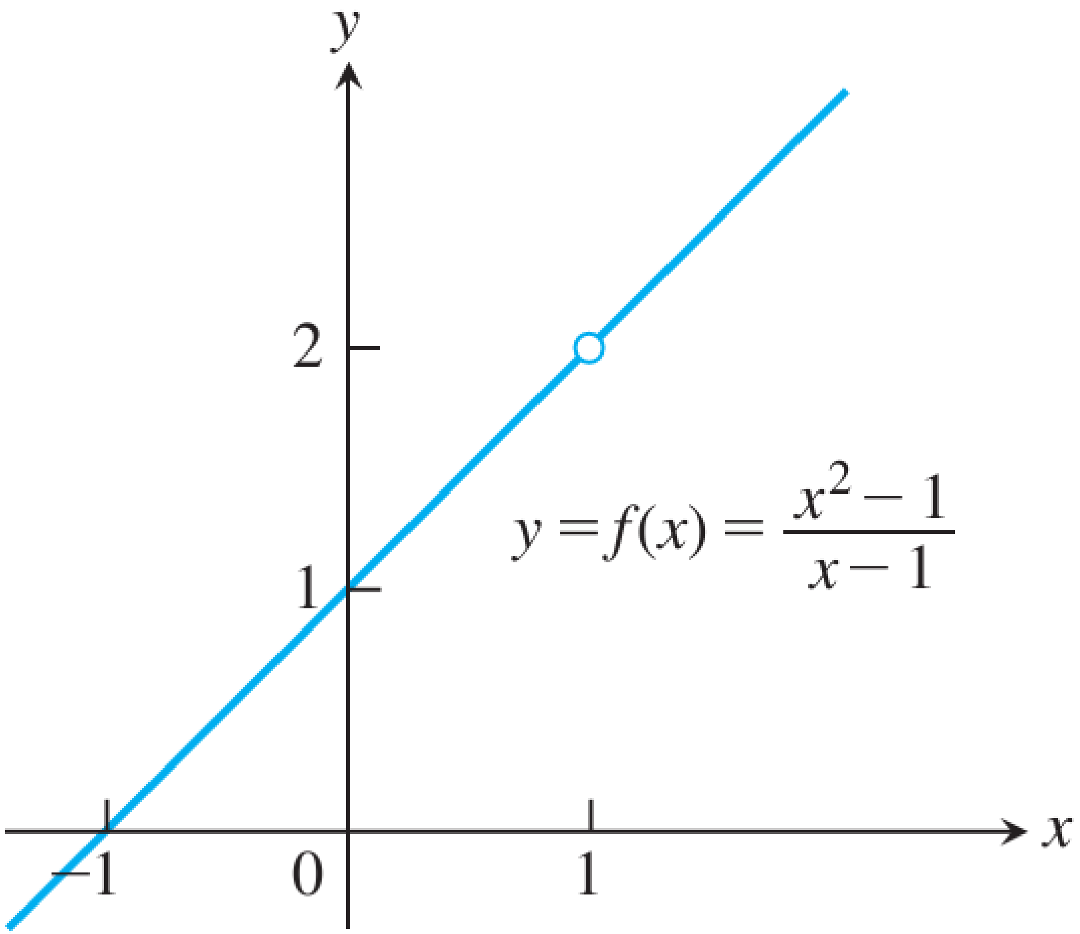
then $\lim_{x\to 1} f(x) = ?$

Solution The given formula defines f for all real numbers x except $x = 1$ (since we cannot divide by zero). For any $x \neq 1$, we can simplify the formula by factoring the numerator and canceling common factors:

$$f(x) = \frac{(x - 1)(x + 1)}{x - 1} = x + 1 \qquad \text{for} \qquad x \neq 1.$$

The graph of f is the line $y = x + 1$ with the point $(1, 2)$ *removed*.

Even though $f(1)$ is not defined, it is clear that we can make the value of $f(x)$ as close as we want to 2 by choosing x close enough to 1.



Thus $\lim_{x \to 1} f(x) = 2,$ or $\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2.$

Problem

Use the given graph of f to state the value of each quantity, if it exists. If it does not exist, explain why.

(a) $\lim_{x \to 2^-} f(x)$

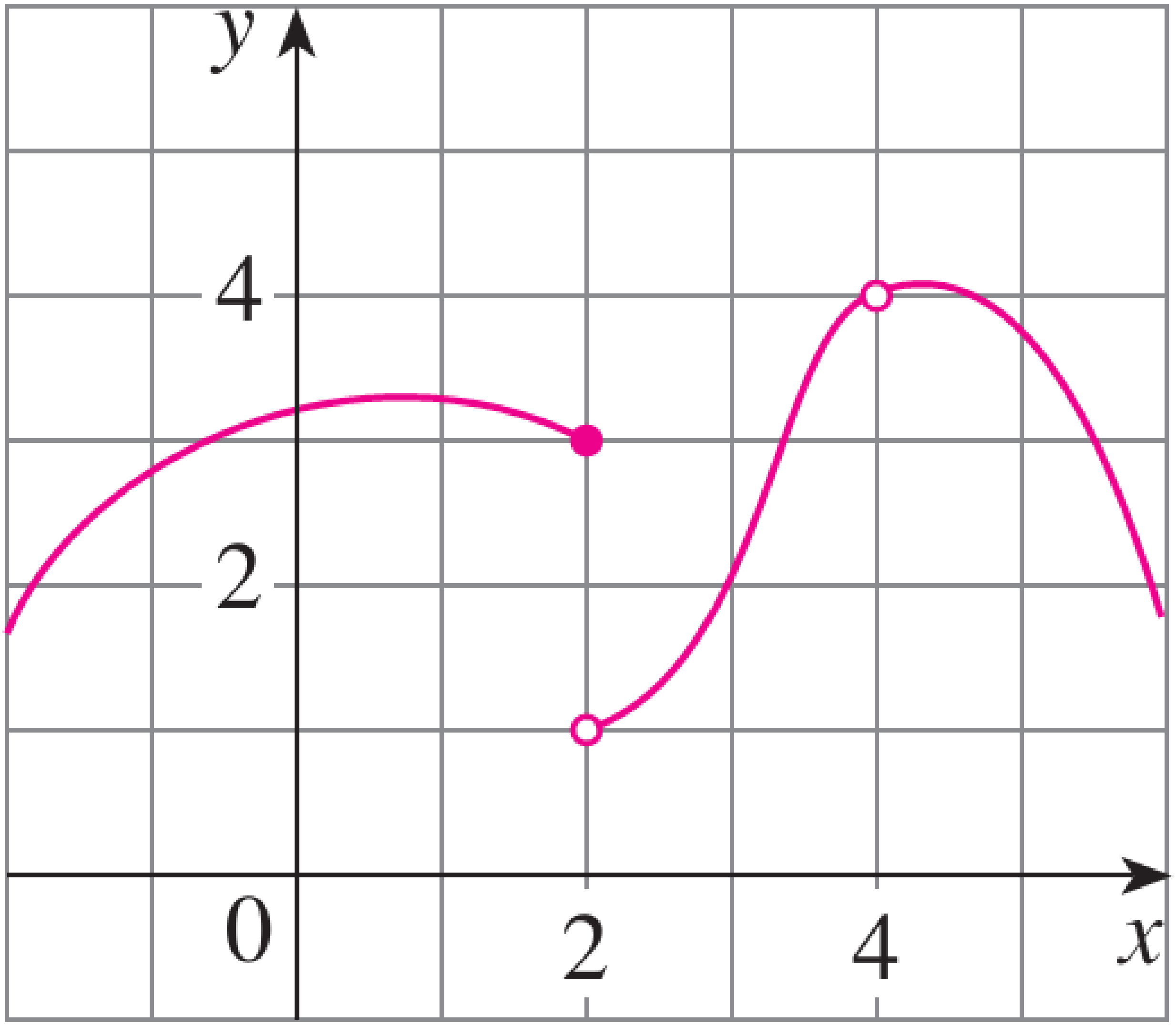
(b) $\lim_{x \to 2^+} f(x)$

(c) $\lim_{x \to 2} f(x)$

(d) $f(2)$

(e) $\lim_{x \to 4} f(x)$

(f) $f(4)$


y
4
2
0
2
4
x

Solution

ⓐ $\lim_{x \to 2^-} f(x) = 3$

ⓑ $\lim_{x \to 2^+} f(x) = 1$

ⓒ $\lim_{x \to 2} f(x)$ does not exist.

$\left[\because \lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x) \right]$

ⓓ $f(2) = 3$

ⓔ $\lim_{x \to 4} f(x) = 4$

ⓕ $f(4)$ is not defined.

Problem

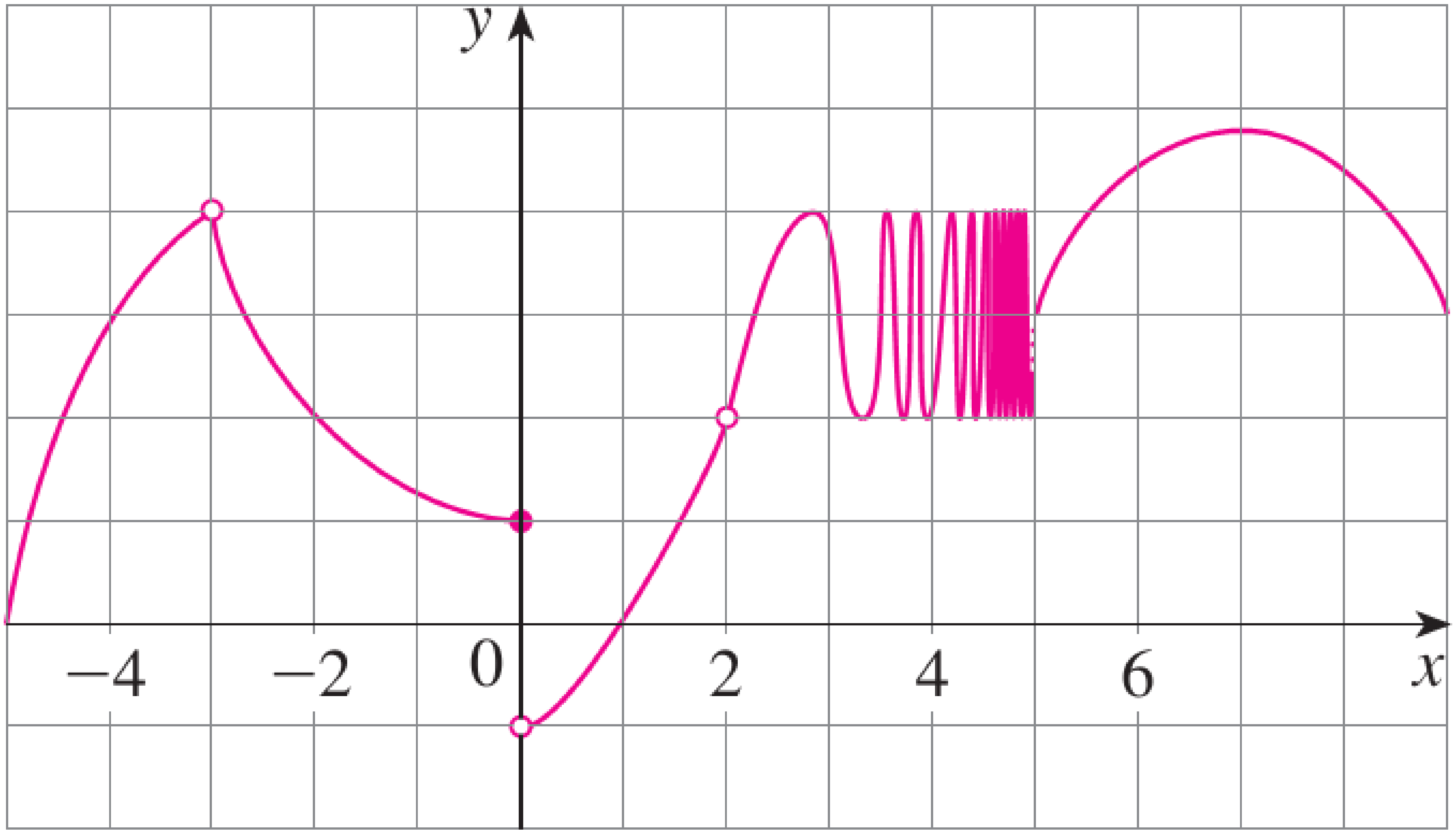
For the function h whose graph is given, state the value of each quantity, if it exists. If it does not exist, explain why.

(a) $\lim_{x \to -3^-} h(x)$ (b) $\lim_{x \to -3^+} h(x)$ (c) $\lim_{x \to -3} h(x)$

(d) $h(-3)$ (e) $\lim_{x \to 0^-} h(x)$ (f) $\lim_{x \to 0^+} h(x)$

(g) $\lim_{x \to 0} h(x)$ (h) $h(0)$ (i) $\lim_{x \to 2} h(x)$

(j) $h(2)$ (k) $\lim_{x \to 5^+} h(x)$ (l) $\lim_{x \to 5^-} h(x)$


y
x
0
−4
−2
2
4
6

Solution

(a) $\lim_{x \to -3^-} h(x) = 4$

(b) $\lim_{x \to -3^+} h(x) = 4$

(c) $\lim_{x \to -3} h(x) = 4$

(d) $h(-3)$ is not defined

(e) $\lim_{x \to 0^-} h(x) = 1$

(f) $\lim_{x \to 0^+} h(x) = -1$

(g) $\lim_{x \to 0} h(x)$ does not exist. $\left[\because \lim_{x \to 0^-} h(x) \neq \lim_{x \to 0^+} h(x) \right]$

(h) $h(0) = 1$

(i) $\lim_{x \to 2} h(x) = 2$

(j) $h(2)$ is not defined

(k) $\lim_{x \to 5^+} h(x) = 3$

(l) $\lim_{x \to 5^-} h(x)$ does not exist because the values of the function h oscillate b/w 2 and 3 as x approaches 5 from the left.

Limit Laws Suppose that c is a constant and the limits

$$\lim_{x \to a} f(x) \qquad \text{and} \qquad \lim_{x \to a} g(x)$$

exist. Then

1. $\displaystyle \lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
2. $\displaystyle \lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$
3. $\displaystyle \lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$
4. $\displaystyle \lim_{x \to a} [f(x)\, g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$
5. $\displaystyle \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \quad \text{if } \lim_{x \to a} g(x) \neq 0$

Power Law

6. $\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$ where n is a positive integer

Root Law

7. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ where n is a positive integer

[If n is even, we assume that $\lim_{x \to a} f(x) > 0$.]

8. $\lim_{x \to a} c = c$

9. $\lim_{x \to a} x = a$

10. $\lim_{x \to a} x^n = a^n$ where n is a positive integer

11. $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer

(If n is even, we assume that $a > 0$.)

Problem Investigate $\lim_{x \to 0} \sin \dfrac{\pi}{x}$.

Solution $f(x) = \sin(\pi/x)$ is undefined at 0.

Evaluating the function for some small values of x, we get

$$f(1) = \sin \pi = 0 \qquad f\left(\tfrac{1}{2}\right) = \sin 2\pi = 0$$

$$f\left(\tfrac{1}{3}\right) = \sin 3\pi = 0 \qquad f\left(\tfrac{1}{4}\right) = \sin 4\pi = 0$$

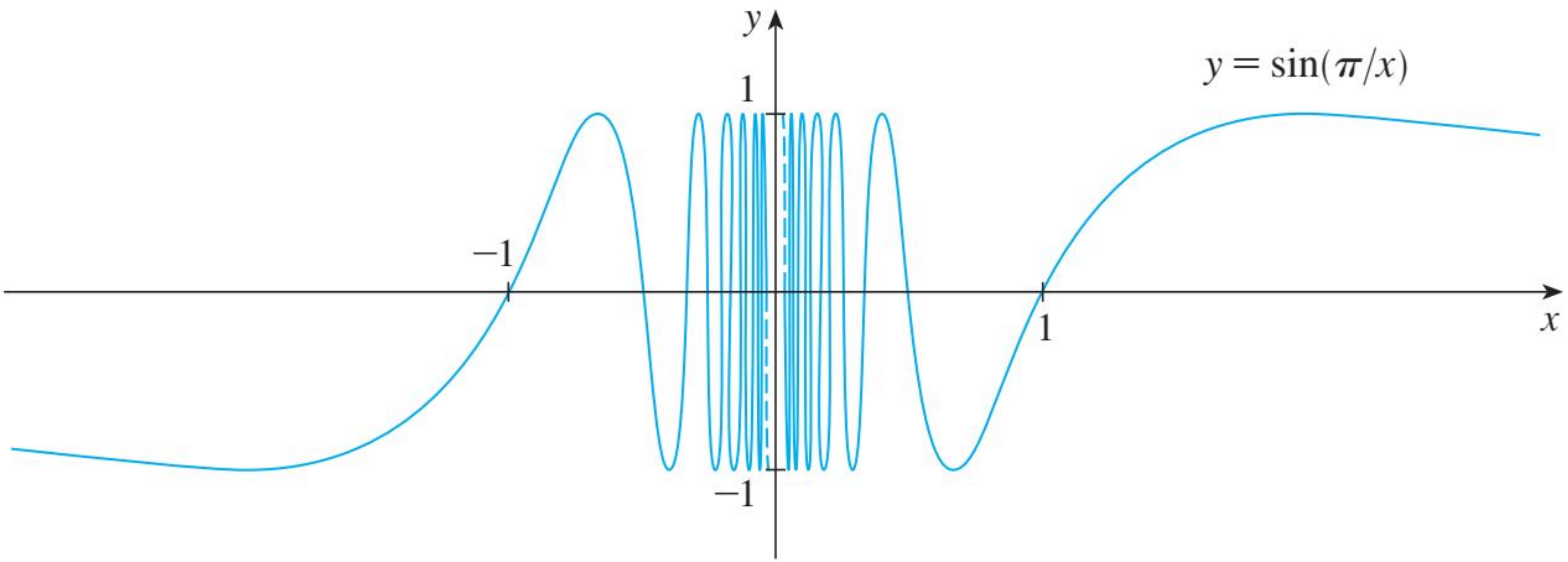
$$f(0.1) = \sin 10\pi = 0 \qquad f(0.01) = \sin 100\pi = 0$$

Similarly, $f(0.001) = f(0.0001) = 0$.

On the basis of this information we might be tempted to guess that

$$\lim_{x \to 0} \sin \frac{\pi}{x} = 0$$

but this time our guess is wrong. Note that although $f\left(\frac{1}{n}\right) = \sin(n\pi)$ for any integer n, it is also true that $f(x) = 1$ for infinitely many values of x (such as 2/5 or 2/9 or 2/11 or 2/49) that approach 0. We can see this from the graph of f shown in the figure.


y
$y = \sin(\pi/x)$
1
−1
1
−1
x

From the graph we can see that the values of $\sin\left(\frac{\pi}{x}\right)$ oscillate between 1 and -1 as x approaches 0. Since the values of $f(x)$ do not approach a fixed number as x approaches 0,

Thus

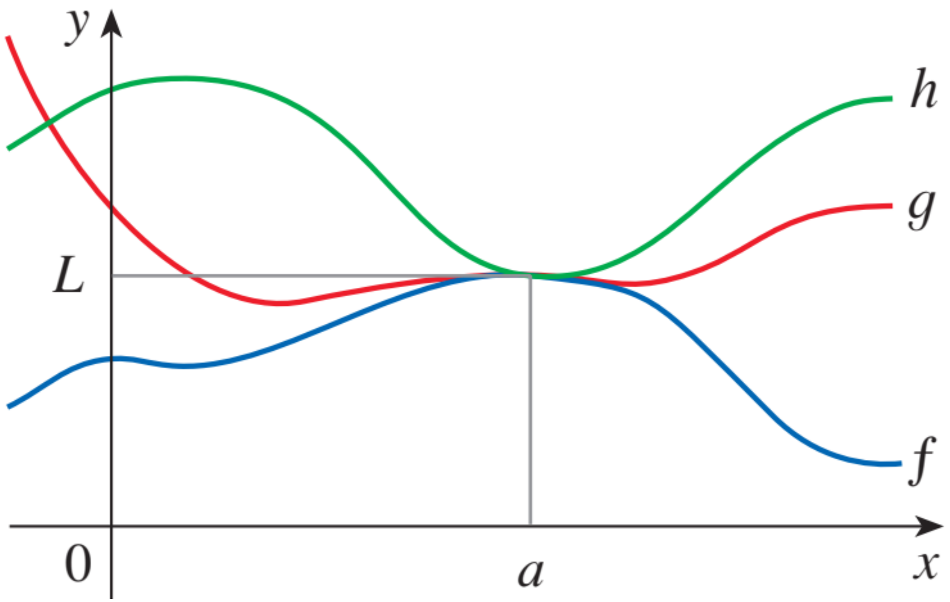
$$\lim_{x\to 0} \sin \frac{\pi}{x} \text{ does not exist}$$

The Squeeze Theorem If $f(x) \leqslant g(x) \leqslant h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then

$$\lim_{x \to a} g(x) = L$$



Problem Show that $\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0.$

Solution

First note that we **cannot** rewrite the limit as the product of the limits $\lim_{x \to 0} x^2$ and $\lim_{x \to 0} \sin(1/x)$ because $\lim_{x \to 0} \sin(1/x)$ does not exist.

We *can* find the limit by using the Squeeze Theorem.

Since

$$-1 \leqslant \sin \frac{1}{x} \leqslant 1$$

$$-x^2 \leqslant x^2 \sin \frac{1}{x} \leqslant x^2$$

We know that

$$\lim_{x \to 0} x^2 = 0 \qquad \text{and} \qquad \lim_{x \to 0} (-x^2) = 0$$

Taking $f(x) = -x^2$, $g(x) = x^2 \sin(1/x)$, and $h(x) = x^2$ in the Squeeze Theorem, we obtain

$$\boxed{\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0}$$

Problem

Evaluate $$\lim_{x\to 0} \frac{\sqrt{x^2 + 100} - 10}{x^2}$$

Solution

$$\frac{\sqrt{x^2 + 100} - 10}{x^2} = \frac{\sqrt{x^2 + 100} - 10}{x^2} \cdot \frac{\sqrt{x^2 + 100} + 10}{\sqrt{x^2 + 100} + 10}$$

$$
\begin{aligned}
\frac{\sqrt{x^2+100}-10}{x^2} &= \frac{x^2+100-100}{x^2\left(\sqrt{x^2+100}+10\right)} \\
&= \frac{x^2}{x^2\left(\sqrt{x^2+100}+10\right)} \\
&= \frac{1}{\sqrt{x^2+100}+10}.
\end{aligned}
$$

Therefore,

$$\begin{aligned}\lim_{x\to 0}\frac{\sqrt{x^2+100}-10}{x^2} &= \lim_{x\to 0}\frac{1}{\sqrt{x^2+100}+10}\\ &= \frac{1}{\sqrt{0^2+100}+10}\\ &= \frac{1}{20} = 0.05.\end{aligned}$$

Problem

Let

$$B(t) = \begin{cases} 4 - \frac{1}{2}t & \text{if } t < 2 \\ \sqrt{t + \mathrm{c}} & \text{if } t \geqslant 2 \end{cases}$$

Find the value of c so that $\lim_{t \to 2} B(t)$ exists.

Problem

Let $g(x) = \dfrac{x^2 + x - 6}{|x - 2|}$.

(a) Find

(i) $\lim_{x \to 2^+} g(x)$ (ii) $\lim_{x \to 2^-} g(x)$

(b) Does $\lim_{x \to 2} g(x)$ exist?

(c) Sketch the graph of g.

Problem

Is there a number a such that

$$\lim_{x \to -2} \frac{3x^2 + ax + a + 3}{x^2 + x - 2}$$

exists? If so, find the value of a and the value of the limit.

Problem

Find a and b such that $\lim_{x \to 2} f(x)$ and $\lim_{x \to 3} f(x)$ both exist.

$$h(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & \text{if } x < 2 \\ ax^2 - bx + 3 & \text{if } 2 \le x < 3 \\ 2x - a + b & \text{if } x \ge 3 \end{cases}$$

Practice Problem 1

Compute the limit (if exist).

(a) $$\lim_{x \to -2^-} (x+3)\frac{|x+2|}{x+2}$$

(c) $$\lim_{x \to 0^+} \frac{|\sin x|}{\sin x}$$

(b) $$\lim_{x \to 1^+} \frac{\sqrt{2x}(x-1)}{|x-1|}$$

(d) $$\lim_{x \to 0^+} \frac{1-\cos x}{|\cos x - 1|}$$

Practice Problem 2

Let

$$g(x) = \begin{cases} x & \text{if } x < 1 \\ 3 & \text{if } x = 1 \\ 2 - x^2 & \text{if } 1 < x \leqslant 2 \\ x - 3 & \text{if } x > 2 \end{cases}$$

(a) Evaluate each of the following, if it exists.

(i) $\lim_{x \to 1^-} g(x)$ (ii) $\lim_{x \to 1} g(x)$ (iii) $g(1)$

(iv) $\lim_{x \to 2^-} g(x)$ (v) $\lim_{x \to 2^+} g(x)$ (vi) $\lim_{x \to 2} g(x)$

(b) Sketch the graph of g.

Practice Problem 3

Evaluate $\displaystyle \lim_{x \to 2} \frac{\sqrt{6 - x} - 2}{\sqrt{3 - x} - 1}$.

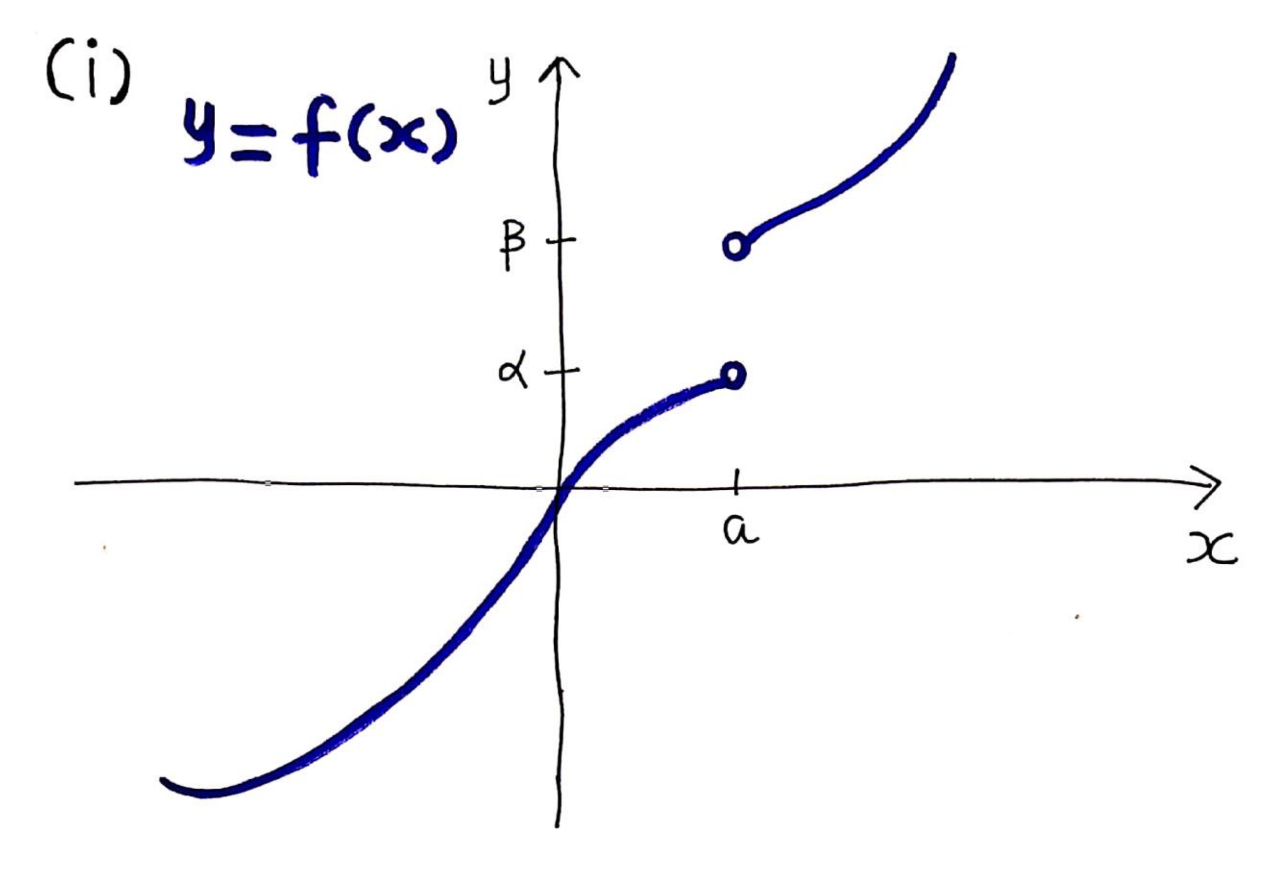
Continuity

A function f is continuous at a number a if

- $f(a)$ is defined (that is, a is in the domain of f)
- $\lim_{x \to a} f(x)$ exist
- $\lim_{x \to a} f(x) = f(a)$

We say that f is **discontinuous at *a*** (or f has a **discontinuity** at a) if f is not continuous at a.

Let's investigate the continuity of the following functions at a.



$f(a)$ is not defined

$\lim_{x \to a^-} f(x) = \alpha$

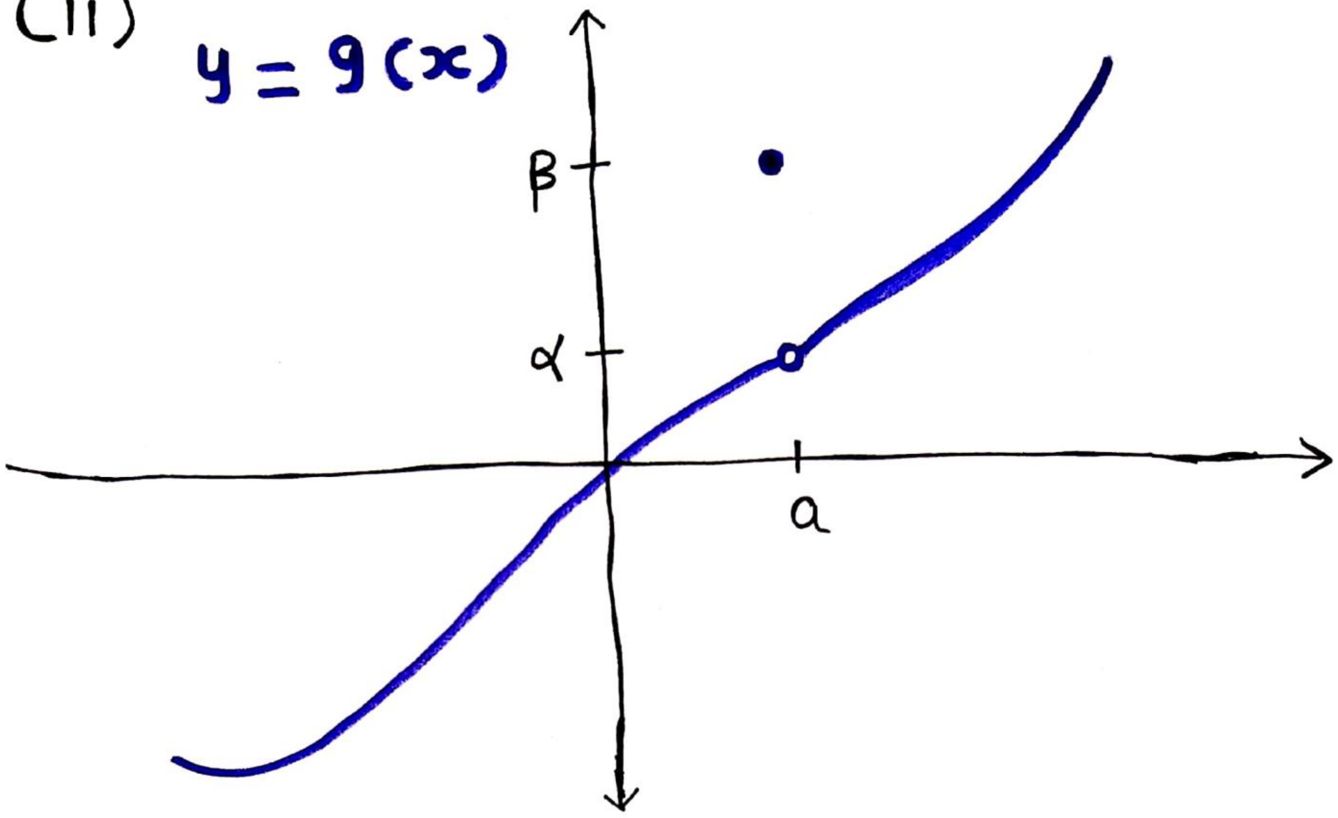
$\lim_{x \to a^+} f(x) = \beta$

$\lim_{x \to a} f(x)$ does not exist.

$\left[\because \lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x) \right]$

f is discontinuous at a.

(ii)



$g(a) = \beta$

$\lim_{x \to a^-} g(x) = \alpha$

$\lim_{x \to a^+} g(x) = \alpha$

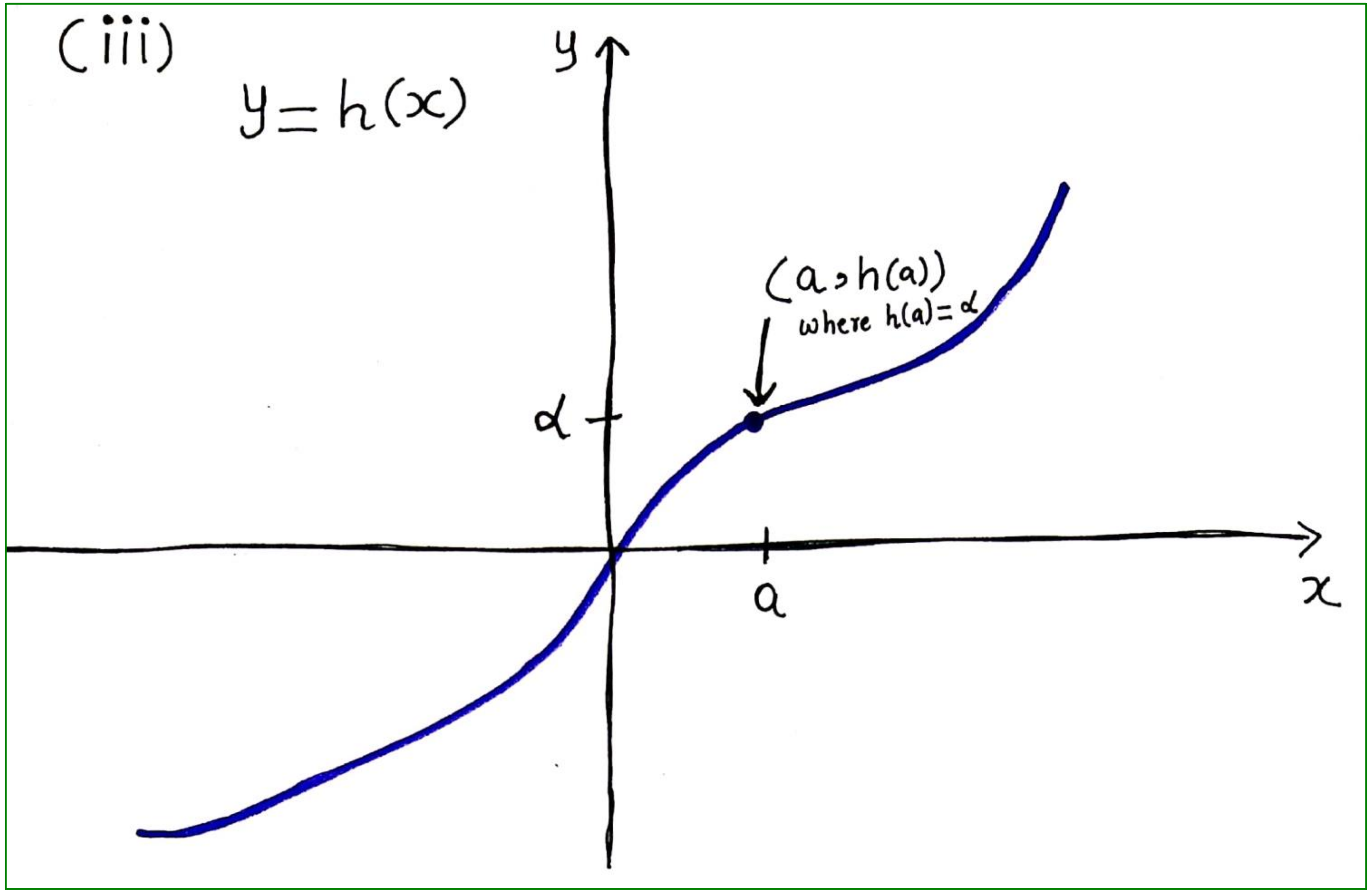
$\lim_{x \to a} g(x) = \alpha$

but

$\lim_{x \to a} g(x) \neq g(a)$

$[\because \alpha \neq \beta]$

f is discontinuous at a.



$h(a) = \alpha$

$\lim_{x \to a^-} h(x) = \alpha$

$\lim_{x \to a^+} h(x) = \alpha$

$\lim_{x \to a} h(x) = \alpha$

$\lim_{x \to a} h(x) = h(a)$

h is continuous at a.

Problem

Check the continuity of the function $g(x) = \dfrac{x^2 + x - 6}{|x - 2|}$ at $a = 2$.

Solution

Since 2 is not in the domain of g therefore $g(2)$ is not define and thus g is discontinuous at 2.

Problem

Check the continuity of the function h at $a = 2$.

$$h(x) = \begin{cases} \dfrac{x^3 - 8}{x - 2} & if\ x \neq 2 \\ 12 & if\ x = 2 \end{cases}$$

Solution $\quad h(2) = 12$

$$\lim_{x \to 2} h(x) = \lim_{x \to 2} \left(\frac{x^3 - 8}{x - 2} \right) = \lim_{x \to 2} \left(\frac{x^3 - 2^3}{x - 2} \right)$$

$$= \lim_{x \to 2} \frac{(x-2)(x^2 + 4 + 2x)}{(x-2)}$$

$$= \lim_{x \to 2} (x^2 + 2x + 4)$$

$$= 2^2 + 2(2) + 4 = 4 + 4 + 4 = 12$$

$$\Rightarrow \lim_{x \to 2} h(x) = 12$$

Since $\lim_{x \to 2} h(x) = h(2)$

$\Rightarrow$ h is continuous at 2.

Practice Problem

Check the continuity of the function h at $a = 2$.

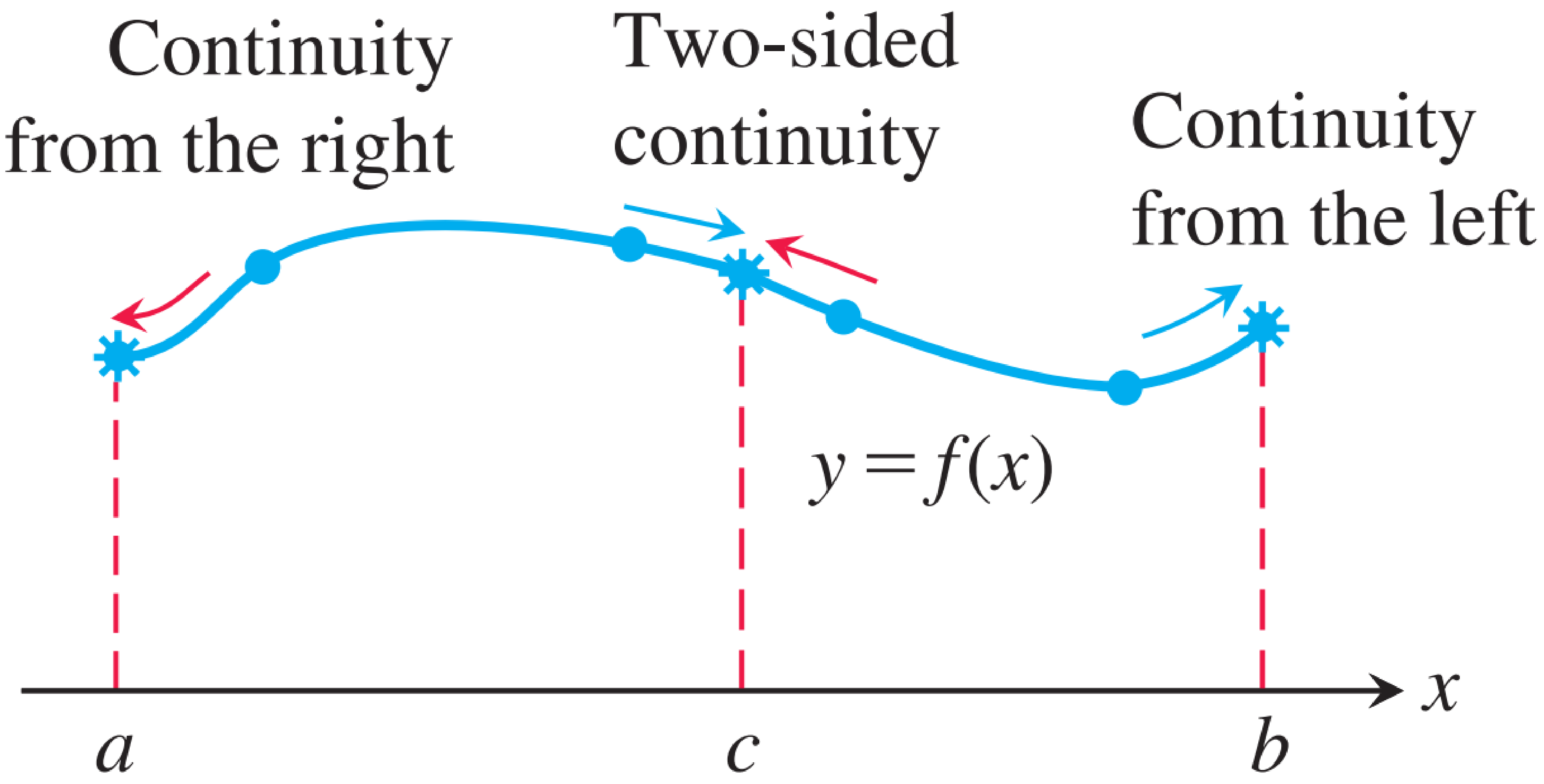
$$h(x) = \begin{cases} \dfrac{x^2 + x - 6}{|x - 2|} & if\ x \neq 2 \\ 5 & if\ x = 2 \end{cases}$$

Definition A function f is **continuous from the right at a number *a*** if

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is **continuous from the left at *a*** if

$$\lim_{x \to a^-} f(x) = f(a)$$


Continuity from the right
Two-sided continuity
Continuity from the left
$y = f(x)$
x
a
c
b

Definition A function f is **continuous on an interval** if it is continuous at every number in the interval. (If f is defined only on one side of an endpoint of the interval, we understand *continuous* at the endpoint to mean *continuous from the right* or *continuous from the left*.)

Theorem If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $f + g$

2. $f - g$

3. cf

4. fg

5. $\dfrac{f}{g}$ if $g(a) \neq 0$

Theorem The following types of functions are continuous at every number in their domains:

- polynomials
- rational functions
- root functions
- trigonometric functions
- inverse trigonometric functions
- exponential functions
- logarithmic functions

Problem

Where is the function $f(x) = \dfrac{\ln x + \tan^{-1}x}{x^2 - 1}$ continuous?

Solution

$y = \ln x$ is continuous for $x > 0$ and $y = \tan^{-1}x$ is continuous on $\mathbb{R}$.

Thus, $y = \ln x + \tan^{-1}x$ is continuous on $(0, \infty)$.

The denominator, $y = x^2 - 1$, is a polynomial, so it is continuous everywhere. Therefore, f is continuous at all positive numbers x except where $x^2 - 1 = 0 \Longleftrightarrow x = \pm 1$.

So f is continuous on the intervals $(0, 1)$ and $(1, \infty)$.

Problem Find $\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x}$.

Solution The function $f(x) = \frac{x^3 + 2x^2 - 1}{5 - 3x}$ is rational,

it is continuous on its domain, which is $\{x \mid x \neq \frac{5}{3}\}$.

Therefore

$$\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} = \lim_{x \to -2} f(x) = f(-2)$$

$$= \frac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)} = \boxed{-\frac{1}{11}}$$

Theorem If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a.

Problem Where are the following functions continuous?

(a) $h(x) = \sin(x^2)$ (b) $F(x) = \ln(1 + \cos x)$

Solution

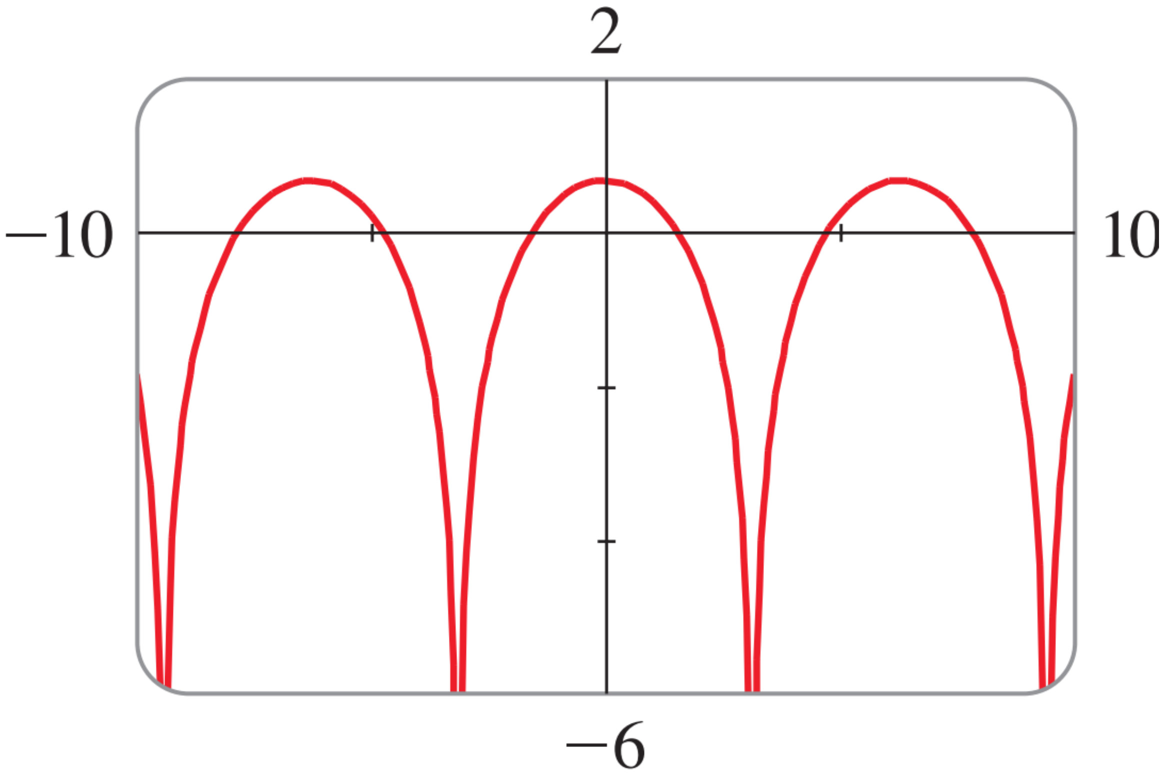
(a) We have $h(x) = f(g(x))$, where

$$g(x) = x^2 \qquad \text{and} \qquad f(x) = \sin x$$

We know that g is continuous on $\mathbb{R}$ since it is a polynomial, and f is also continuous everywhere. Thus $h = f \circ g$ is continuous on $\mathbb{R}$.

(b) We know that $f(x) = \ln x$ is continuous and $g(x) = 1 + \cos x$ is continuous (because both $y = 1$ and $y = \cos x$ are continuous). Therefore, $F(x) = f(g(x))$ is continuous wherever it is defined. The expression $\ln(1 + \cos x)$ is defined when $1 + \cos x > 0$, so it is undefined when $\cos x = -1$, and this happens when $x = \pm\pi, \pm 3\pi, \ldots$.

Thus F has discontinuities when x is an odd multiple of π and is continuous on the intervals between these values (see Figure).



$y = \ln(1 + \cos x)$

Problem

Show that the function

$$f(x) = \begin{cases} x^4 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is continuous on $(-\infty, \infty)$.

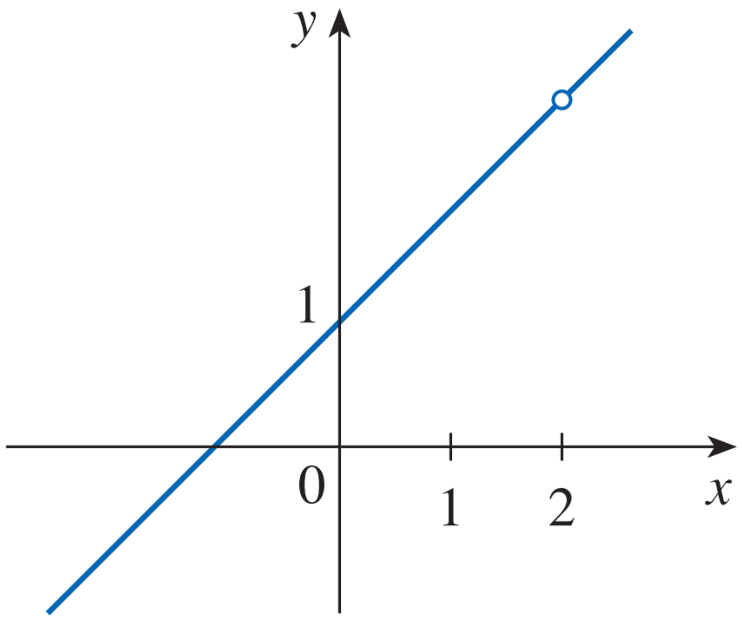
Practice Problem

Compute the values of a and b that makes $h(x)$ continuous everywhere.

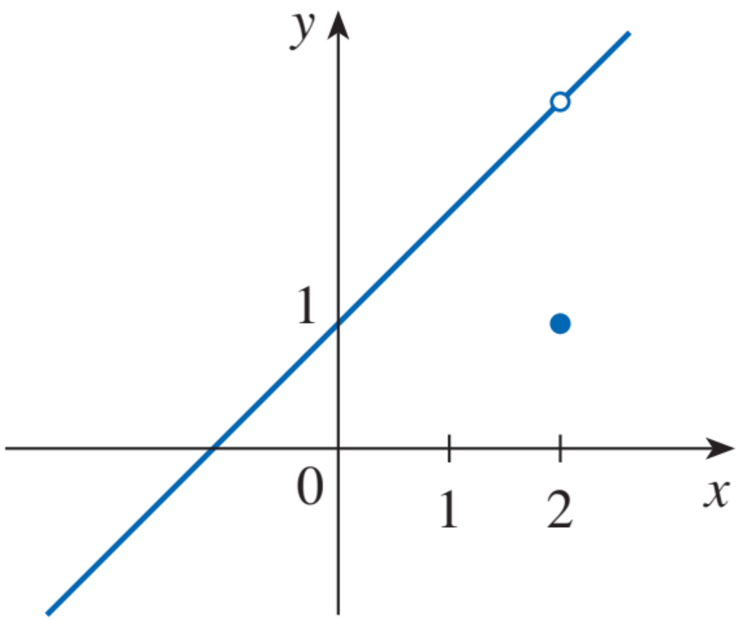
$$h(x)=\begin{cases} \dfrac{x^2-4}{x-2} & \text{if } x<2 \\ ax^2-bx+3 & \text{if } 2\le x<3 \\ 2x-a+b & \text{if } x\ge 3 \end{cases}$$

Types of Discontinuity

Removable Discontinuity

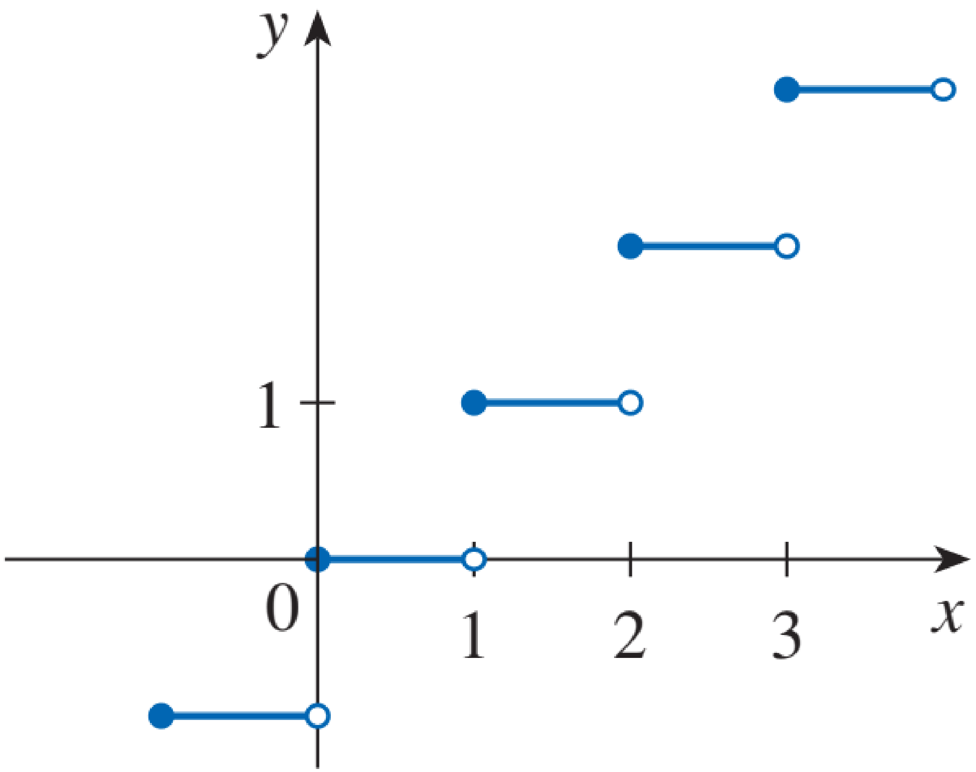


(a) A removable discontinuity

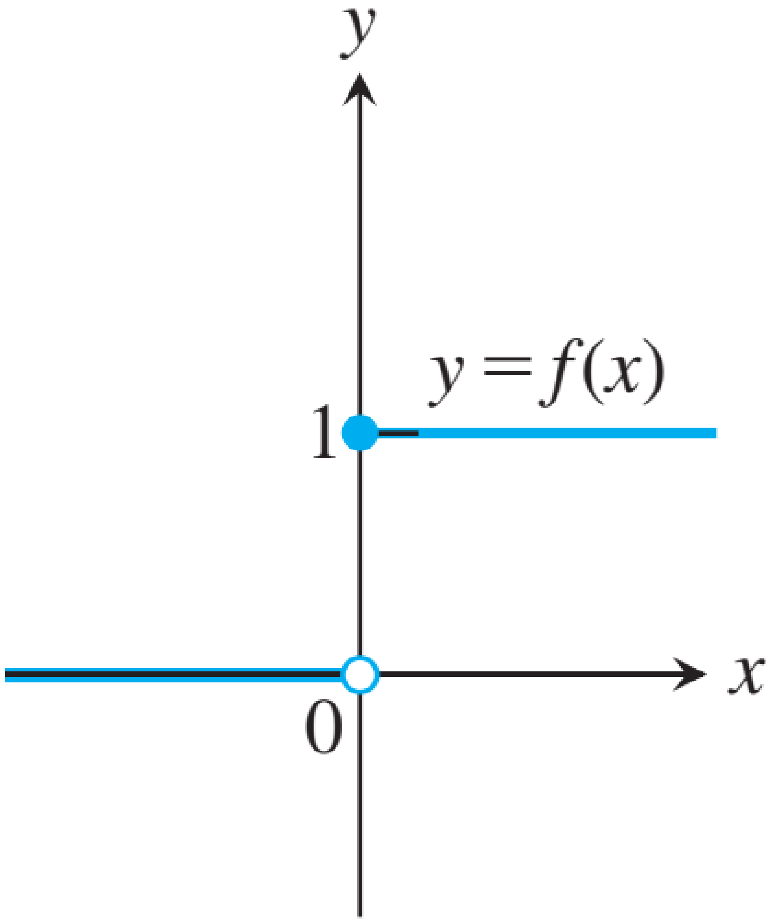


(b) A removable discontinuity

Jumping Discontinuity

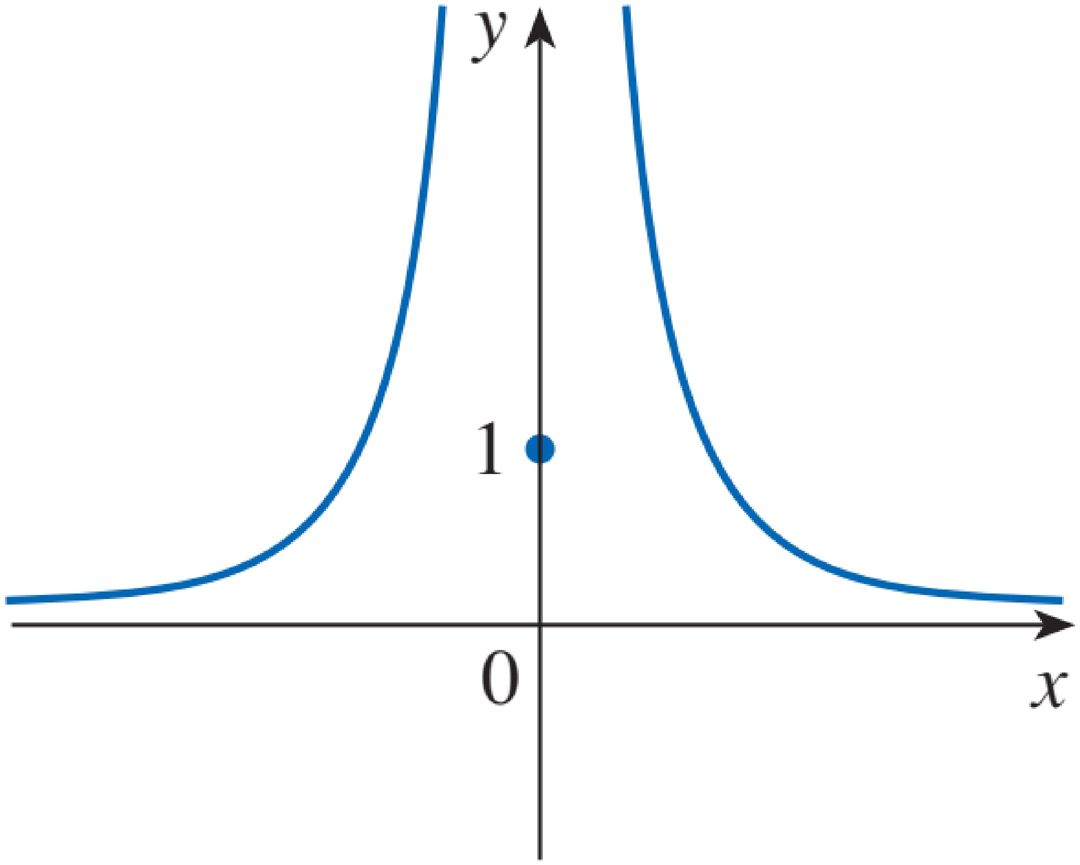


Jump discontinuities



Jump discontinuities

Infinite Discontinuity



An infinite discontinuity

Problem

Determine whether the piecewise define function defined by

$$h(x) = \begin{cases} \dfrac{x^3 - 1}{x - 1} & \text{if } x < 1 \\ 5 & \text{if } x = 1 \\ x + 2 & \text{if } x > 1 \end{cases}$$

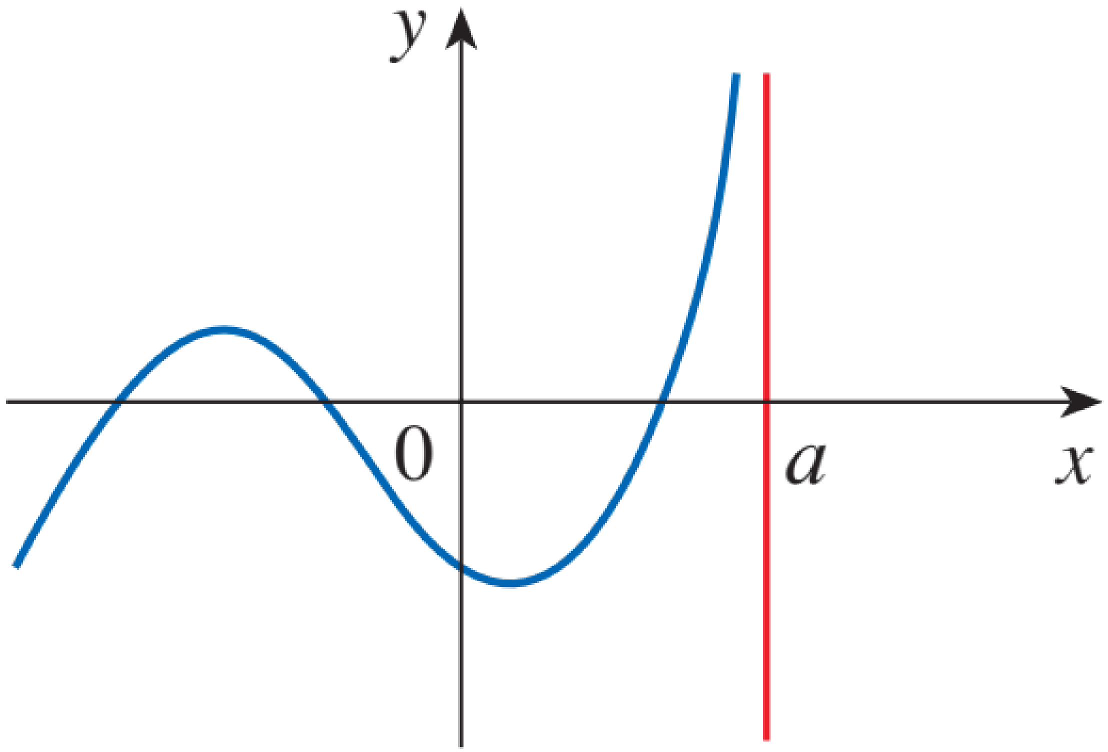
is continuous at 1. If the function is discontinuous, then mention the type of discontinuity and remove it.

Vertical Asymptote

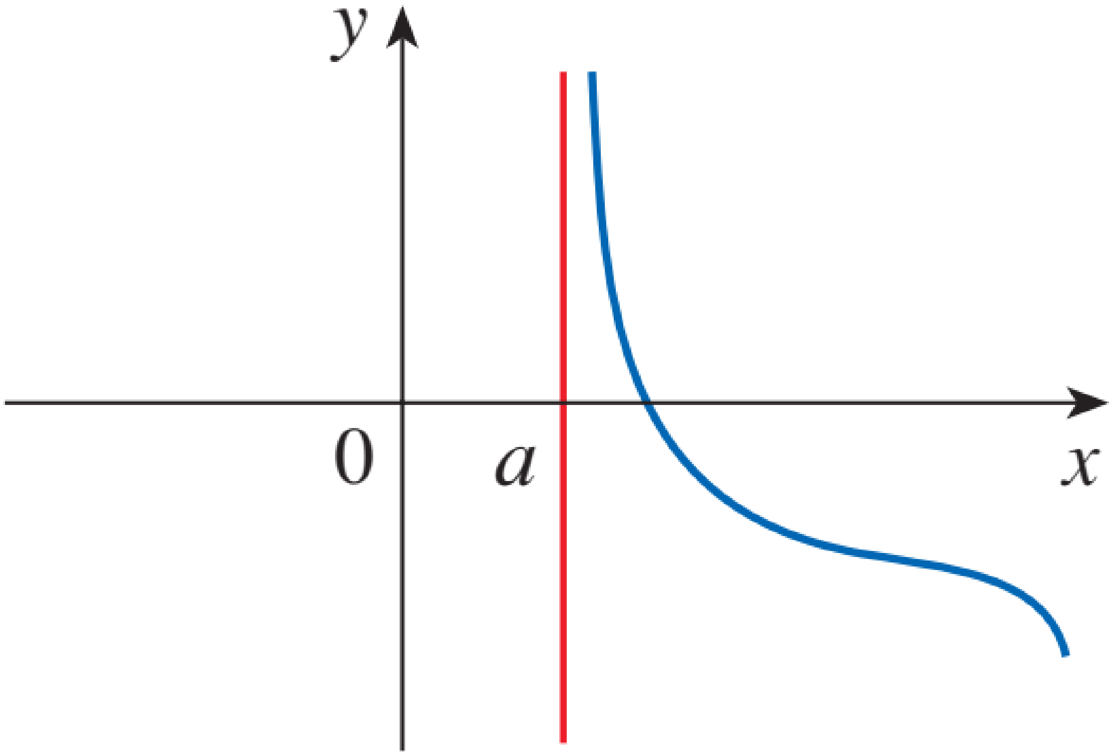
Definition The vertical line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x \to a} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

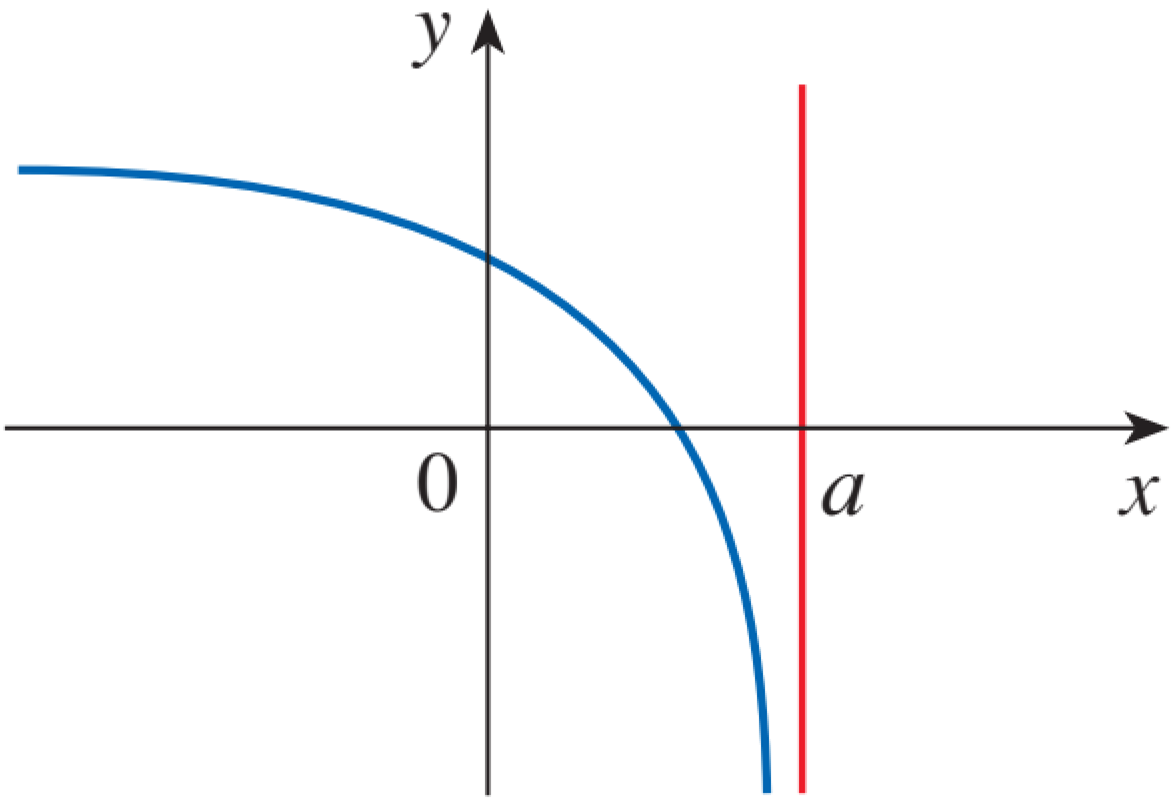
$$\lim_{x \to a} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$



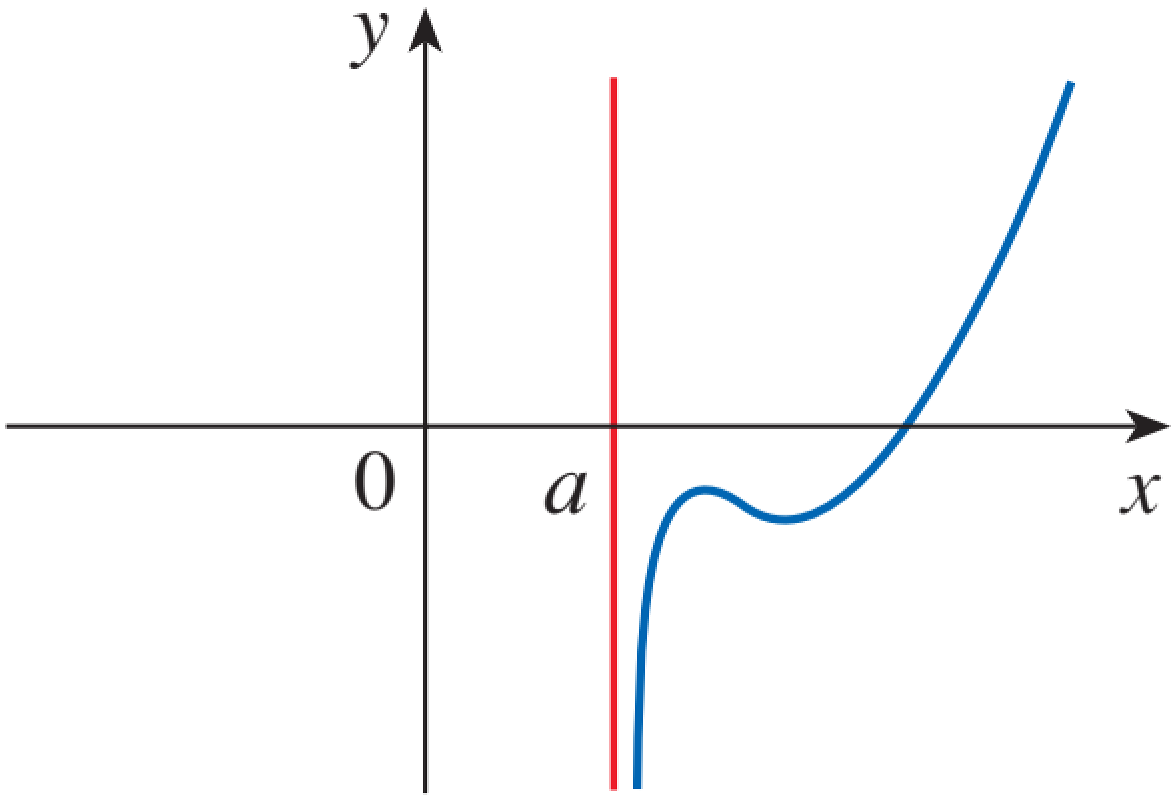
(a) $\lim_{x \to a^-} f(x) = \infty$



(b) $\lim_{x \to a^+} f(x) = \infty$



(c) $\lim_{x \to a^-} f(x) = -\infty$



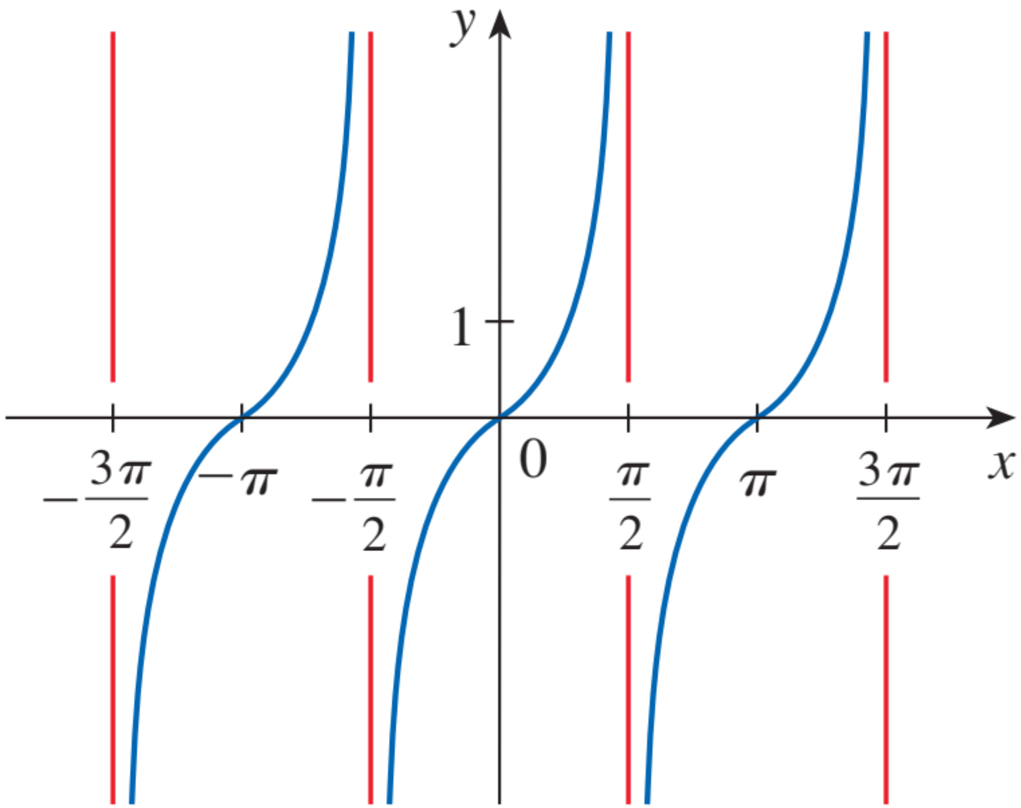
(d) $\lim_{x \to a^+} f(x) = -\infty$

Problem Find the vertical asymptotes of $f(x) = \tan x$.

Solution

$$\lim_{x \to (\pi/2)^-} \tan x = \infty$$

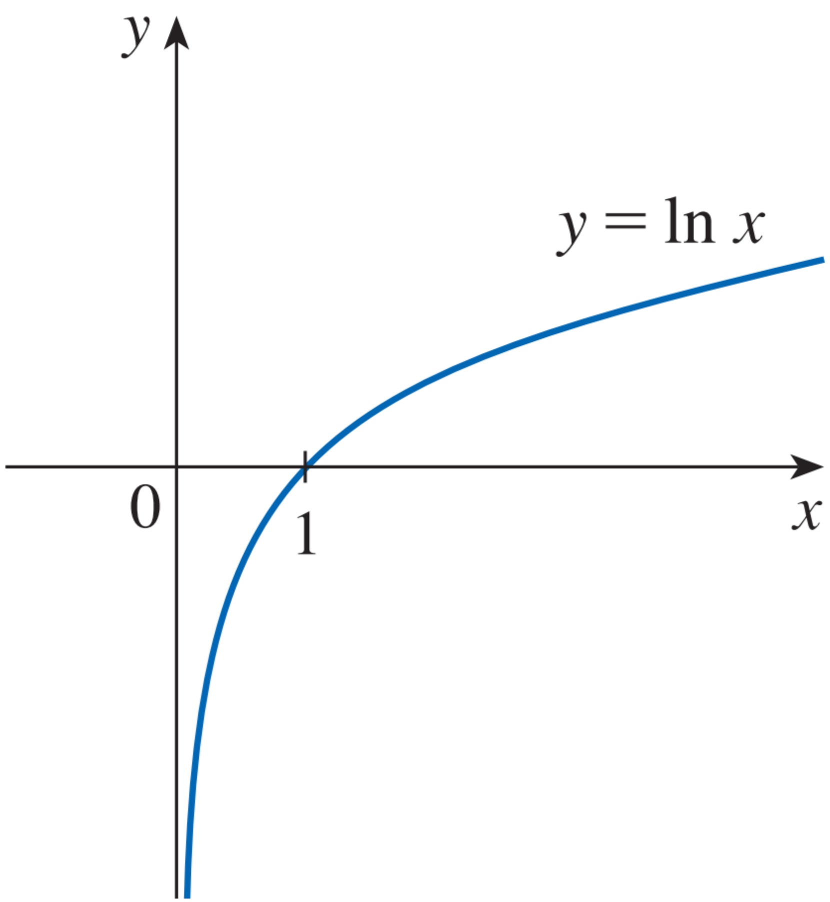
and $$\lim_{x \to (\pi/2)^+} \tan x = -\infty$$



This shows that the line $x = \pi/2$ is a vertical asymptote.

Also the lines $x = \frac{\pi}{2} + n\pi$, where n is an integer, are all vertical asymptotes of $f(x) = \tan x$.

Another example of a function whose graph has a vertical asymptote is the natural logarithmic function $y = \ln x$.



$$\boxed{\lim_{x \to 0^+} \ln x = -\infty}$$

The *y*-axis is a vertical asymptote of the natural logarithmic function.

Problem

Does the curve $y = \dfrac{2x}{x - 3}$ have a vertical asymptote?

Solution

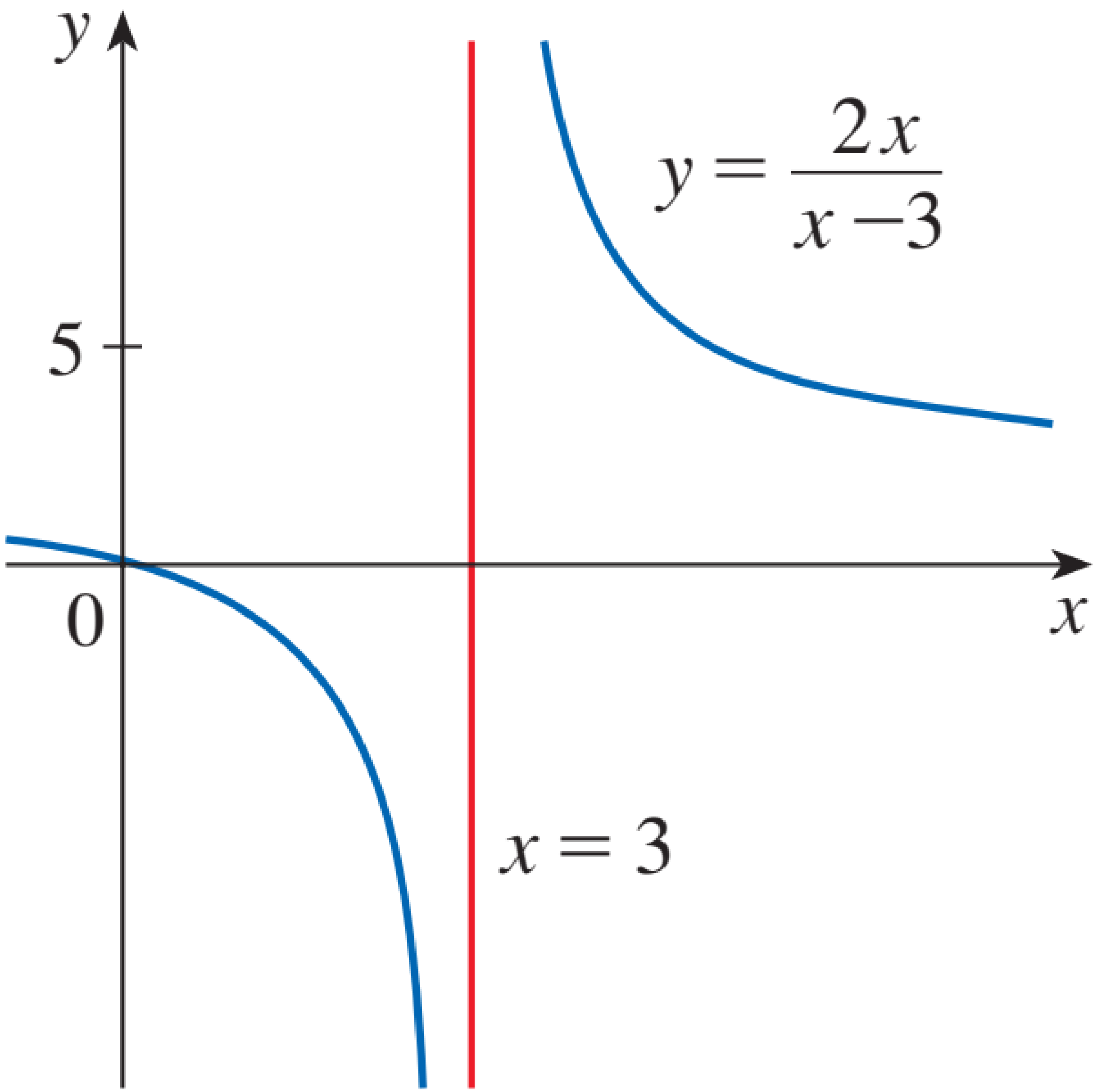
If x is close to 3 but larger than 3, then the denominator $x - 3$ is a small positive number and $2x$ is close to 6. So the quotient $2x/(x - 3)$ is a large *positive* number. [For instance, if $x = 3.01$ then $2x/(x - 3) = 6.02/0.01 = 602$.] Thus, intuitively, we see that

$$\lim_{x \to 3^+} \frac{2x}{x - 3} = \infty$$

Likewise, if x is close to 3 but smaller than 3, then $x - 3$ is a small negative number but $2x$ is still a positive number (close to 6). So $2x/(x - 3)$ is a numerically large *negative* number. Thus

$$\lim_{x \to 3^-} \frac{2x}{x - 3} = -\infty$$

The graph of the curve $y = 2x/(x - 3)$ is the line $x = 3$ is a vertical asymptote.



y
5
0
x
$y = \frac{2x}{x-3}$
$x = 3$

Problem

Determine all vertical asymptotes of the graph of

$$h(x) = \frac{x^2 + 2x - 8}{x^2 - 4}.$$

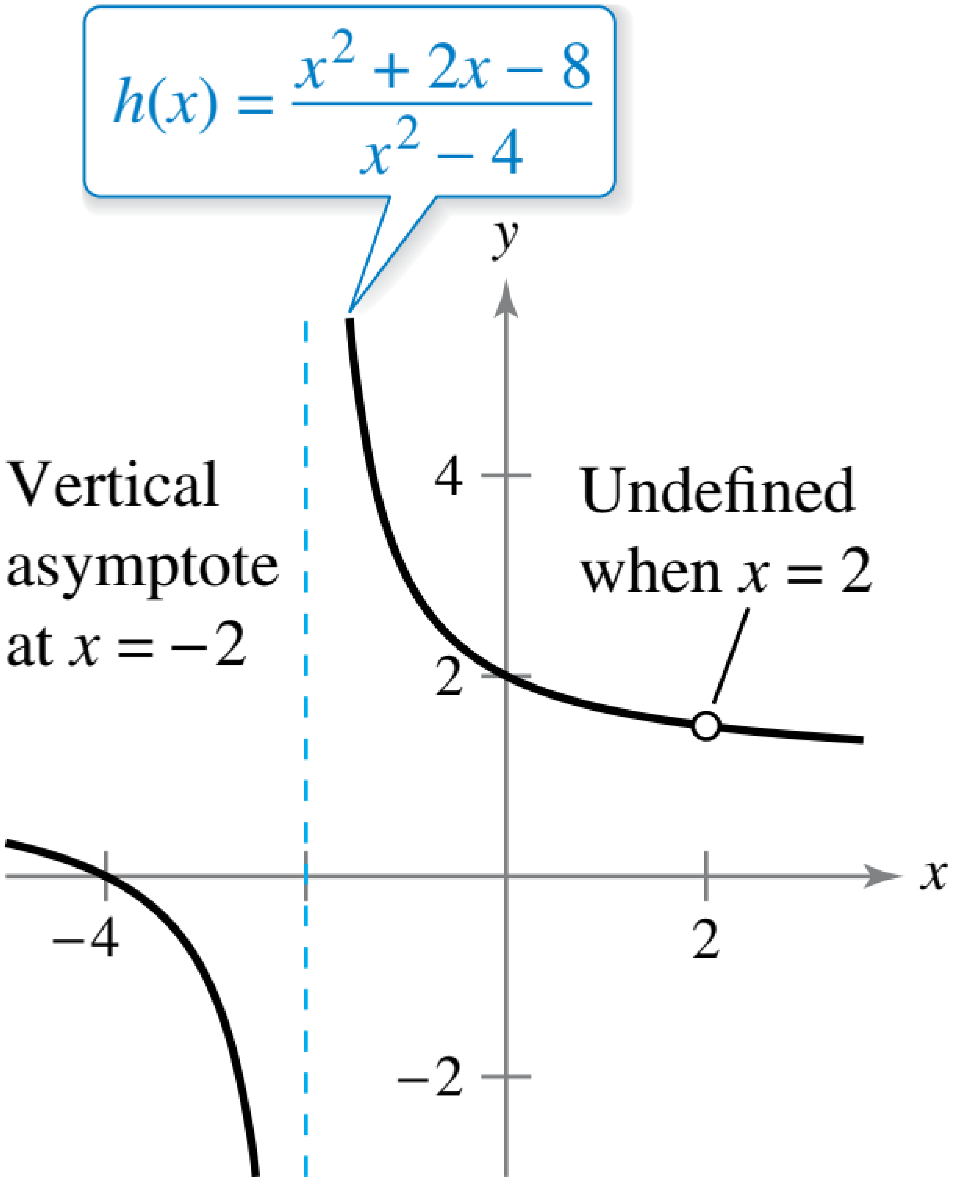
Solution

$$\begin{aligned} h(x) &= \frac{x^2 + 2x - 8}{x^2 - 4} \\ &= \frac{(x + 4)\cancel{(x - 2)}}{(x + 2)\cancel{(x - 2)}} \\ &= \frac{x + 4}{x + 2}, \quad x \neq 2 \end{aligned}$$

At all x-values other than $x = 2$, the graph of h coincides with the graph of $k(x) = (x + 4)/(x + 2)$.

$$\lim_{x \to -2^-} \frac{x^2 + 2x - 8}{x^2 - 4} = -\infty \quad \text{and} \quad \lim_{x \to -2^+} \frac{x^2 + 2x - 8}{x^2 - 4} = \infty.$$

Thus h has a vertical asymptote at $x = -2$.


$h(x) = \frac{x^2 + 2x - 8}{x^2 - 4}$
y
Vertical asymptote at $x = -2$
4
2
-2
Undefined when $x = 2$
x
-4
2

Practice Problem

Find the vertical asymptote(s) of the function.

(i) $f(x) = \frac{x^2+7x+12}{x^2-16}$

(ii) $g(x) = \frac{(x+2)(x+3)}{x^2-16}$

(iii) $h(x) = \frac{1}{x^2+1}$

(iv) $r(x) = \frac{1}{x^2-9}$

(v) $s(x) = \csc x$

(vi) $u(x) = \sec x$

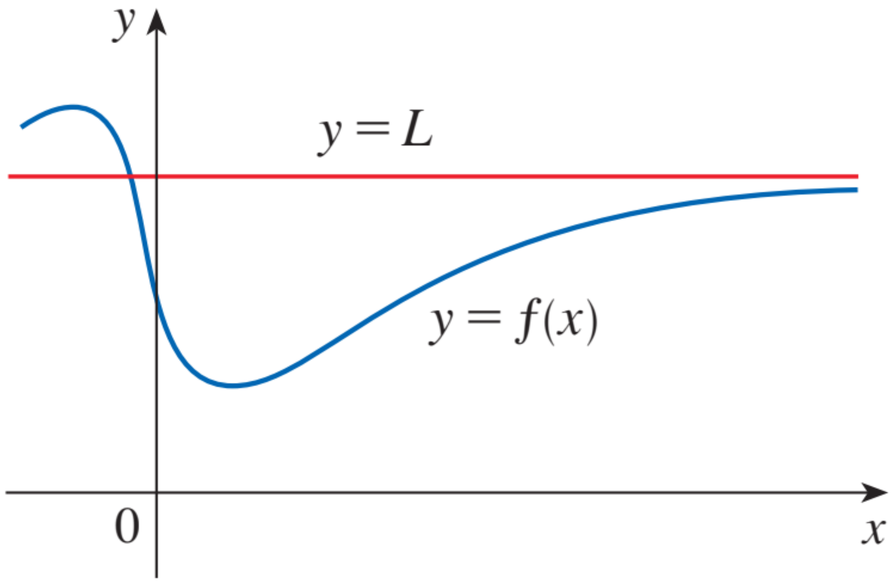
Limit at Infinity & Horizontal Asymptote

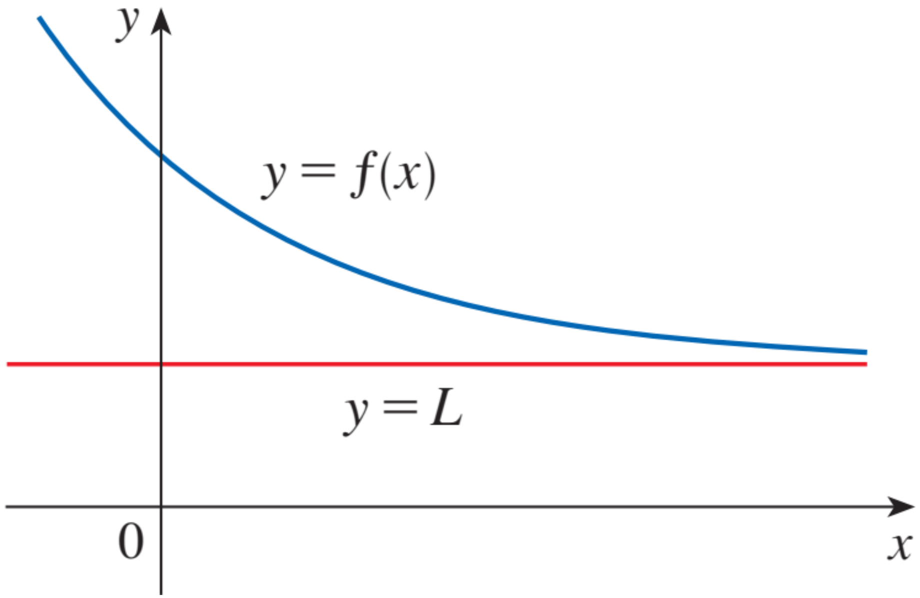
Definition of a Limit at Infinity Let f be a function defined on some interval (a, ∞). Then

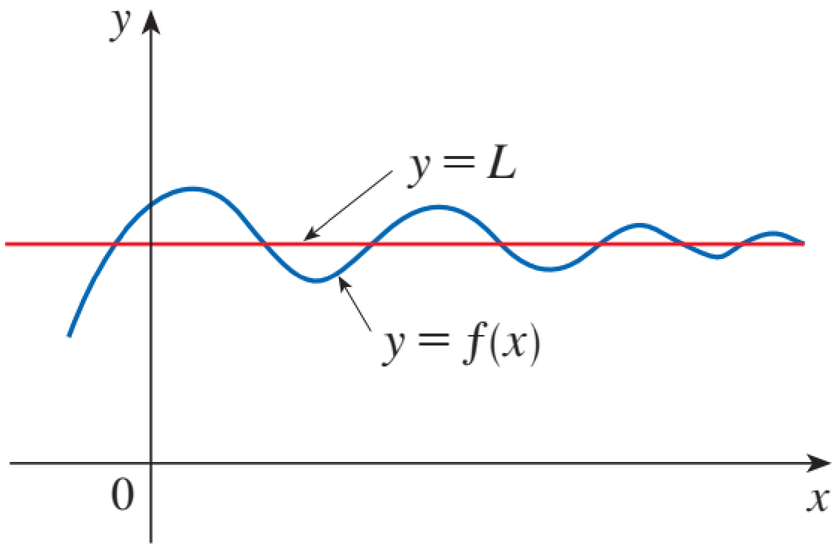
$$\lim_{x \to \infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large.

Another notation for $\lim_{x \to \infty} f(x) = L$ is $f(x) \to L$ as $x \to \infty$


y
y = L
y = f(x)
0
x

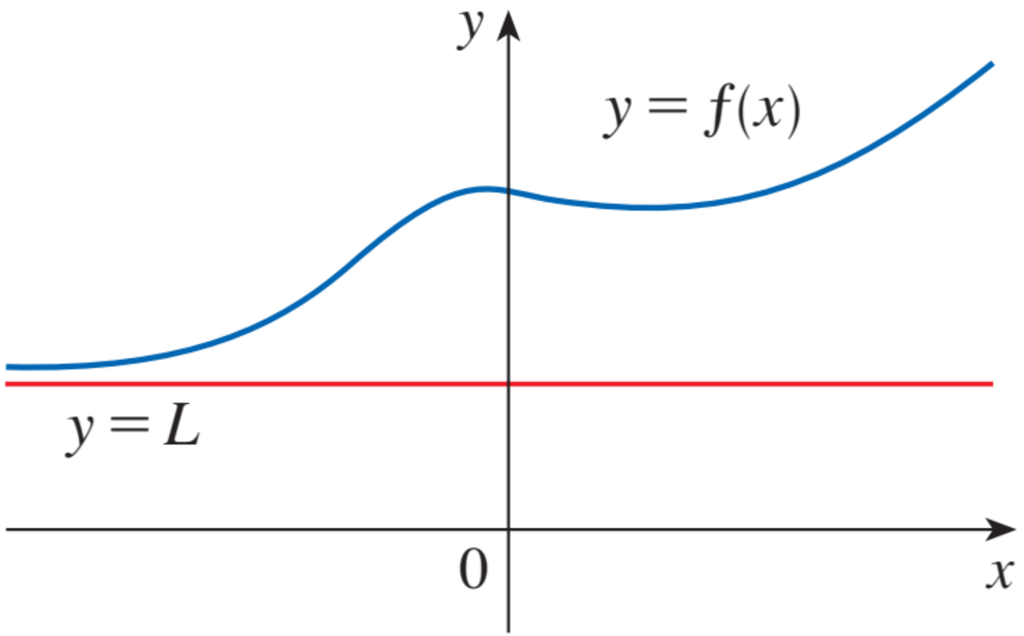

y
y = f(x)
y = L
0
x


y
y = L
y = f(x)
0
x

Definition Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large negative.



Horizontal Asymptote

Definition The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

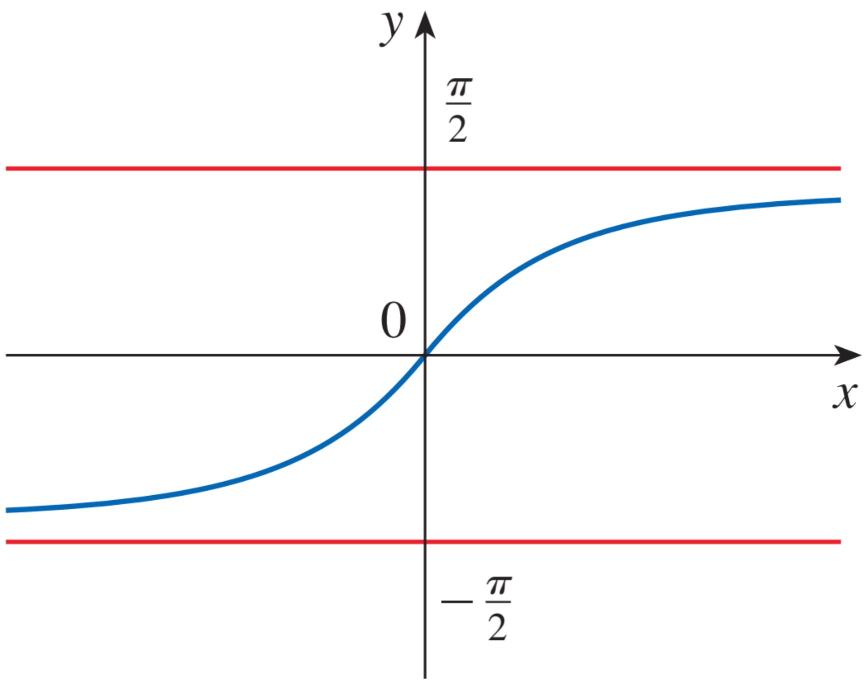
$$\lim_{x \to \infty} f(x) = L \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = L$$

Cosider $y = \tan^{-1}x$

Both of the lines $y = -\pi/2$ and $y = \pi/2$ are horizontal asymptotes.

Because

$$\boxed{\lim_{x \to -\infty} \tan^{-1}x = -\frac{\pi}{2} \qquad \lim_{x \to \infty} \tan^{-1}x = \frac{\pi}{2}}$$



Evaluating Limits at Infinity

Theorem If $r > 0$ is a rational number, then

$$\lim_{x \to \infty} \frac{1}{x^r} = 0$$

If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x \to -\infty} \frac{1}{x^r} = 0$$

THEOREM

For a polynomial of degree $n > 0$, $p_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, we have

$$\lim_{x\to\infty} p_n(x) = \begin{cases} \infty, & \text{if } a_n > 0 \\ -\infty, & \text{if } a_n < 0 \end{cases}.$$

Problem

Evaluate the following limit $\lim_{x \to \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$

Solution

As x becomes large, both numerator and denominator become large, so it isn't obvious what happens to their ratio.

To evaluate the limit at infinity of any rational function, we first divide both the numerator and denominator by the highest power of x that occurs in the denominator.

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \lim_{x\to\infty} \frac{\dfrac{3x^2 - x - 2}{x^2}}{\dfrac{5x^2 + 4x + 1}{x^2}} = \lim_{x\to\infty} \frac{3 - \dfrac{1}{x} - \dfrac{2}{x^2}}{5 + \dfrac{4}{x} + \dfrac{1}{x^2}}$$

$$= \frac{\displaystyle\lim_{x\to\infty}\left(3 - \frac{1}{x} - \frac{2}{x^2}\right)}{\displaystyle\lim_{x\to\infty}\left(5 + \frac{4}{x} + \frac{1}{x^2}\right)}$$

$$= \frac{\lim\limits_{x\to\infty} 3 - \lim\limits_{x\to\infty} \frac{1}{x} - 2\lim\limits_{x\to\infty} \frac{1}{x^2}}{\lim\limits_{x\to\infty} 5 + 4\lim\limits_{x\to\infty} \frac{1}{x} + \lim\limits_{x\to\infty} \frac{1}{x^2}}$$

$$= \frac{3 - 0 - 0}{5 + 0 + 0}$$

$$= \boxed{\frac{3}{5}}$$

Problem Find the horizontal asymptotes of the graph of the function

$$f(x) = \frac{\sqrt{2x^2 + 1}}{3x - 5}$$

Solution Dividing both numerator and denominator by x (which is the highest power of x in the denominator) and using the properties of limits, we have

$$\lim_{x \to \infty} \frac{\sqrt{2x^2 + 1}}{3x - 5} = \lim_{x \to \infty} \frac{\dfrac{\sqrt{2x^2 + 1}}{x}}{\dfrac{3x - 5}{x}} = \lim_{x \to \infty} \frac{\sqrt{\dfrac{2x^2 + 1}{x^2}}}{\dfrac{3x - 5}{x}} \quad \left(\text{since } \sqrt{x^2} = x \text{ for } x > 0\right)$$

$$= \frac{\displaystyle\lim_{x \to \infty} \sqrt{2 + \frac{1}{x^2}}}{\displaystyle\lim_{x \to \infty} \left(3 - \frac{5}{x}\right)} = \frac{\sqrt{\displaystyle\lim_{x \to \infty} 2 + \lim_{x \to \infty} \frac{1}{x^2}}}{\displaystyle\lim_{x \to \infty} 3 - 5 \lim_{x \to \infty} \frac{1}{x}} = \frac{\sqrt{2 + 0}}{3 - 5 \cdot 0} = \frac{\sqrt{2}}{3}$$

Therefore the line $y = \sqrt{2}/3$ is a horizontal asymptote of the graph of f.

In computing the limit as $x \to -\infty$, we must remember that for $x < 0$, we have $\sqrt{x^2} = |x| = -x$. So when we divide the numerator by x, for $x < 0$ we get

$$\frac{\sqrt{2x^2+1}}{x} = \frac{\sqrt{2x^2+1}}{-\sqrt{x^2}} = -\sqrt{\frac{2x^2+1}{x^2}} = -\sqrt{2+\frac{1}{x^2}}$$

Therefore

$$\lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to-\infty} \frac{-\sqrt{2+\dfrac{1}{x^2}}}{3-\dfrac{5}{x}} = \frac{-\sqrt{2+\displaystyle\lim_{x\to-\infty}\frac{1}{x^2}}}{3-5\displaystyle\lim_{x\to-\infty}\frac{1}{x}} = -\frac{\sqrt{2}}{3}$$

Thus the line $y = -\sqrt{2}/3$ is also a horizontal asymptote.

Infinite Limits at Infinity

The notation

$$\lim_{x \to \infty} f(x) = \infty$$

is used to indicate that the values of $f(x)$ become large as x becomes large. Similar meanings are attached to the following symbols:

$$\lim_{x \to -\infty} f(x) = \infty \qquad \lim_{x \to \infty} f(x) = -\infty \qquad \lim_{x \to -\infty} f(x) = -\infty$$

Problem Find $\lim_{x \to \infty} x^3$ and $\lim_{x \to -\infty} x^3$.

Solution

When x becomes large, x^3 also becomes large. For instance,

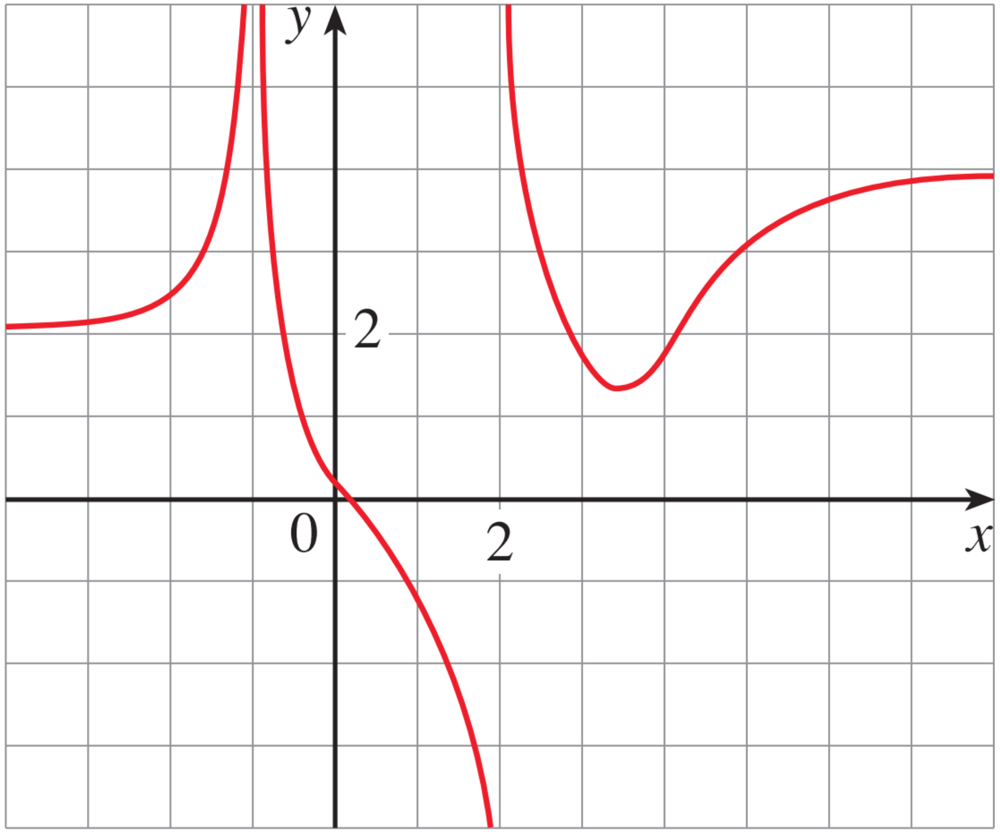
$$10^3 = 1000 \qquad 100^3 = 1{,}000{,}000 \qquad 1000^3 = 1{,}000{,}000{,}000$$

Thus we can write $\lim_{x \to \infty} x^3 = \infty$

Similarly, when x is large negative, so is x^3. Thus $\lim_{x \to -\infty} x^3 = -\infty$

Problem

Find the infinite limits, limits at infinity, and asymptotes for the function f whose graph is shown in the figure.



Solution

We see that the values of $f(x)$ become large as $x \to -1$ from both sides, so

$$\lim_{x \to -1} f(x) = \infty$$

Notice that $f(x)$ becomes large negative as x approaches 2 from the left, but large positive as x approaches 2 from the right. So

$$\lim_{x \to 2^-} f(x) = -\infty \qquad \text{and} \qquad \lim_{x \to 2^+} f(x) = \infty$$

Thus both of the lines $x = -1$ and $x = 2$ are vertical asymptotes.

As x becomes large, it appears that $f(x)$ approaches 4. But as x decreases through negative values, $f(x)$ approaches 2. So

$$\lim_{x \to \infty} f(x) = 4 \qquad \text{and} \qquad \lim_{x \to -\infty} f(x) = 2$$

This means that both $y = 4$ and $y = 2$ are horizontal asymptotes.

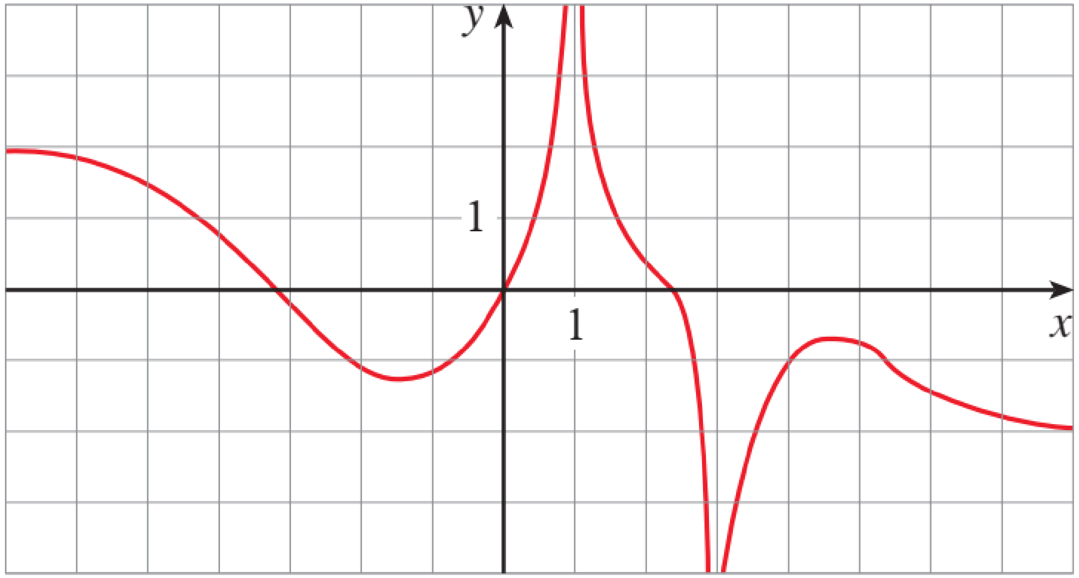
Practice Problem 1

For the function f whose graph is given, state the following.

(a) $\lim_{x \to \infty} f(x)$ (b) $\lim_{x \to -\infty} f(x)$

(c) $\lim_{x \to 1} f(x)$ (d) $\lim_{x \to 3} f(x)$

(e) The equations of the asymptotes



Practice Problem 2

For the function g whose graph is given, state the following.

(a) $\lim_{x \to \infty} g(x)$

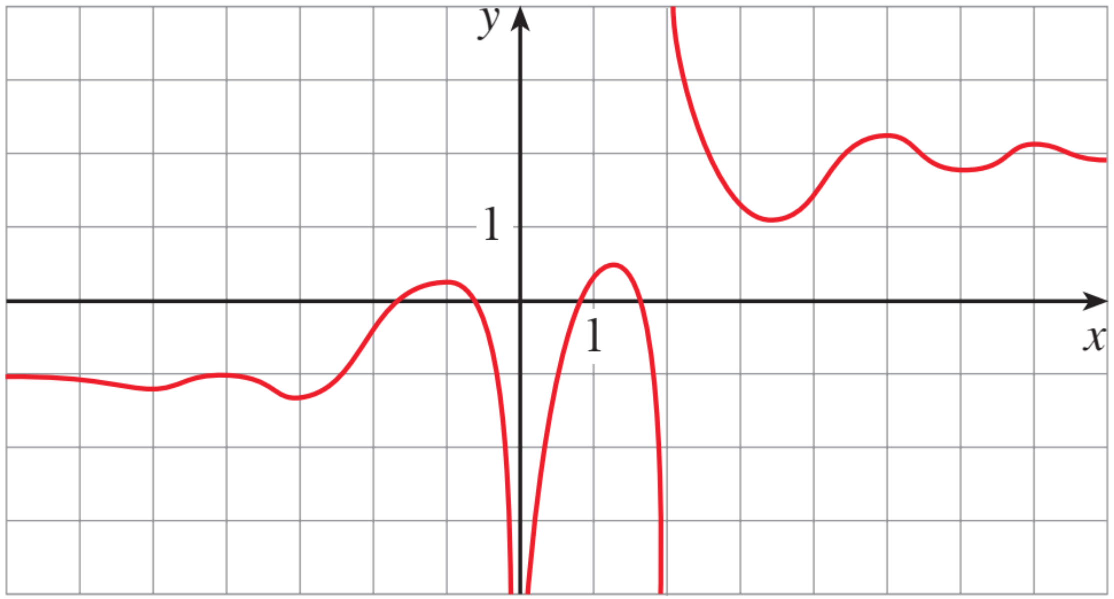
(b) $\lim_{x \to -\infty} g(x)$

(c) $\lim_{x \to 0} g(x)$

(d) $\lim_{x \to 2^-} g(x)$

(e) $\lim_{x \to 2^+} g(x)$

(f) The equations of the asymptotes



Practice Problem 3

Find the limit or show that it does not exist.

(i) $$\lim_{t \to \infty} \frac{t - t\sqrt{t}}{2t^{3/2} + 3t - 5}$$

(iii) $$\lim_{x \to -\infty} (x^2 + 2x^7)$$

(ii) $$\lim_{x \to \infty} \frac{(2x^2 + 1)^2}{(x - 1)^2(x^2 + x)}$$

(iv) $$\lim_{x \to \infty} (e^{-x} + 2\cos 3x)$$

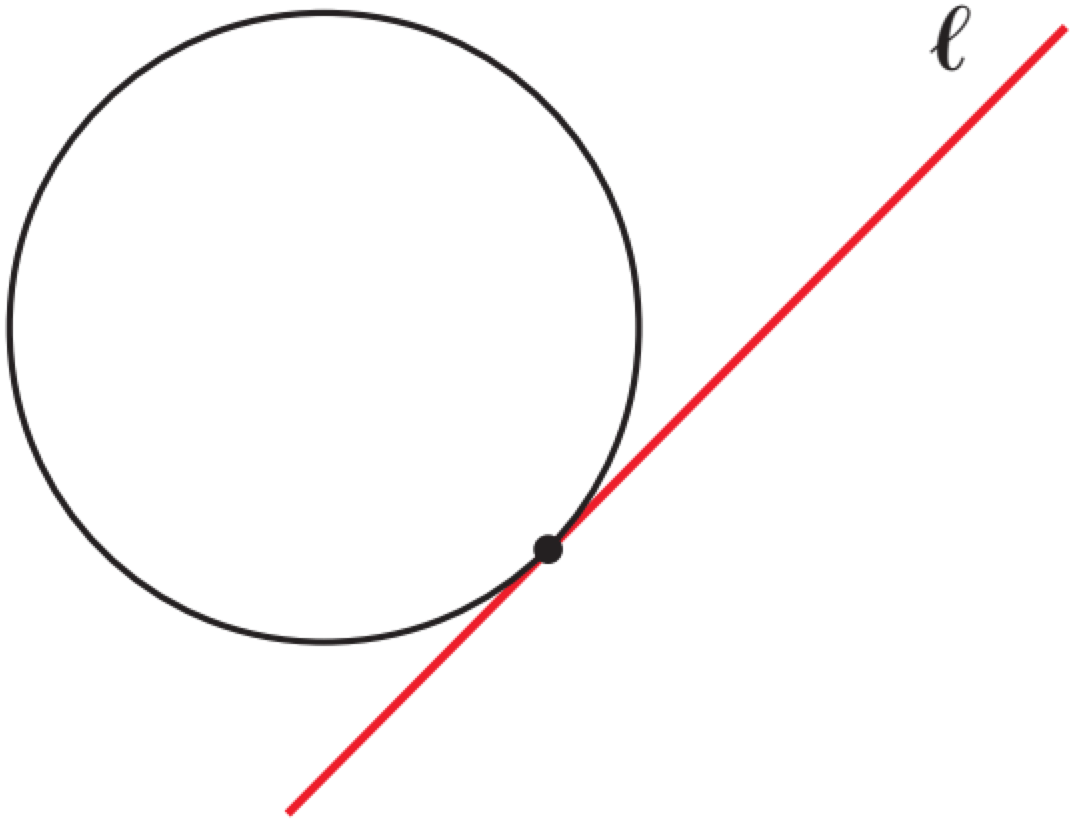
Derivatives

The Tangent Line

The word *tangent* is derived from the Latin word *tangens*, which means "touching." We can think of a tangent to a curve as a line that touches the curve and follows the same direction as the curve at the point of contact.

How to find tangent ?

The tangent at a point on a circle is a line that intersects the circle once and only once.



ℓ

For curves different from the circle this definition is inadequate. Figure 1 shows a line that appears to be a tangent to the curve C at point P, but it intersects C twice.

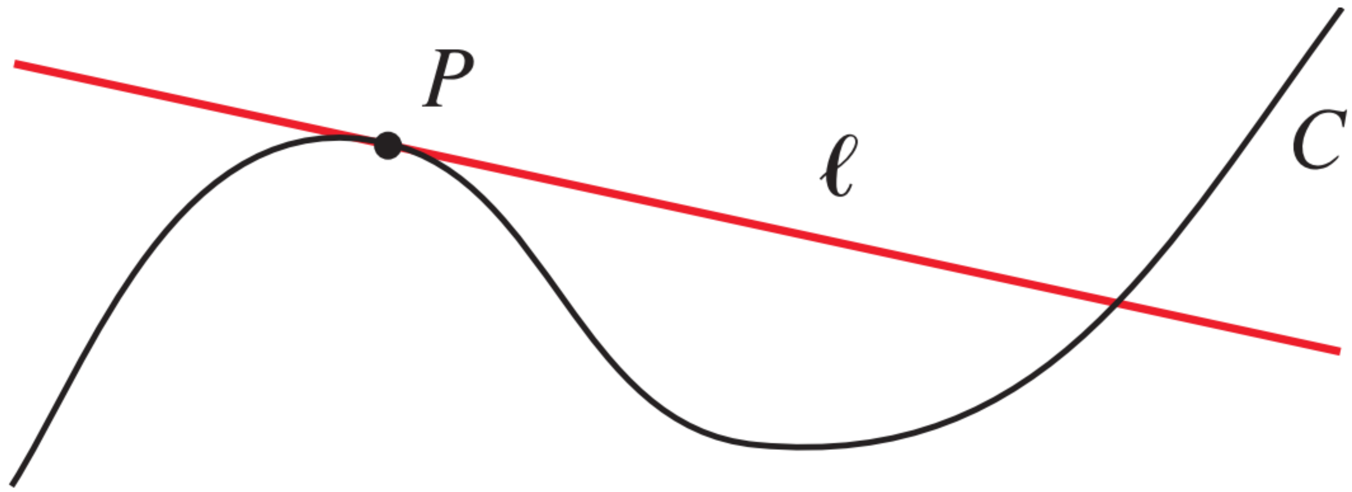


Figure 1

Problem

Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1,1)$.

Solution

We will be able to find an equation of the tangent line as soon as we know its slope m. The difficulty is that we know only one point, P, on l, whereas we need two points to compute the slope. But observe that we can compute an approximation to m by choosing a near by point $Q(x, x^2)$ on the parabola and computing the slope m_{PQ} of the secant line PQ .

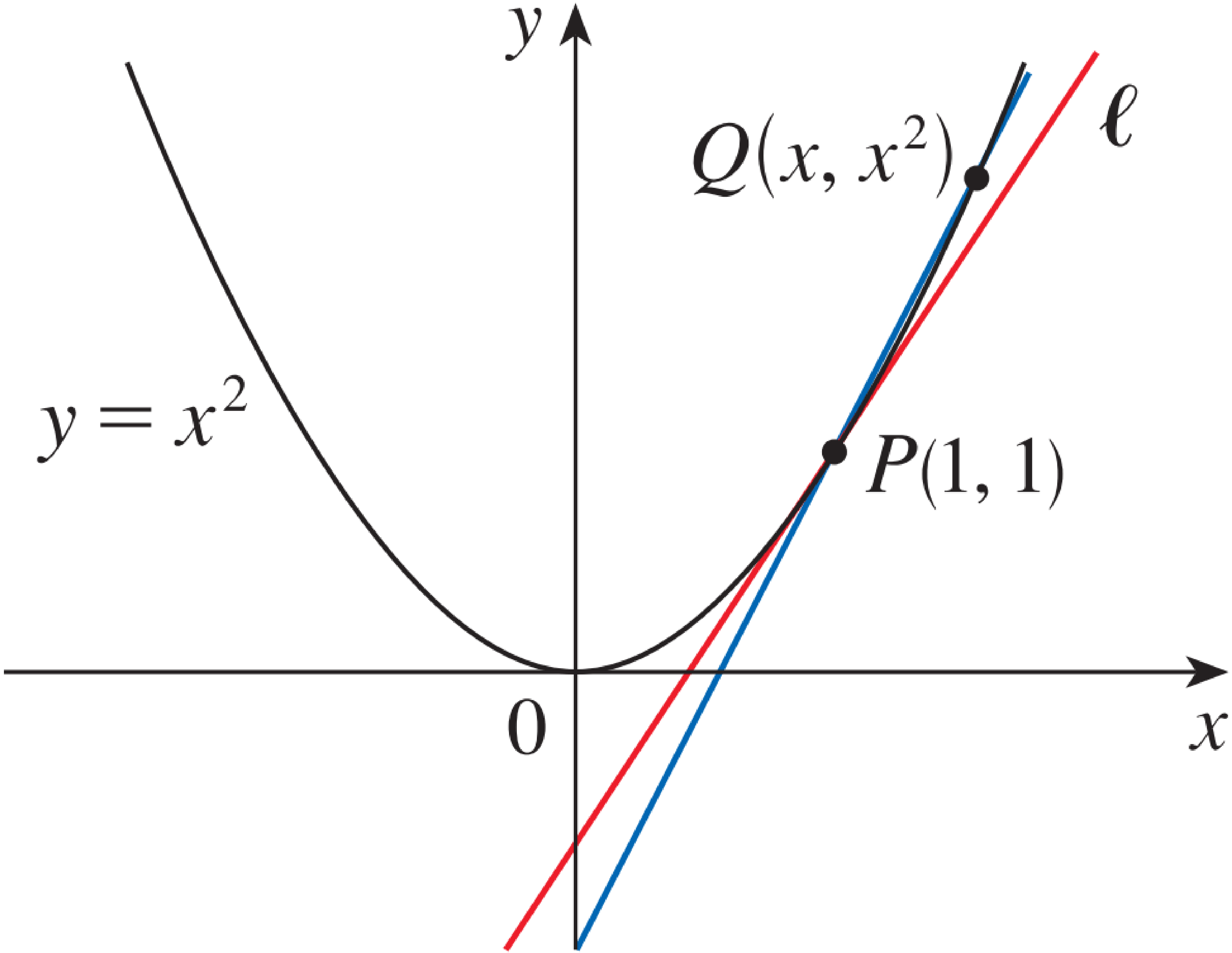
A **secant line**, from the Latin word *secans*, meaning cutting, is a line that cuts [intersects] a curve more than once.)

We choose $x \neq 1$ so that $Q \neq P$. Then

$$m_{PQ} = \frac{x^2 - 1}{x - 1}$$

The slope of the tangent line is the *limit* of the slopes of the secant lines, and we express this symbolically.

$$\lim_{Q \to P} m_{PQ} = m \quad \text{and} \quad \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$$



y
$Q(x, x^2)$
ℓ
$y = x^2$
$P(1, 1)$
0
x

Using point-slope formula, the equation of tangent line at (1,1) is

$$y - 1 = 2(x - 1)$$

or

$$\boxed{y = 2x - 1}$$

Practice Problem

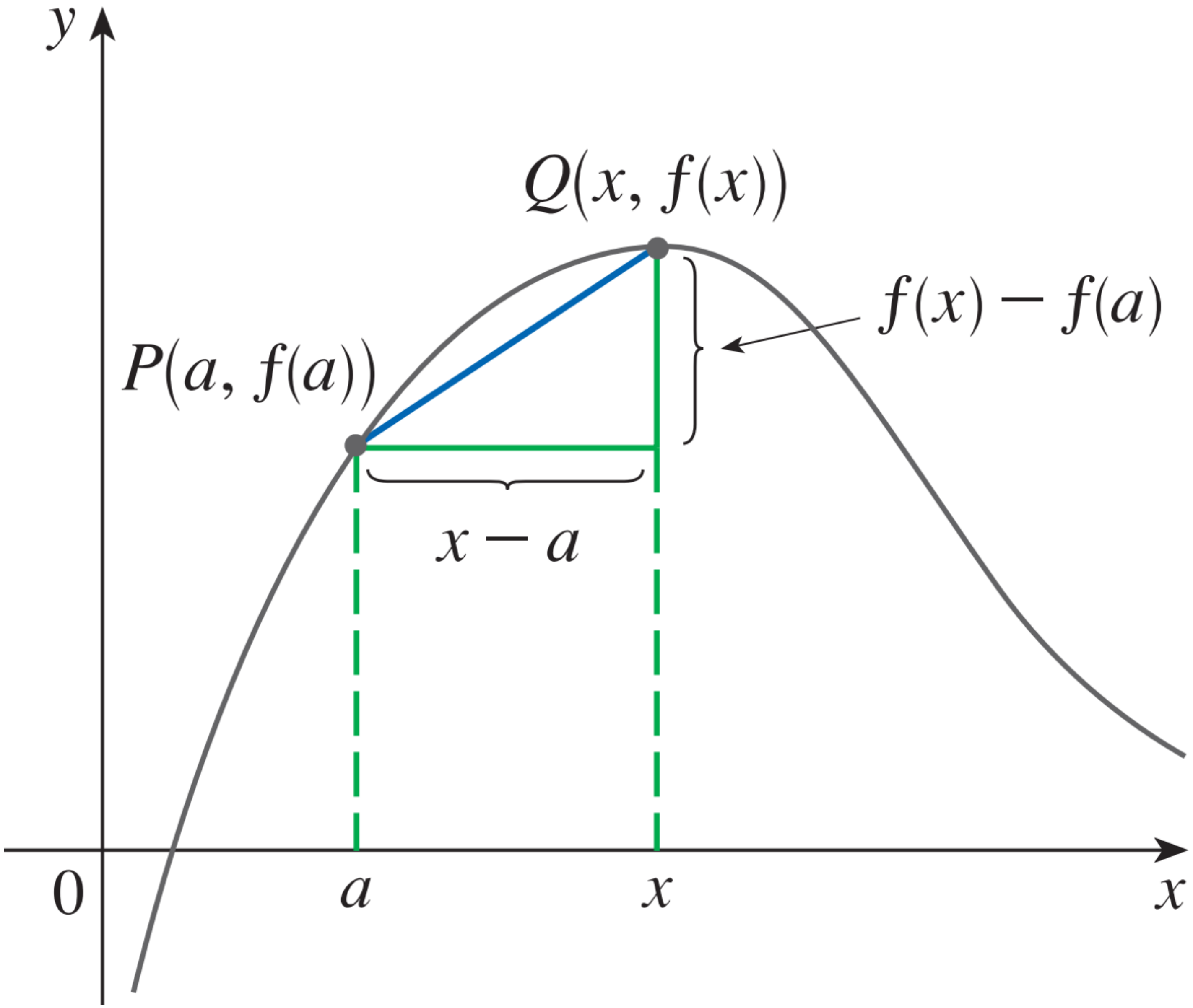
Find an equation of tangent line to the cubic function $y = x^3$ at the point $(2,8)$.

Definition The **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.

We sometimes refer to the slope of the tangent line to a curve at a point as the **slope of the curve** at the point.

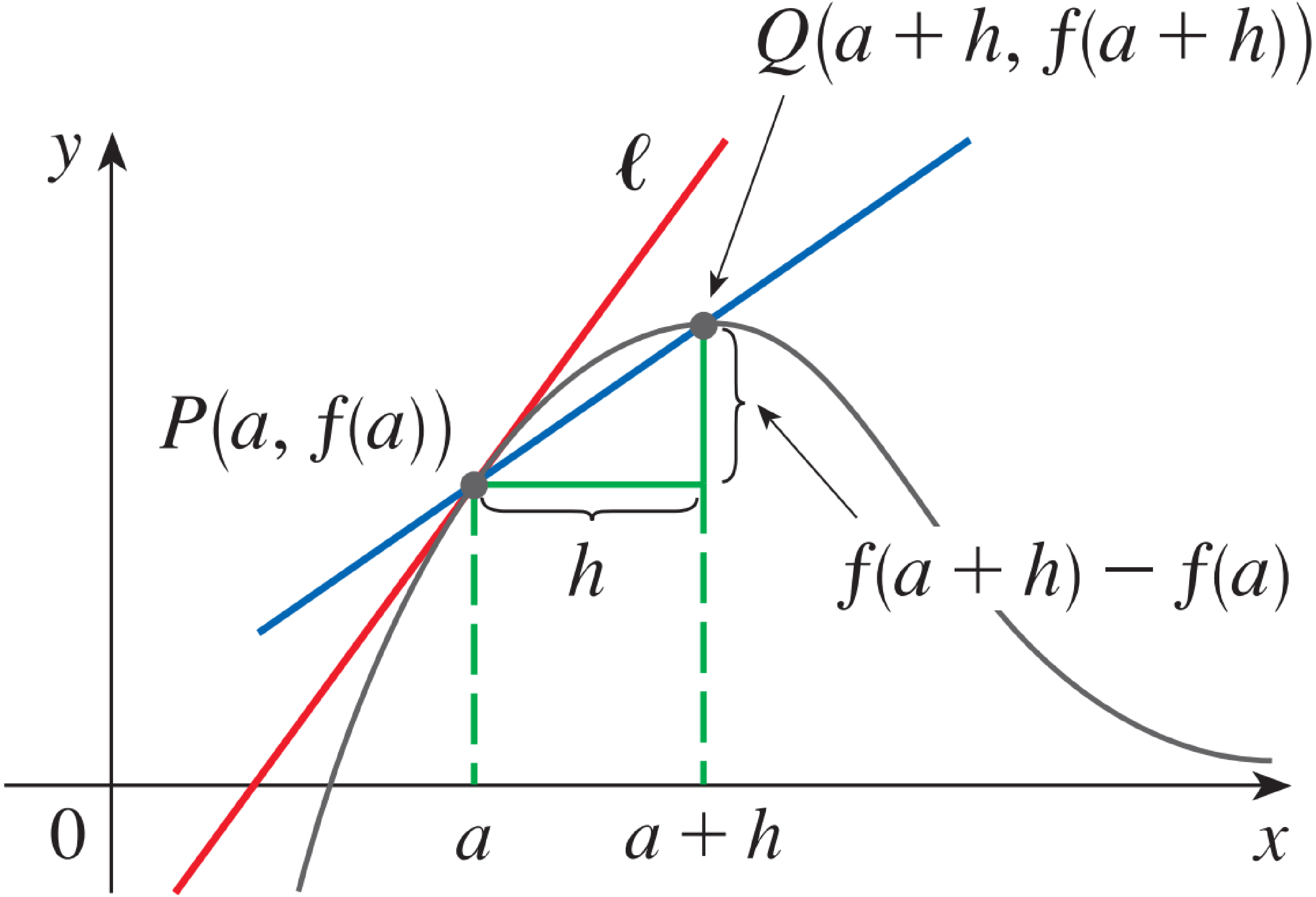

y
$Q(x, f(x))$
$f(x) - f(a)$
$P(a, f(a))$
$x - a$
0
a
x
x

There is another expression for the slope of a tangent line that is sometimes easier to use. If $h = x - a$, then $x = a + h$ and so the slope of the secant line PQ is

$$m_{PQ} = \frac{f(a + h) - f(a)}{h}$$

and

$$\boxed{m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}}$$


$Q(a+h, f(a+h))$
y
ℓ
$P(a, f(a))$
h
$f(a+h) - f(a)$
0
a
$a+h$
x

Problem

Find an equation of the tangent line to the hyperbola $y = 3/x$ at the point $(3, 1)$.

Solution

Let $f(x) = 3/x$.

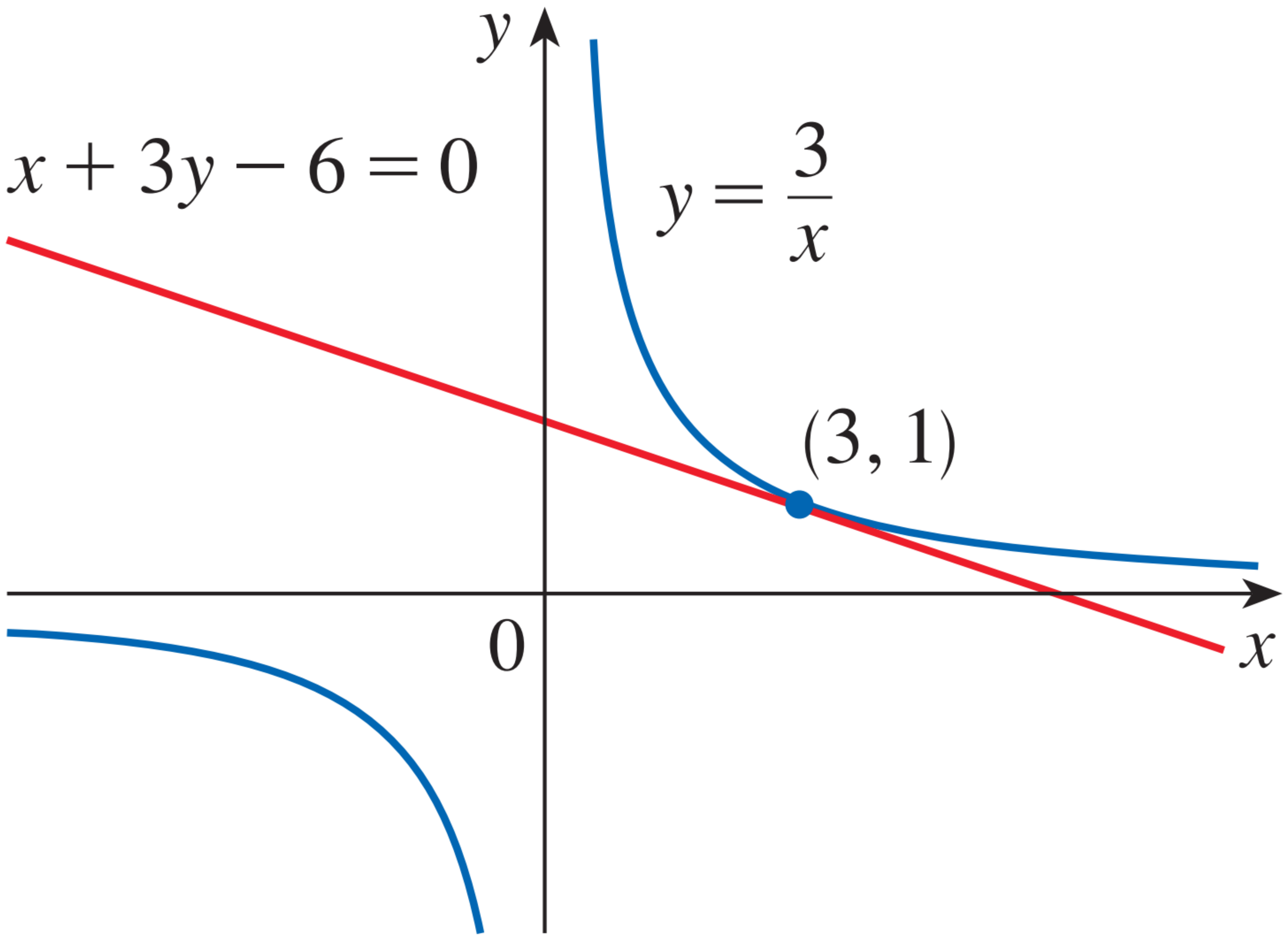
$$m = \lim_{h \to 0} \frac{f(3 + h) - f(3)}{h} = \lim_{h \to 0} \frac{\dfrac{3}{3 + h} - 1}{h} = \lim_{h \to 0} \frac{\dfrac{3 - (3 + h)}{3 + h}}{h}$$

$$m = \lim_{h \to 0} \frac{-h}{h(3 + h)} = \lim_{h \to 0} -\frac{1}{3 + h} = -\frac{1}{3}$$

Therefore an equation of the tangent at the point (3, 1) is

$$y - 1 = -\tfrac{1}{3}(x - 3)$$

$$\boxed{x + 3y - 6 = 0}$$


y
x + 3y − 6 = 0
y = 3/x
(3, 1)
0
x

Definition The **derivative of a function f at a number a**, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

OR

$$\boxed{f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}}$$

The tangent line to $y = f(x)$ at $(a, f(a))$ is the line through $(a, f(a))$ whose slope is equal to $f'(a)$, the derivative of f at a.

Problem

Find the derivative of the function $f(x) = x^2 - 8x + 9$ at the numbers (a) 2 and (b) a.

Solution

(a)

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{(2+h)^2 - 8(2+h) + 9 - (-3)}{h}$$

$$= \lim_{h \to 0} \frac{4 + 4h + h^2 - 16 - 8h + 9 + 3}{h}$$

$$f'(2) = \lim_{h\to 0} \frac{h^2 - 4h}{h} = \lim_{h\to 0} \frac{h(h-4)}{h} = \lim_{h\to 0} (h-4) = \boxed{-4}$$

(b)

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h\to 0} \frac{[(a+h)^2 - 8(a+h) + 9] - [a^2 - 8a + 9]}{h}$$

$$f'(a) = \lim_{h \to 0} \frac{a^2 + 2ah + h^2 - 8a - 8h + 9 - a^2 + 8a - 9}{h}$$

$$= \lim_{h \to 0} \frac{2ah + h^2 - 8h}{h} = \lim_{h \to 0} (2a + h - 8) = \boxed{2a - 8}$$

As a check on our work in part (a), notice that if we let $a = 2$, then $f'(2) = 2(2) - 8 = -4$.

Rate of Change

Suppose y is a quantity that depends on another quantity x. Thus y is a function of x and we write $y = f(x)$. If x changes from x_1 to x_2, then the change in x (also called the **increment** of x) is

$$\Delta x = x_2 - x_1$$

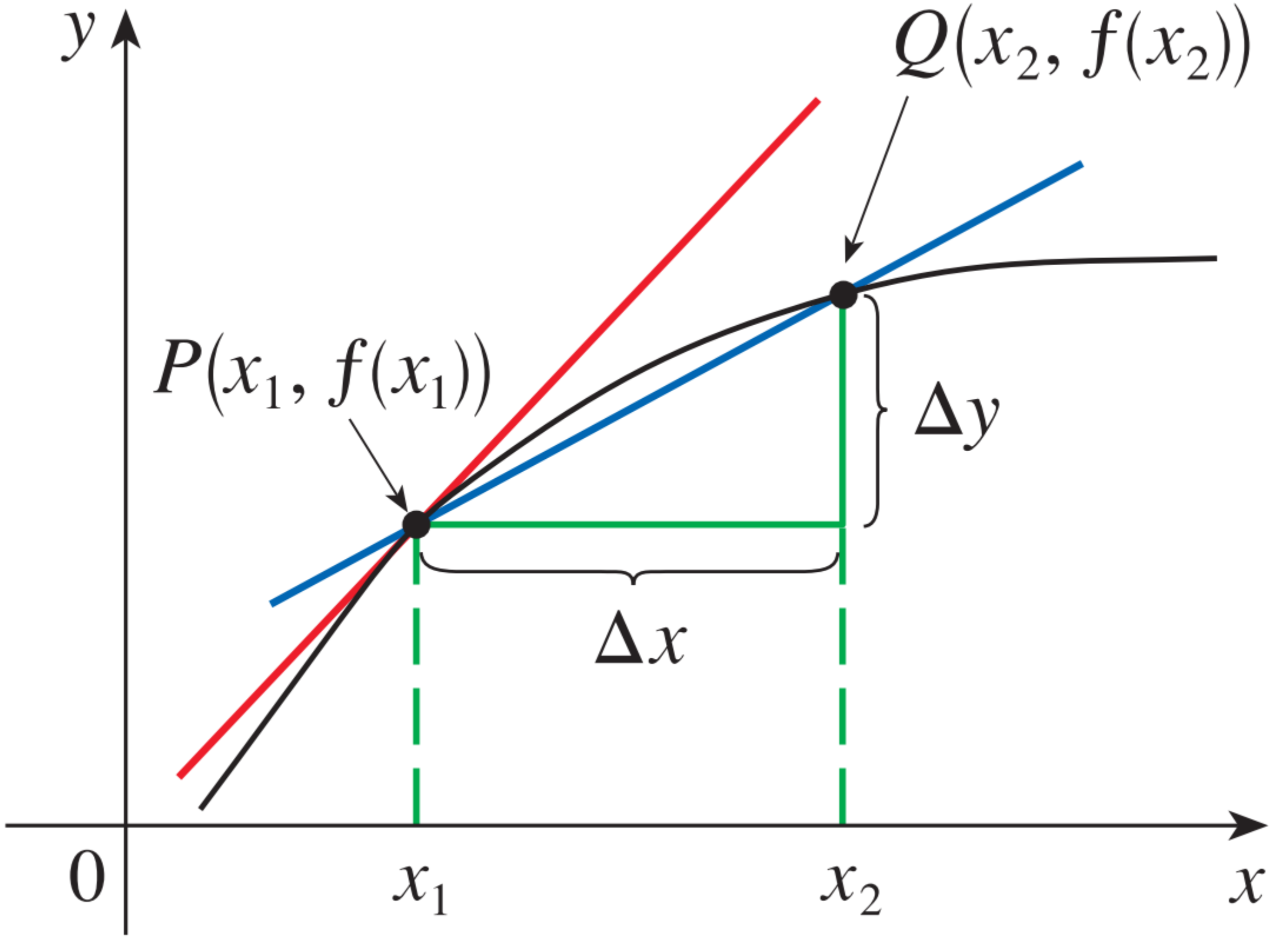
and the corresponding change in y is

$$\Delta y = f(x_2) - f(x_1)$$

The difference quotient

$$\boxed{\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}}$$

is called the **average rate of change of *y* with respect to *x*** over the interval $[x_1, x_2]$ and can be interpreted as the slope of the secant line PQ in the figure in the next slide.



average rate of change $= m_{PQ}$

$$\text{instantaneous rate of change} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

We recognize this limit as being the derivative $f'(x_1)$.

We know that one interpretation of the derivative $f'(a)$ is as the slope of the tangent line to the curve $y = f(x)$ when $x = a$. We now have a second interpretation:

The derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$.

In particular, if $s = f(t)$ is the position function of a particle that moves along a straight line, then $f'(a)$ is the rate of change of the displacement s with respect to the time t. In other words, $f'(a)$ *is the velocity of the particle at time* $t = a$. The **speed** of the particle is the absolute value of the velocity, that is, $|f'(a)|$.

Problem

For $y = f(x) = x^2$, find the average rate of change as:

x changes from 1 to 3.

Solution

a) When $x_1 = 1$,

$$y_1 = f(x_1) = f(1) = 1^2 = 1;$$

and when $x_2 = 3$,

$$y_2 = f(x_2) = f(3) = 3^2 = 9.$$

The average rate of change is

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$= \frac{9 - 1}{3 - 1}$$

$$= \frac{8}{2} = 4.$$

The Mean Value Theorem Let f be a function that satisfies the following hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).

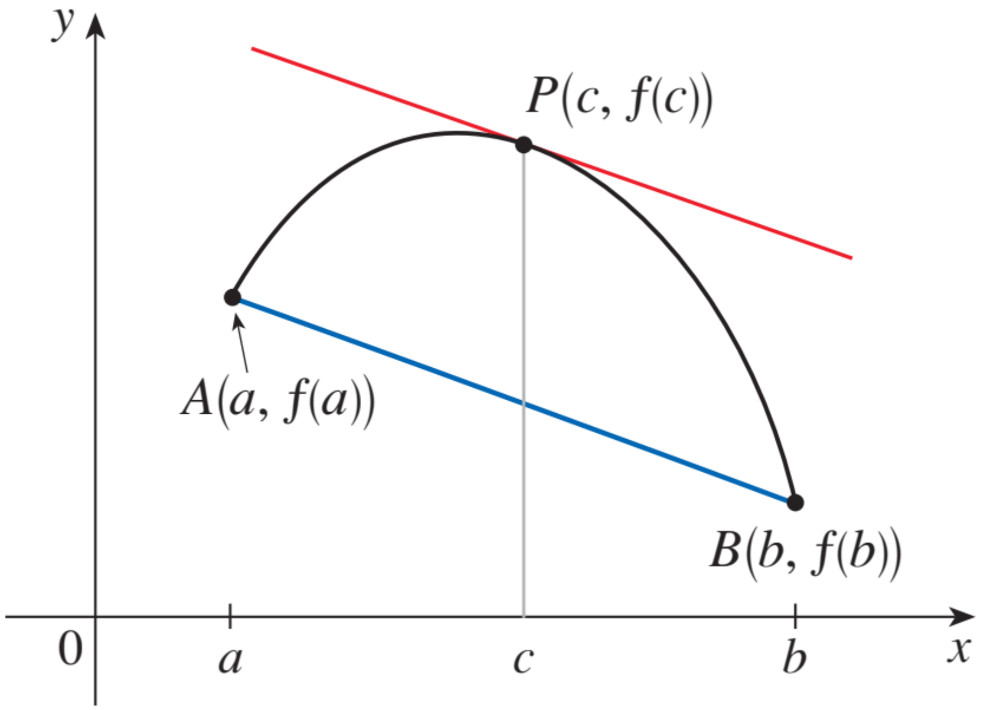
Then there is a number c in (a, b) such that

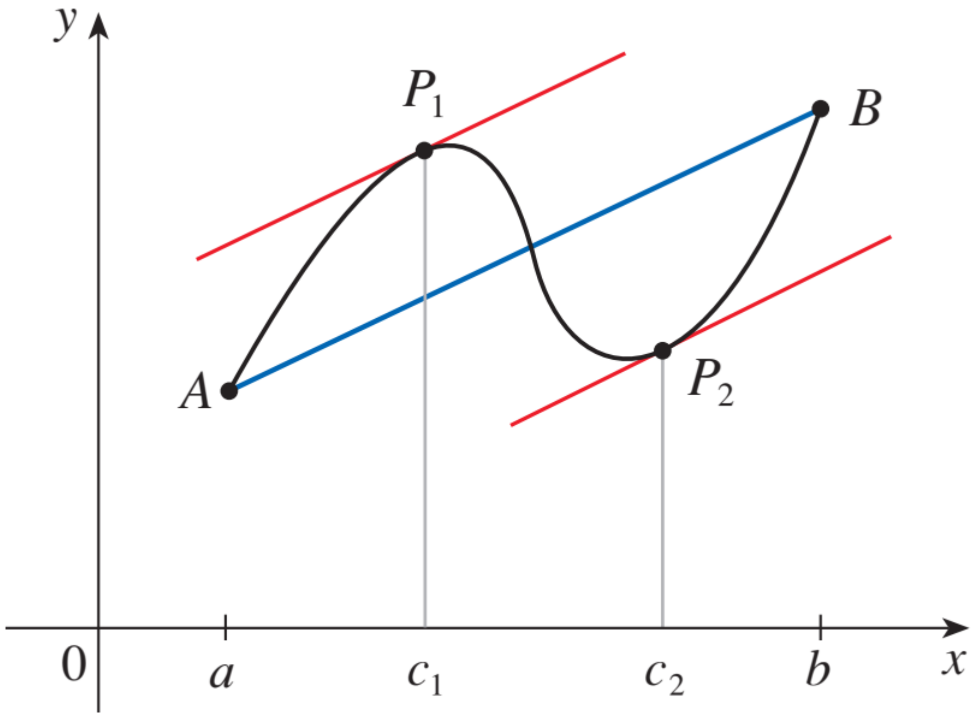
$$\boxed{1} \qquad f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

$$\boxed{2} \qquad f(b) - f(a) = f'(c)(b - a)$$

Mean Value Theorem tell us that if a function f is continuous on the closed interval [a, b] and differentiable on the open interval (a,b) , then there will exist atleast one point c between a and b such that the tangent line at (c,f(c)) is parallel to the secant line through (a, f(a)) and (b, f(b)).





The *Mean Value Theorem* says that the average rate of change for a smooth function is achieved as an “instantaneous rate of change" at some point within the interval.

Speed cameras used by road traffic authorities use this principle: they calculate the average speed of a vehicle over some time interval. If the average is more than the speed limit, then this theorem guarantees that the speedometer of the vehicle (which in principle displays the instantaneous speed from moment to moment) was indicating that at some point of time the driver exceeded the speed limit.

Problem

The distance from Town A to Town B is 37 km. The speed limit is 70 kph (kilometres per hour) throughout, along the road between the two towns.

A person drives from Town A to Town B in the morning, leaving Town A at 10:32 am and arriving in Town B at 11:06 am.

The person returns in the afternoon, leaving Town B at 2:11 pm, arriving back in Town A at 2:40 pm.

The problem is to determine the average speed for each of the journeys, and to make any relevant deductions about the driver exceeding the speed limit.

Solution

The forward journey took 34 minutes, which is $\frac{34}{60}$ hours. The average speed for that leg was

$$\frac{37}{34/60} = \frac{37 \times 60}{34} \approx 65 \text{ kph} ,$$

to the nearest unit, which is safely below the speed limit. However, that does not guarantee the driver did not exceed the speed limit at some moment in the forward journey.

The return journey took 29 minutes, which is $\frac{29}{60}$ hours. The average speed for that leg was

$$\frac{37}{29/60} = \frac{37 \times 60}{29} \approx 77 \text{ kph} ,$$

to the nearest unit, which is well above the speed limit. This appears to guarantee that at some moment during the return journey the driver exceeded the speed limit.

Problem A dynamite blast blows a heavy rock straight up with a launch velocity of 160 ft/sec (about 109 mph) • It reaches a height of $s = 160t - 16t^2$ ft after t sec.

(a) How high does the rock go?

(b) What are the velocity and speed of the rock when it is 256 ft above the ground on the way up? On the way down?

(c) What is the acceleration of the rock at any time t during its flight (after the blast)?

(d) When does the rock hit the ground again?

Solution **(a)** At any time t during the rock's motion, its velocity is

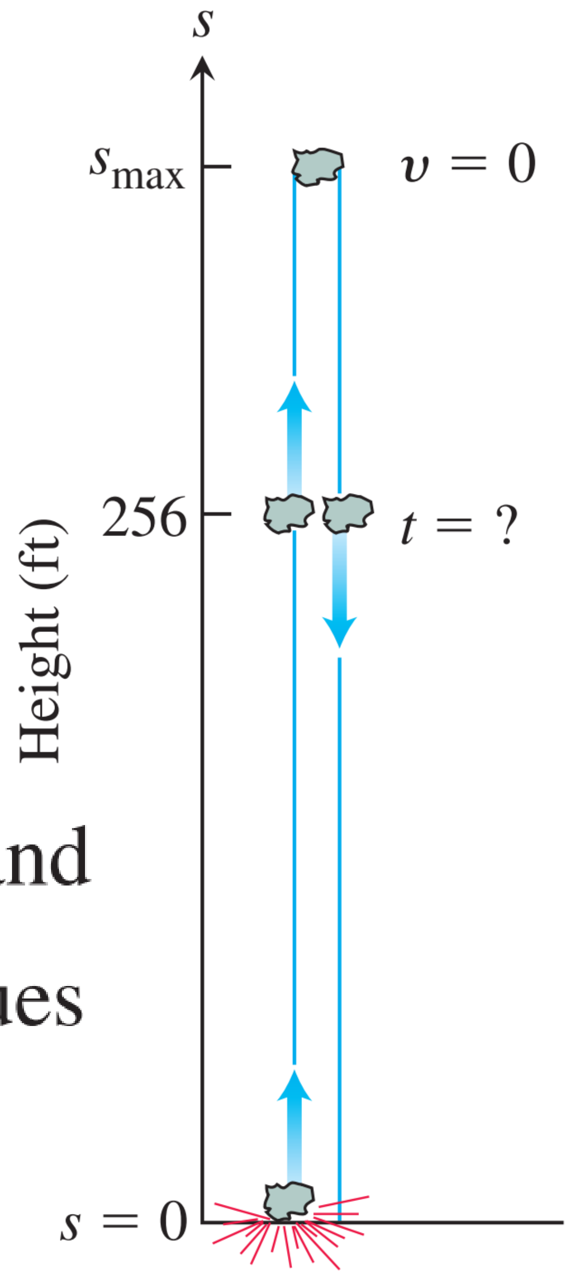
$$v = \frac{ds}{dt} = \frac{d}{dt}(160t - 16t^2) = 160 - 32t \text{ ft/sec}.$$

The velocity is zero when

$$160 - 32t = 0 \qquad \text{or} \qquad t = 5 \text{ sec.}$$

The rock's height at $t = 5$ sec is

$$s_{max} = s(5) = 160(5) - 16(5)^2 = 800 - 400 = \boxed{400 \text{ ft.}}$$



(b) To find the rock's velocity at 256 ft on the way up and again on the way down, we first find the two values of t for which $s(t) = 160t - 16t^2 = 256.$

To solve this equation, we write

$$\begin{aligned} 16t^2 - 160t + 256 &= 0 \\ 16(t^2 - 10t + 16) &= 0 \\ (t - 2)(t - 8) &= 0 \end{aligned}$$

$t = 2\,\text{sec}, t = 8\,\text{sec}.$

The rock is 256 ft above the ground 2 sec after the explosion and again 8 sec after the explosion. The rock's velocities at these times are

$$v(2) = 160 - 32(2) = 160 - 64 = 96\ \text{ft/sec}.$$

$$v(8) = 160 - 32(8) = 160 - 256 = -96\ \text{ft/sec}.$$

At both instants, the rock's speed is 96 ft/sec. Since $v(2) > 0$, the rock is moving upward (s is increasing) at $t = 2$ sec; it is moving downward (s is decreasing) at $t = 8$ because $v(8) < 0$.

(c) At any time during its flight following the explosion, the rock's acceleration is a constant

$$a = \frac{dv}{dt} = \frac{d}{dt}(160 - 32t) = -32 \text{ ft/sec}^2.$$

(d) The rock hits the ground at the positive time t for which $s = 0$. The equation $160t - 16t^2 = 0$ factors to give $16t(10 - t) = 0$, so it has solutions $t = 0$ and $t = 10$. At $t = 0$, the blast occurred and the rock was thrown upward. It returns to the ground 10 sec later.

Practice Problem

The position of a particle moving along a straight line is given by

$$s = f(t) = 2t^3 - 15t^2 + 24t \qquad t \geq 0$$

where t is measured in seconds and s in feet.

a. Find an expression giving the velocity of the particle at any time t. What are the velocity and speed of the particle when $t = 2$?

b. Determine the position of the particle when it is stationary.

c. When is the particle moving in the positive direction? In the negative direction?

Derivative as a function

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

Problem

Show that the function $f(x) = |x|$ is differentiable everywhere except at 0. Also sketch the graph of the derivative function $f'(x)$.

Solution

For $x = 0$ we have to investigate

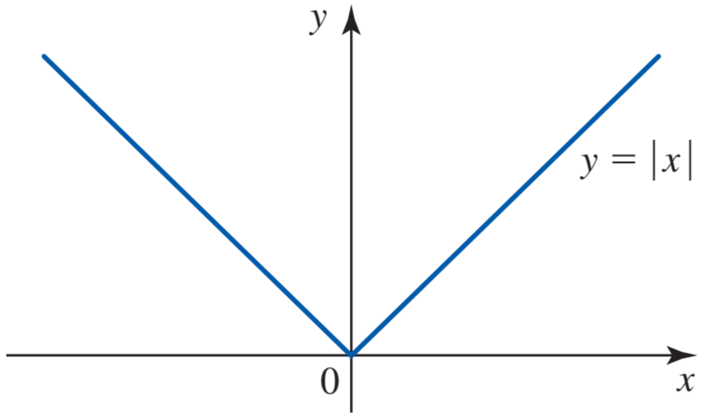
$$f'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{|0 + h| - |0|}{h} = \lim_{h \to 0} \frac{|h|}{h}$$

Let's compute the left and right limits separately:

$$\lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = \lim_{h\to 0^+} 1 = 1$$

$$\lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = \lim_{h\to 0^-} (-1) = -1$$



Since these limits are different, $f'(0)$ does not exist.

To show that f is differentiable at all other numbers, we rewrite $f(x)$ in the form

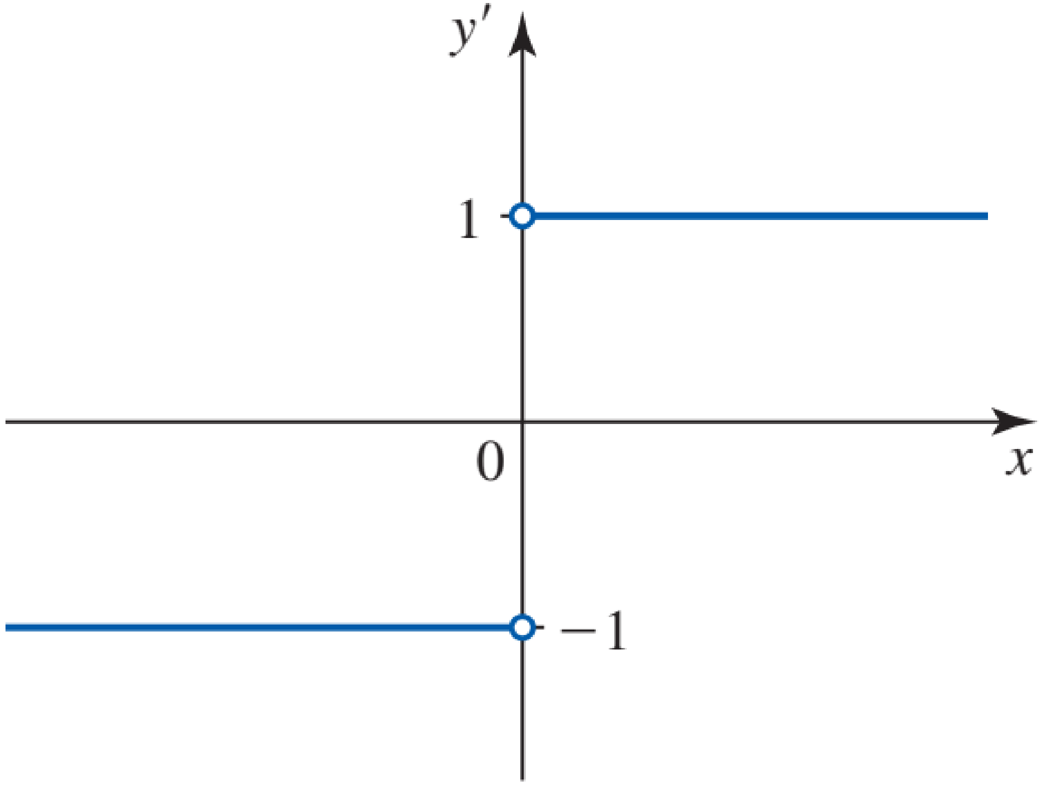
$$f(x) = |x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

and then differentiate $f(x)$ to obtain

$$f'(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$

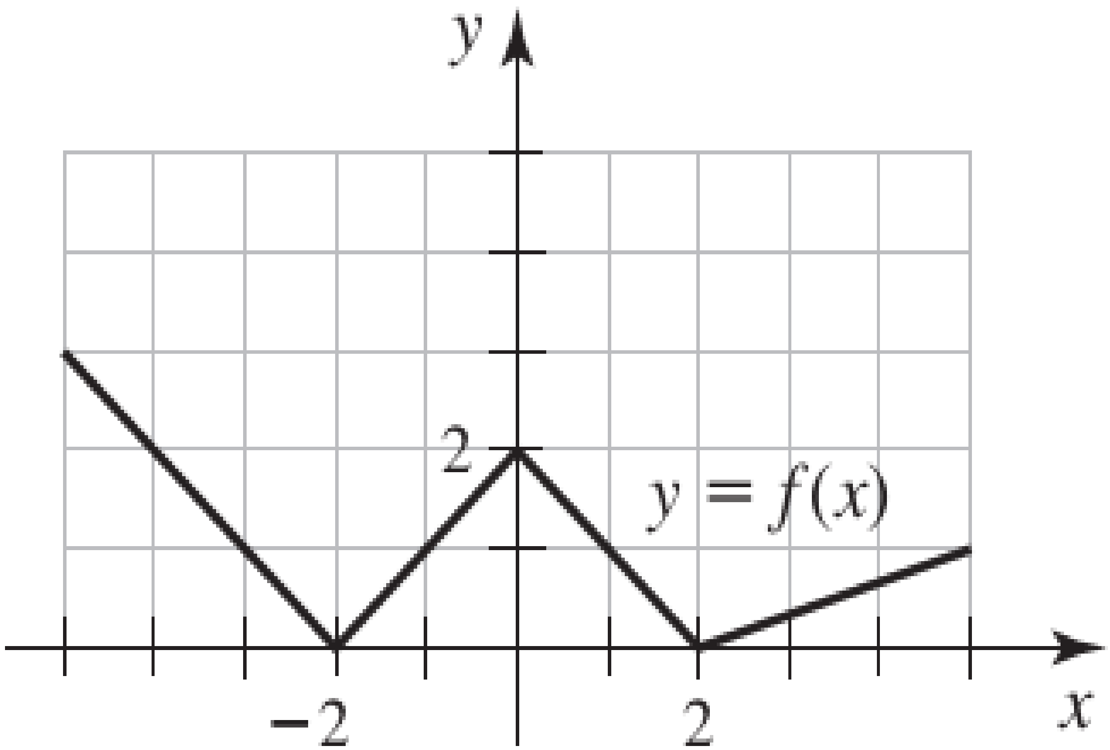
Geometrically, this result is evident if we consider the graph of , which consists of two rays. The slope of the half-line to the left of the origin is -1 , and the slope of the half-line to the right of the origin is 1 .

The graph of f^{-1} is



Practice Problem

Use the graph of f to sketch the graph of f'.



Common Notations of Derivative

If we use the traditional notation $y = f(x)$ to indicate that the independent variable is x and the dependent variable is y, then some common alternative notations for the derivative are as follows:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$$

The symbols D and d/dx are called **differentiation operators** because they indicate the operation of **differentiation**, which is the process of calculating a derivative.

The symbol dy/dx, was introduced by Leibniz.

If we want to indicate the value of a derivative dy/dx in Leibniz notation at a number a, we use the notation

$$\left.\frac{dy}{dx}\right|_{x=a} \qquad \text{or} \qquad \left.\frac{dy}{dx}\right]_{x=a}$$

which is a synonym for $f'(a)$. The vertical bar means "evaluate at."

Theorem If f is differentiable at a, then f is continuous at a.

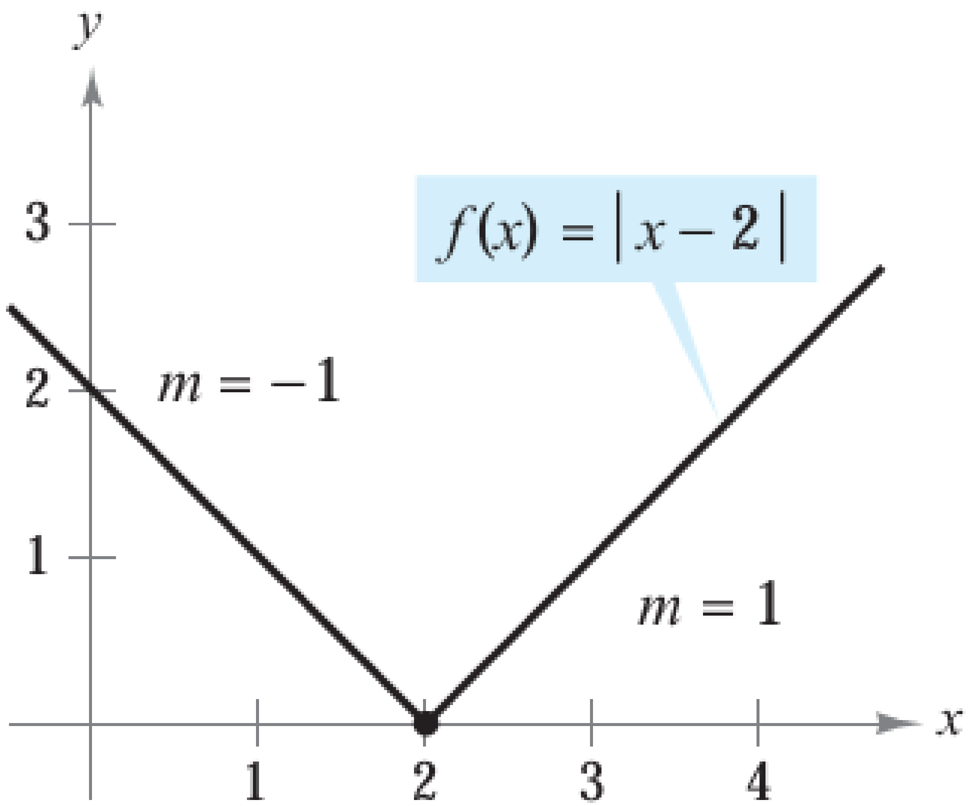
The converse of above theorem is false; that is, there are functions that are continuous but not differentiable. e.g. the function $f(x) = |x|$ is continuous at $x = 0$ but not differentiable at $x = 0$.

Where can a function fail to be differentiable?

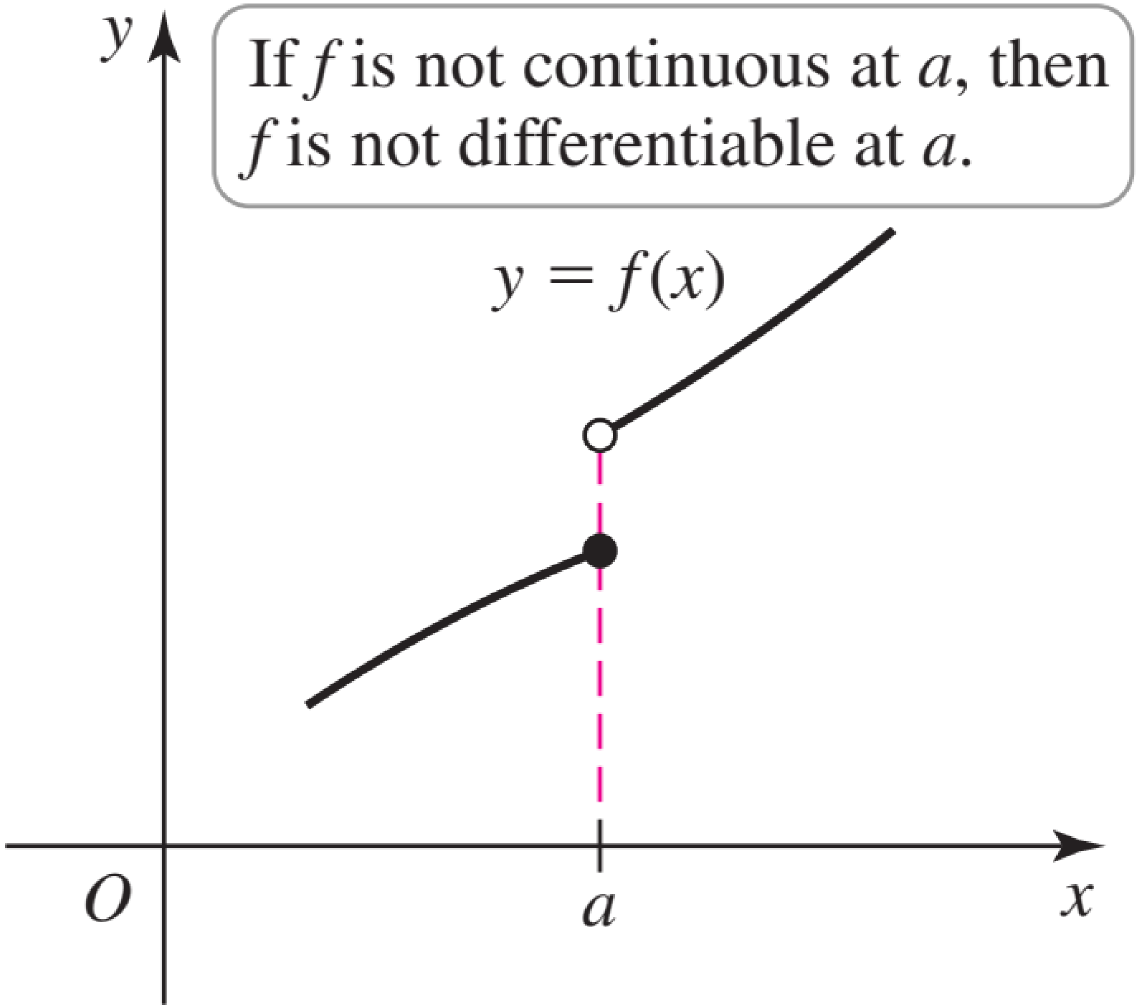
1. In general, if the graph of a function f has a "corner" or "kink" in it, then the graph of f has no tangent at this point and f is not differentiable there. [In trying to compute $f'(a)$, we find that the left and right limits are different.]

2. If f is not continuous at a, then f is not differentiable at a. So at any discontinuity (for instance, a jump discontinuity) f fails to be differentiable.

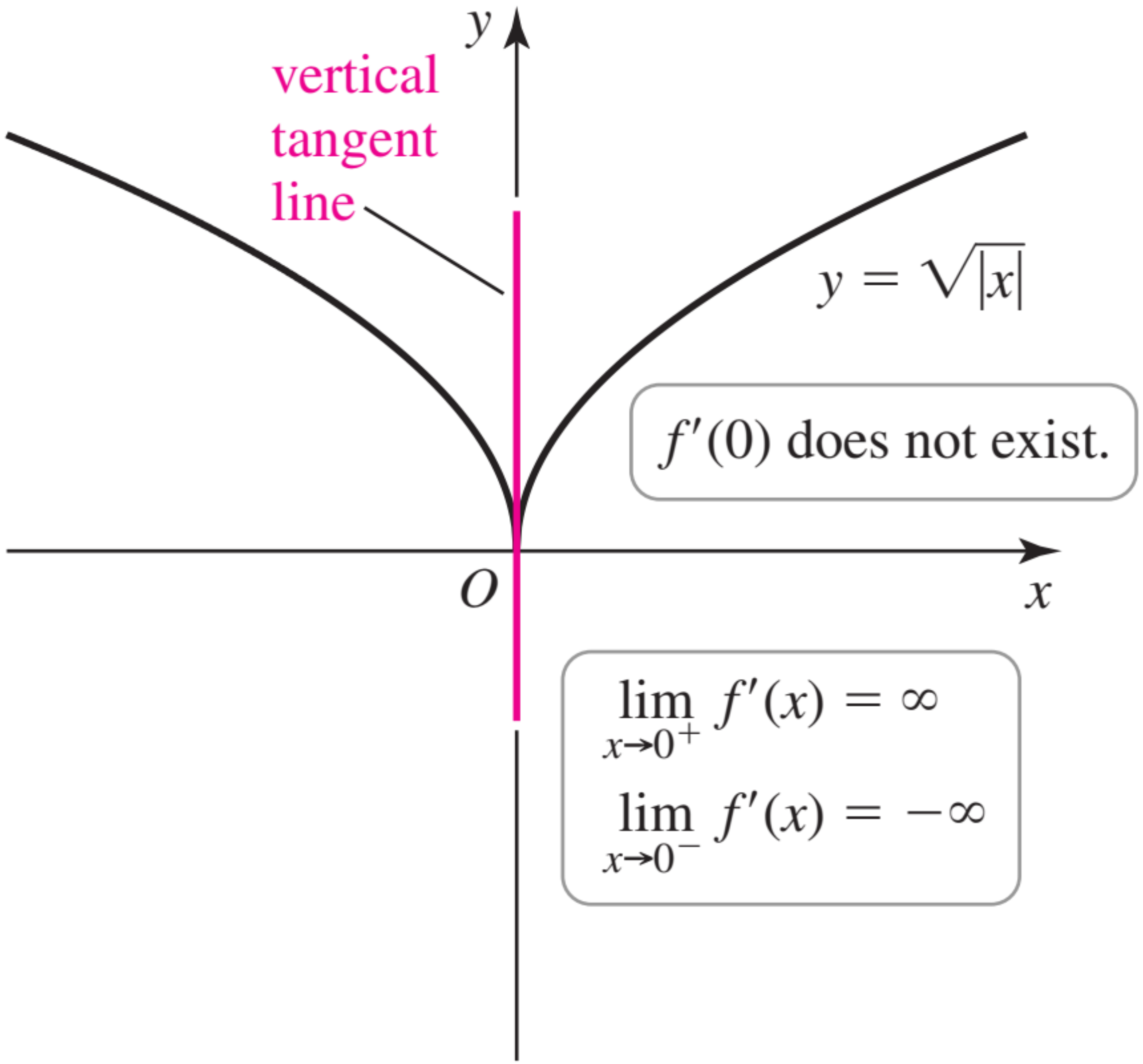
3. A third possibility is that the curve has a **vertical tangent line** when $x = a$; that is, f is continuous at a and $\lim_{x \to a} |f'(x)| = \infty$.



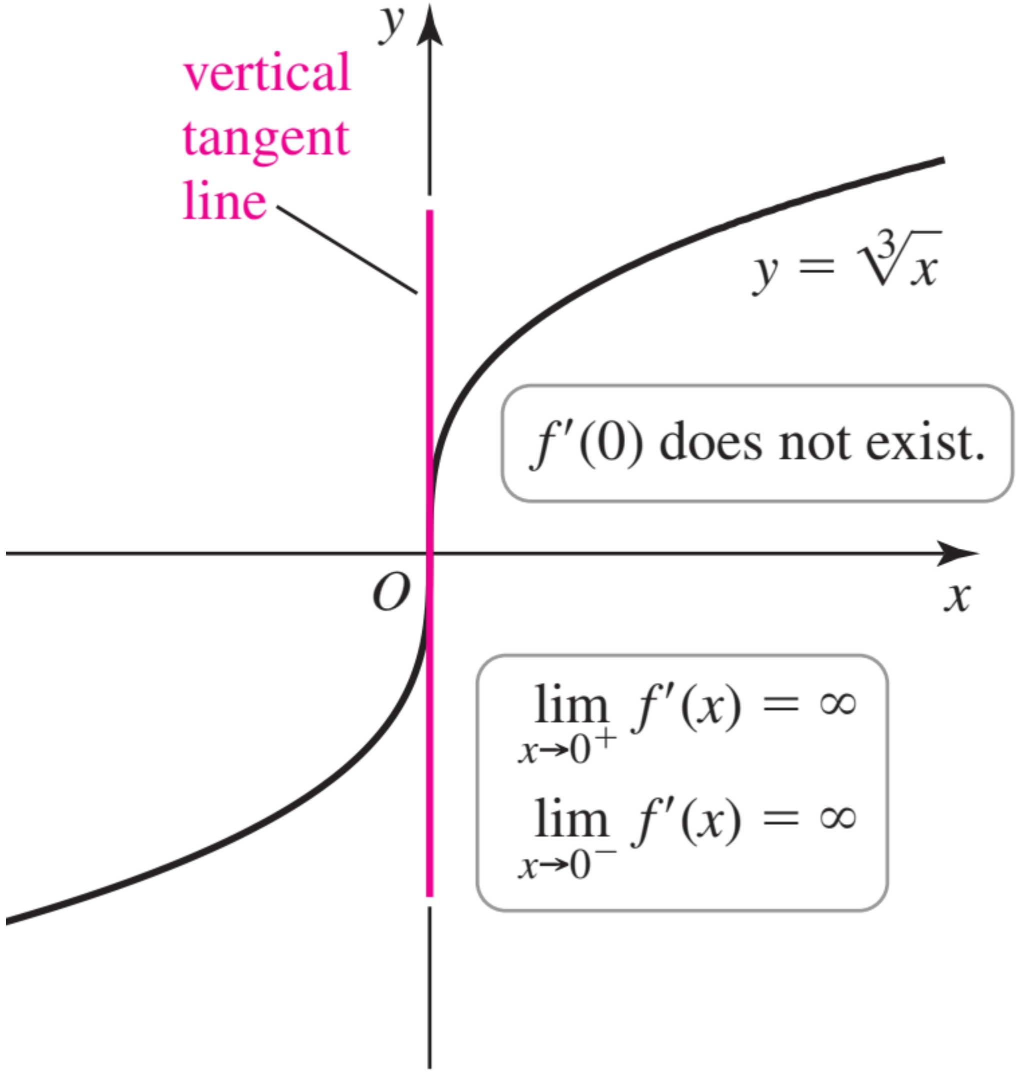
f is not differentiable at $x = 2$, because the derivatives from the left and from the right are not equal.



We know from a theorem that, differentiability implies continuity. Now from its contrapositive we can say that if a function is not continuous then it must not be differentiable.



vertical tangent line
$y = \sqrt{|x|}$
$f'(0)$ does not exist.
$\lim_{x \to 0^+} f'(x) = \infty$
$\lim_{x \to 0^-} f'(x) = -\infty$
O
x
y



vertical tangent line
$y = \sqrt[3]{x}$
$f'(0)$ does not exist.
$\lim_{x \to 0^+} f'(x) = \infty$
$\lim_{x \to 0^-} f'(x) = \infty$
O
x
y

Higher Derivatives

If f is a differentiable function, then its derivative f' is also a function, so f' may have a derivative of its own, denoted by $(f')' = f''$. This new function f'' is called the **second derivative** of f because it is the derivative of the derivative of f. Using Leibniz notation, we write the second derivative of $y = f(x)$ as

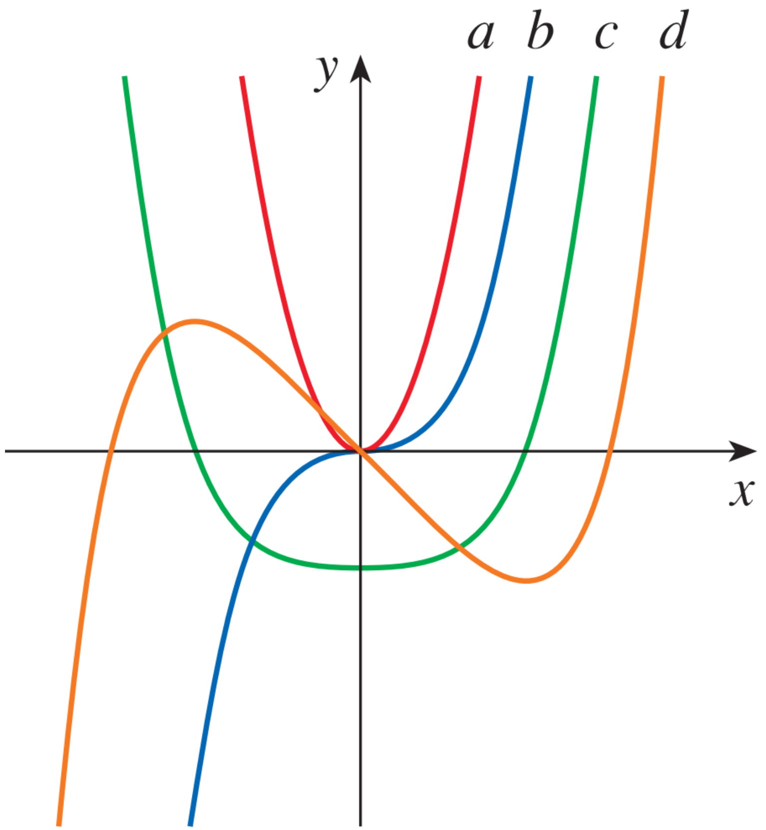
$$\underbrace{\frac{d}{dx}}_{\text{derivative of}} \underbrace{\left(\frac{dy}{dx}\right)}_{\text{first derivative}} = \underbrace{\frac{d^2y}{dx^2}}_{\text{second derivative}}$$

The differentiation process can be continued. The fourth derivative f'''' is usually denoted by $f^{(4)}$. In general, the nth derivative of f is denoted by $f^{(n)}$ and is obtained from f by differentiating n times. If $y = f(x)$, we write

$$y^{(n)} = f^{(n)}(x) = \frac{d^n y}{dx^n}$$

Practice Problem

The figure shows graphs of f, f', f'', and f'''. Identify each curve, and explain your choices.



Differentiation Rules

Derivative of a Constant Function

$$\frac{d}{dx}(c) = 0$$

The Power Rule (General Version) If n is any real number, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

The Constant Multiple Rule If c is a constant and f is a differentiable function, then

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$$

The Sum and Difference Rules If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

The Product Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

The Quotient Rule If f and g are differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\dfrac{d}{dx}[f(x)] - f(x)\dfrac{d}{dx}[g(x)]}{[g(x)]^2}$$

Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

When you memorize this table, it is helpful to notice that the minus signs go with the derivatives of the "cofunctions," that is, cosine, cosecant, and cotangent.

The Chain Rule

The Chain Rule If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x) \tag{1}$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \tag{2}$$

Problem Find $F'(x)$ if $F(x) = \sqrt{x^2 + 1}$.

Solution 1 (Using Formula 1)

$F(x) = (f \circ g)(x) = f(g(x))$ where $f(u) = \sqrt{u}$ and $g(x) = x^2 + 1$. Since

$$f'(u) = \tfrac{1}{2}u^{-1/2} = \frac{1}{2\sqrt{u}} \qquad \text{and} \qquad g'(x) = 2x$$

we have

$$F'(x) = f'(g(x)) \cdot g'(x) = \frac{1}{2\sqrt{x^2 + 1}} \cdot 2x$$

$$= \boxed{\frac{x}{\sqrt{x^2 + 1}}}$$

Solution 2 (Using Formula 2)

If we let $u = x^2 + 1$ and $y = \sqrt{u}$, then

$$F'(x) = \frac{dy}{du}\frac{du}{dx} = \frac{1}{2\sqrt{u}}(2x) = \frac{1}{2\sqrt{x^2 + 1}}(2x) = \boxed{\frac{x}{\sqrt{x^2 + 1}}}$$

NOTE In using the Chain Rule we work from the outside to the inside. Formula 1 says that *we differentiate the outer function* f [*at the inner function* $g(x)$] *and then we multiply by the derivative of the inner function.*

$$\frac{d}{dx}\underbrace{f}_{\substack{\text{outer}\\\text{function}}}\underbrace{(g(x))}_{\substack{\text{evaluated}\\\text{at inner}\\\text{function}}} = \underbrace{f'}_{\substack{\text{derivative}\\\text{of outer}\\\text{function}}}\underbrace{(g(x))}_{\substack{\text{evaluated}\\\text{at inner}\\\text{function}}} \cdot \underbrace{g'(x)}_{\substack{\text{derivative}\\\text{of inner}\\\text{function}}}$$

Problem

Differentiate $y = \sin(x^2)$

Solution

$$\frac{dy}{dx} = \frac{d}{dx}\underbrace{\sin}_{\substack{\text{outer}\\ \text{function}}}\quad\underbrace{(x^2)}_{\substack{\text{evaluated}\\ \text{at inner}\\ \text{function}}} = \underbrace{\cos}_{\substack{\text{derivative}\\ \text{of outer}\\ \text{function}}}\quad\underbrace{(x^2)}_{\substack{\text{evaluated}\\ \text{at inner}\\ \text{function}}} \cdot \underbrace{2x}_{\substack{\text{derivative}\\ \text{of inner}\\ \text{function}}}$$

$$= \boxed{2x\cos(x^2)}$$

Extended Power Rule

The Power Rule Combined with the Chain Rule If n is any real number and $u = g(x)$ is differentiable, then

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

Alternatively, $$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} \cdot g'(x)$$

Problem Find the derivative of the function

$$g(t) = \left(\frac{t-2}{2t+1}\right)^9$$

Solution

Combining the Power Rule, Chain Rule, and Quotient Rule, we get

$$g'(t) = 9\left(\frac{t-2}{2t+1}\right)^8 \frac{d}{dt}\left(\frac{t-2}{2t+1}\right)$$

$$= 9\left(\frac{t-2}{2t+1}\right)^8 \frac{(2t+1)\cdot 1 - 2(t-2)}{(2t+1)^2} = \frac{45(t-2)^8}{(2t+1)^{10}}$$

The reason for the name "Chain Rule" becomes clear when we make a longer chain by adding another link. Suppose that $y = f(u)$, $u = g(x)$, and $x = h(t)$ where f, g, and h are differentiable functions. Then, to compute the derivative of y with respect to t, we use

$$\frac{dy}{dt} = \frac{dy}{du}\frac{du}{dx}\frac{dx}{dt}$$

Problem

Compute the derivative of $f(x) = \sin(\cos(tanx))$.

Solution

$$f'(x) = \cos(\cos(\tan x))\frac{d}{dx}\cos(\tan x)$$

$$= \cos(\cos(\tan x))[-\sin(\tan x)]\frac{d}{dx}(\tan x)$$

$$= -\cos(\cos(\tan x))\sin(\tan x)\sec^2 x$$

Derivative of General Exponential Function

$$\boxed{\frac{d}{dx}(b^x) = b^x \ln b}$$

Problem

Find the derivative of each of the functions.

(a) $g(x) = 2^x$ (b) $h(x) = 5^{x^2}$

Solution

$$g'(x) = \frac{d}{dx}(2^x) = 2^x \ln 2$$

(b)

The outer function is an exponential function and the inner function is the squaring function, thus

$$h'(x) = \frac{d}{dx}\left(5^{x^2}\right) = 5^{x^2} \ln 5 \cdot \frac{d}{dx}(x^2) = 2x \cdot 5^{x^2} \ln 5$$

Practice Problem 1

Find the derivative of the function.

(a) $f(x) = e^{\sin^2(x^2)}$

(b) $y = (3^{\cos(x^2)} - 1)^4$

(c) $y = \sin(\theta + \tan(\theta + \cos\theta))$

(d) $y = \cos\sqrt{\sin(\tan \pi x)}$

(e) $y = 2^{3^{4^x}}$

(f) $y = \sin^3(\cos(x^2))$

Practice Problem 2

Suppose $g(1)=2$, $g'(1)=3$, $g''(1)=1$, $f'(2)=4$, and $f''(2)=3$. Find $\left.\dfrac{d^2}{dx^2}f(g(x))\right|_{x=1}$.

Implicit Differentiation

The functions that we have met so far can be described by expressing one variable explicitly in terms of another variable—for example,

$$y = \sqrt{x^3 + 1} \qquad \text{or} \qquad y = x \sin x$$

or, in general, $y = f(x)$. Some functions, however, are defined implicitly by a relation

between x and y such as

$$\boxed{1} \qquad x^2 + y^2 = 25$$

or

$$\boxed{2} \qquad x^3 + y^3 = 6xy$$

When we say that f is a function defined implicitly by Equation 2, we mean that the equation

$$x^3 + [f(x)]^3 = 6xf(x)$$

is true for all values of x in the domain of f.

Fortunately, we don't need to solve an equation for y in terms of x in order to find the derivative of y. Instead we can use the method of **implicit differentiation.** This consists of differentiating both sides of the equation with respect to x and then solving the resulting equation for $\frac{dy}{dx}$.

Problem

If $x^2 + y^2 = 25$, find $\dfrac{dy}{dx}$. Then find an equation of the tangent to the circle $x^2 + y^2 = 25$ at the point (3, 4).

Solution

Differentiate both sides of the equation $x^2 + y^2 = 25$:

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25) \longrightarrow \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$$

Remembering that y is a function of x and using the Chain Rule, we have

$$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2)\frac{dy}{dx} = 2y\frac{dy}{dx}$$

So We have

$$2x + 2y\frac{dy}{dx} = 0$$

Now we solve this equation for dy/dx:

$$\frac{dy}{dx} = -\frac{x}{y}$$

At the point (3, 4) we have $x = 3$ and $y = 4$, so

$$\frac{dy}{dx} = -\frac{3}{4}$$

An equation of the tangent to the circle at (3, 4) is therefore

$$y - 4 = -\tfrac{3}{4}(x - 3) \qquad \text{or} \qquad 3x + 4y = 25$$

Problem

Find y' if $\sin(x + y) = y^2 \cos x$.

Solution

Differentiating implicitly with respect to x and remembering that y is a function of x, we get

$$\cos(x + y) \cdot (1 + y') = y^2(-\sin x) + (\cos x)(2yy')$$

(Note that we have used the Chain Rule on the left side and the Product Rule and Chain Rule on the right side.) If we collect the terms that involve y', we get

$$\cos(x + y) + y^2 \sin x = (2y \cos x)y' - \cos(x + y) \cdot y'$$

$$y' = \frac{y^2 \sin x + \cos(x + y)}{2y \cos x - \cos(x + y)}$$

Practice Problems

1. If $g(x) + x \sin g(x) = x^2$, find $g'(0)$.

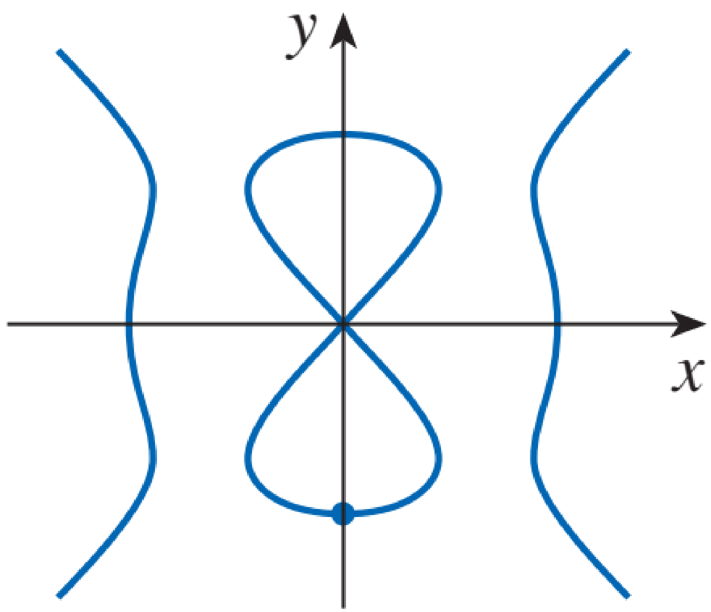
2. **Determine** an equation of tangent to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

at the point (x_0, y_0).

3. Use implicit differentiation to find an equation of the tangent line to the curve at the given point.

$$y^2(y^2 - 4) = x^2(x^2 - 5), \quad (0, -2) \quad \text{(devil's curve)}$$



Derivative of Logarithmic Function

$$\boxed{\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}}$$

If we put $b = e$ then we get

$$\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}}$$

Problem

Differentiate $y = \ln(x^3 + 1)$.

Solution

To use the Chain Rule, we let $u = x^3 + 1$. Then $y = \ln u$, so

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{u}\frac{du}{dx} = \frac{1}{x^3 + 1}(3x^2) = \frac{3x^2}{x^3 + 1}$$

$$\boxed{\frac{d}{dx}[\ln g(x)] = \frac{g'(x)}{g(x)}}$$

Problem

Differentiate $f(x) = \log_{10}(2 + \sin x)$.

Solution

$$f'(x) = \frac{d}{dx}\log_{10}(2 + \sin x)$$

$$= \frac{1}{(2 + \sin x)\ln 10}\frac{d}{dx}(2 + \sin x)$$

$$= \frac{\cos x}{(2 + \sin x)\ln 10}$$

Problem Find $f'(x)$ if $f(x) = \ln|x|$.

Solution Since

$$f(x) = \begin{cases} \ln x & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases}$$

it follows that

$$f'(x) = \begin{cases} \dfrac{1}{x} & \text{if } x > 0 \\ \dfrac{1}{-x}(-1) = \dfrac{1}{x} & \text{if } x < 0 \end{cases}$$

Thus $f'(x) = 1/x$ for all $x \neq 0$.

Logarithmic Differentiation

The calculation of derivatives of complicated functions involving products, quotients, or powers can often be simplified by taking logarithms. The method used in the following example is called **logarithmic differentiation.**

Say we want to differentiate the following function.

$$y = \frac{x^{3/4}\sqrt{x^2 + 1}}{(3x + 2)^5}.$$

We take logarithms of both sides of the equation and use the Laws of Logarithms to simplify:

$$\ln y = \tfrac{3}{4} \ln x + \tfrac{1}{2} \ln(x^2 + 1) - 5 \ln(3x + 2)$$

Differentiating implicitly with respect to x gives

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{4} \cdot \frac{1}{x} + \frac{1}{2} \cdot \frac{2x}{x^2 + 1} - 5 \cdot \frac{3}{3x + 2}$$

Solving for dy/dx, we get

$$\frac{dy}{dx} = y\left(\frac{3}{4x} + \frac{x}{x^2 + 1} - \frac{15}{3x + 2}\right)$$

Because we have an explicit expression for y, we can substitute and write

$$\boxed{\frac{dy}{dx} = \frac{x^{3/4}\sqrt{x^2 + 1}}{(3x + 2)^5}\left(\frac{3}{4x} + \frac{x}{x^2 + 1} - \frac{15}{3x + 2}\right)}$$

Steps in Logarithmic Differentiation

1. Take natural logarithms of both sides of an equation $y = f(x)$ and use the Laws of Logarithms to expand the expression.
2. Differentiate implicitly with respect to x.
3. Solve the resulting equation for y' and replace y by $f(x)$.

In general there are four cases for exponents and bases:

Constant base, constant exponent

1. $\dfrac{d}{dx}(b^n) = 0 \qquad (b \text{ and } n \text{ are constants})$

Variable base, constant exponent

2. $\dfrac{d}{dx}[f(x)]^n = n[f(x)]^{n-1}f'(x)$

Constant base, variable exponent

3. $\dfrac{d}{dx}[b^{g(x)}] = b^{g(x)}(\ln b)g'(x)$

Variable base, variable exponent

4. To find $(d/dx)[f(x)]^{g(x)}$,

logarithmic differentiation can be used .

Problem Differentiate $y = x^{\sqrt{x}}$.

Solution

Since both the base and the exponent are variable, we use logarithmic differentiation:

$$\ln y = \ln x^{\sqrt{x}} = \sqrt{x}\,\ln x$$

$$\frac{y'}{y} = \sqrt{x} \cdot \frac{1}{x} + (\ln x)\,\frac{1}{2\sqrt{x}}$$

$$y' = y\left(\frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}}\right) = x^{\sqrt{x}}\left(\frac{2 + \ln x}{2\sqrt{x}}\right)$$

Practice Problem

Find the derivative of the following functions.

(i) $y = sinx^{cosx}$ (ii) $y = x^{x^x}$

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1 - x^2}} \qquad \frac{d}{dx}(\csc^{-1}x) = -\frac{1}{x\sqrt{x^2 - 1}}$$

$$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1 - x^2}} \qquad \frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2 - 1}}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1 + x^2} \qquad \frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1 + x^2}$$

Problem

Recall that $\arctan x$ is an alternative notation for $\tan^{-1} x$.

Differentiate (a) $y = \dfrac{1}{\sin^{-1} x}$ and (b) $f(x) = x \arctan \sqrt{x}$.

Solution

(a) $$\frac{dy}{dx} = \frac{d}{dx} (\sin^{-1} x)^{-1} = -(\sin^{-1} x)^{-2} \frac{d}{dx} (\sin^{-1} x)$$

$$= -\frac{1}{(\sin^{-1} x)^2 \sqrt{1 - x^2}}$$

(b) $$f'(x) = x\,\frac{1}{1+\left(\sqrt{x}\right)^2}\left(\tfrac{1}{2}x^{-1/2}\right) + \arctan\sqrt{x}$$

$$= \frac{\sqrt{x}}{2(1+x)} + \arctan\sqrt{x}$$

Practice Problems

Use Implicit Differentiation to find $\frac{dy}{dx}$ if $x^4 + x^2 y^3 - y^5 = 2x + 1$.

Find the derivative of $y = \dfrac{\sqrt[3]{x^4 + 6x^2}\,(8x+3)^5}{(2x^2+7)^{2/3}}$.

Find the 50th derivative of $y = \cos(2x)$.

Find the 1000th derivative of $y = xe^{-x}$.

Find the derivative of the function $y = \sqrt{x + \sqrt{x + \sqrt{x + \cdots}}}$

Related Rates

If we are pumping air into a balloon, both the volume and the radius of the balloon are increasing and their rates of increase are related to each other. But it is much easier to measure directly the rate of increase of the volume than the rate of increase of the radius.

In a related rates problem the idea is to compute the rate of change of one quantity in terms of the rate of change of another quantity (which may be more easily measured). The procedure is to find an equation that relates the two quantities and then use the Chain Rule to differentiate both sides with respect to time.

Problem Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm^3/s. How fast is the radius of the balloon increasing when the diameter is 50 cm?

Solution We start by identifying two things:

the *given information:*

the rate of increase of the volume of air is 100 cm^3/s

and the *unknown:*

the rate of increase of the radius when the diameter is 50 cm

Let V be the volume of the balloon and let r be its radius.

The key thing to remember is that rates of change are derivatives. In this problem, the volume and the radius are both functions of the time t. The rate of increase of the volume with respect to time is the derivative dV/dt, and the rate of increase of the radius is dr/dt. We can therefore restate the given and the unknown as follows:

Given: $$\frac{dV}{dt} = 100 \text{ cm}^3/\text{s}$$

Unknown: $$\frac{dr}{dt} \quad \text{when } r = 25 \text{ cm}$$

In order to connect dV/dt and dr/dt, we first relate V and r by a formula—in this case, the formula for the volume of a sphere:

$$V = \frac{4}{3}\pi r^3$$

In order to use the given information, we differentiate each side of this equation with respect to t. To differentiate the right side, we need to use the Chain Rule:

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Now we solve for the unknown quantity:

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}$$

If we put $r = 25$ and $dV/dt = 100$ in this equation, we obtain

$$\frac{dr}{dt} = \frac{1}{4\pi(25)^2}100 = \frac{1}{25\pi}$$

The radius of the balloon is increasing at the rate of $1/(25\pi) \approx 0.0127$ cm/s when the diameter is 50 cm.

Practice Problems

1. Each side of a square is increasing at a rate of 6 cm/s. At what rate is the area of the square increasing when the area of the square is 16 cm^2?

2. The radius of a spherical ball is increasing at a rate of 2 cm/min. At what rate is the surface area of the ball increasing when the radius is 8 cm?

3. A cylindrical tank with radius 5 m is being filled with water at a rate of 3 m^3/min. How fast is the height of the water increasing?

4. If the length of the edge of a cube is increasing at a rate of 4 cm/s, how fast is the volume of the cube increasing when the edge length is 15 cm?

5. Suppose $4x^2 + 9y^2 = 25$, where x and y are functions of t.
(a) If $dy/dt = \frac{1}{3}$, find dx/dt when $x = 2$ and $y = 1$.
(b) If $dx/dt = 3$, find dy/dt when $x = -2$ and $y = 1$.

Linearization

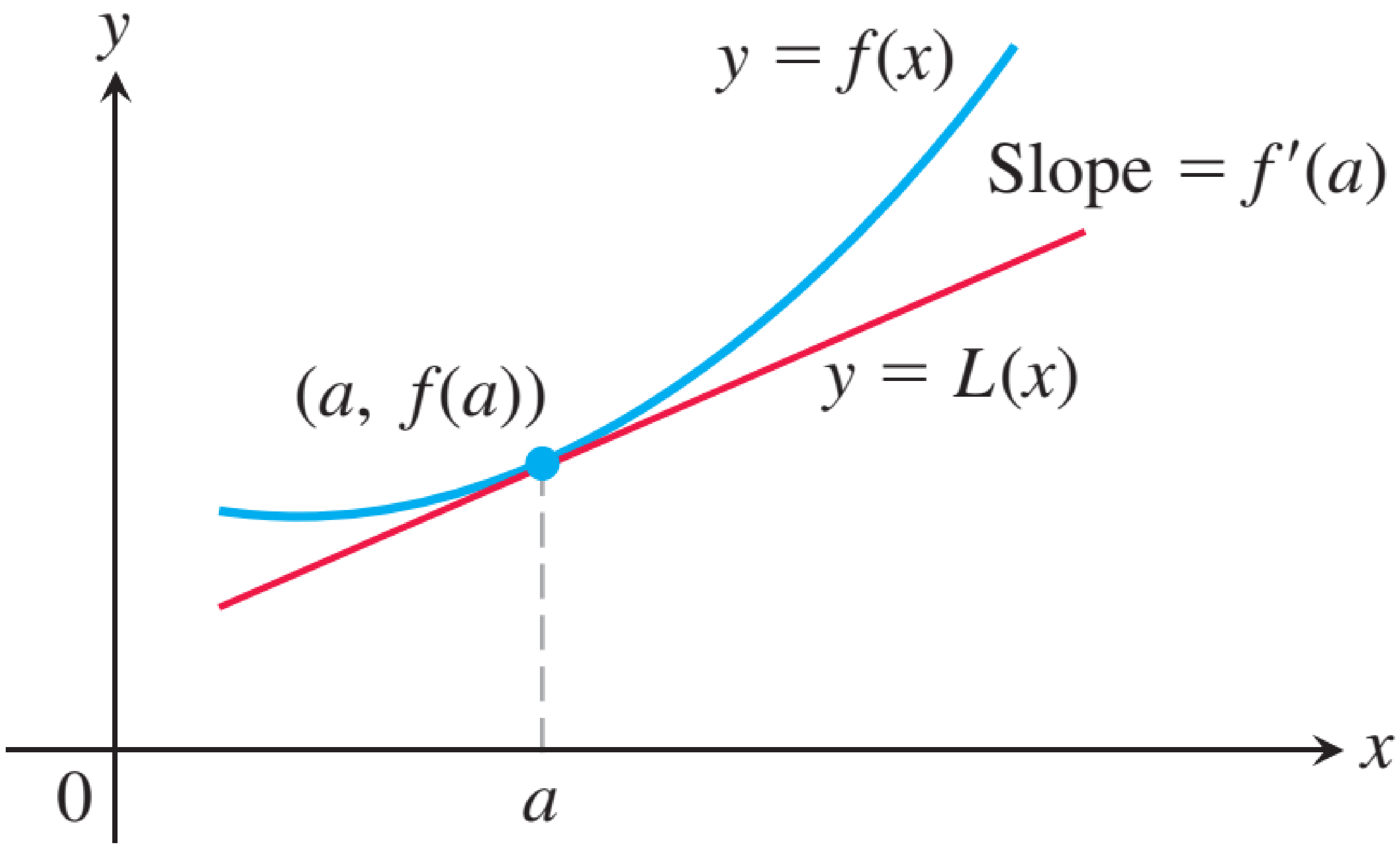
DEFINITIONS If f is differentiable at $x = a$, then the approximating function

$$L(x) = f(a) + f'(a)(x - a)$$

is the **linearization** of f at a. The approximation

$$f(x) \approx L(x)$$

of f by L is the **standard linear approximation** of f at a. The point $x = a$ is the **center** of the approximation.



The tangent to the curve $y = f(x)$ at $x = a$ is the line

$$L(x) = f(a) + f'(a)(x - a).$$

Problem Find the linearization of the function $f(x) = \sqrt{x+3}$ at $a = 1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$.

Solution The derivative of $f(x) = (x+3)^{1/2}$ is

$$f'(x) = \tfrac{1}{2}(x+3)^{-1/2} = \frac{1}{2\sqrt{x+3}}$$

and so we have $f(1) = 2$ and $f'(1) = \tfrac{1}{4}$.

$$L(x) = f(1) + f'(1)(x - 1) = 2 + \tfrac{1}{4}(x - 1) = \frac{7}{4} + \frac{x}{4}$$

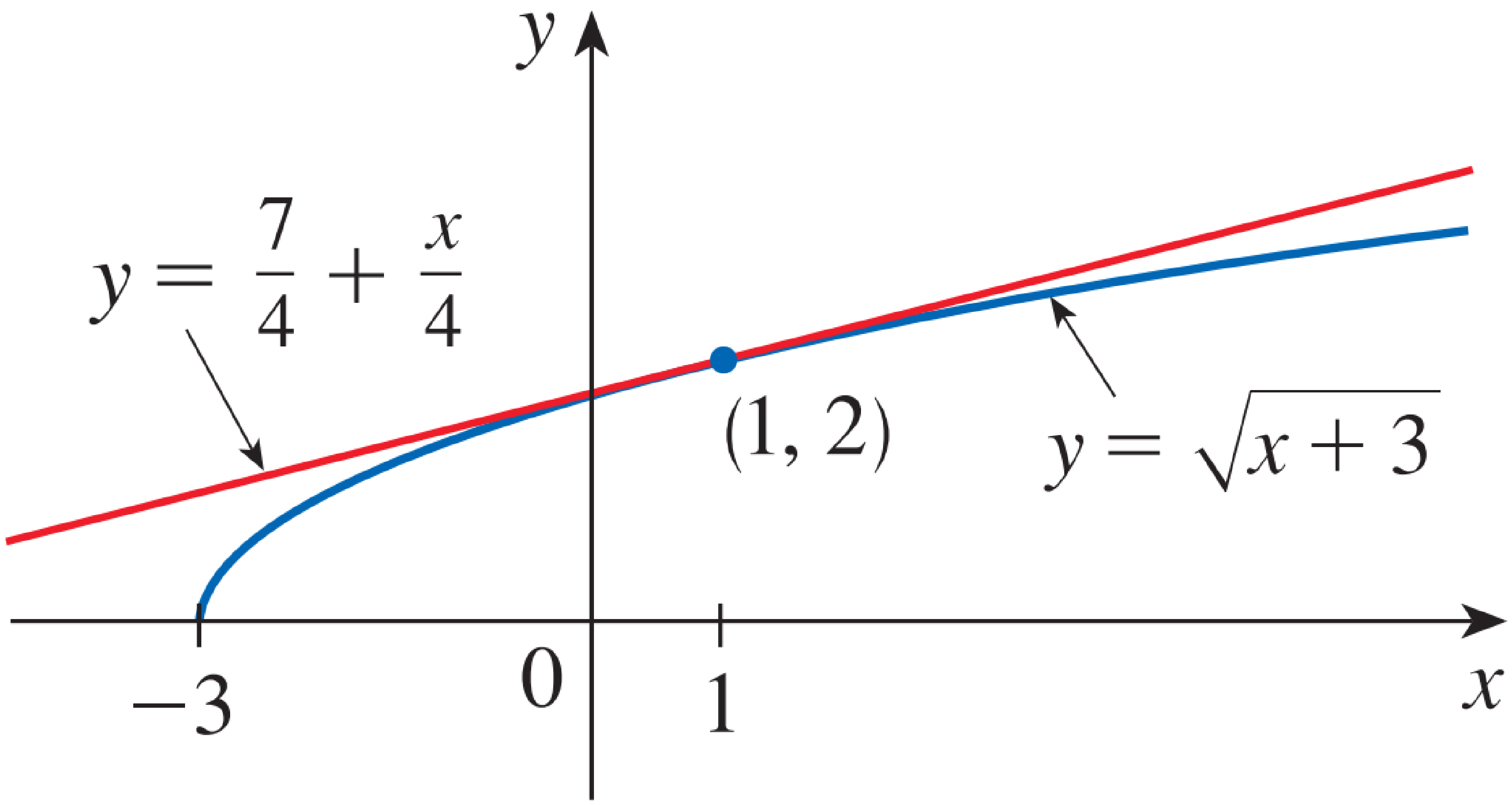
The corresponding linear approximation is

$$\sqrt{x + 3} \approx \frac{7}{4} + \frac{x}{4} \qquad \text{(when } x \text{ is near 1)}$$

In particular, we have

$$\sqrt{3.98} \approx \tfrac{7}{4} + \frac{0.98}{4} = 1.995$$

and $\quad \sqrt{4.05} \approx \frac{7}{4} + \frac{1.05}{4} = 2.0125$



Differentials

The ideas behind linear approximations are sometimes formulated in the terminology and notation of *differentials*. If $y = f(x)$, where f is a differentiable function, then the **differential** dx is an independent variable; that is, dx can be given the value of any real number. The **differential** dy is then defined in terms of dx by the equation

$$dy = f'(x)\,dx$$

So dy is a dependent variable; it depends on the values of x and dx. If dx is given a specific value and x is taken to be some specific number in the domain of f, then the numerical value of dy is determined.

The geometric meaning of differentials is shown in Figure 1

Let $x = a$ and set $dx = \Delta x$. The corresponding change in $y = f(x)$ is

$$\Delta y = f(a + dx) - f(a)$$

The corresponding change in the tangent line L is

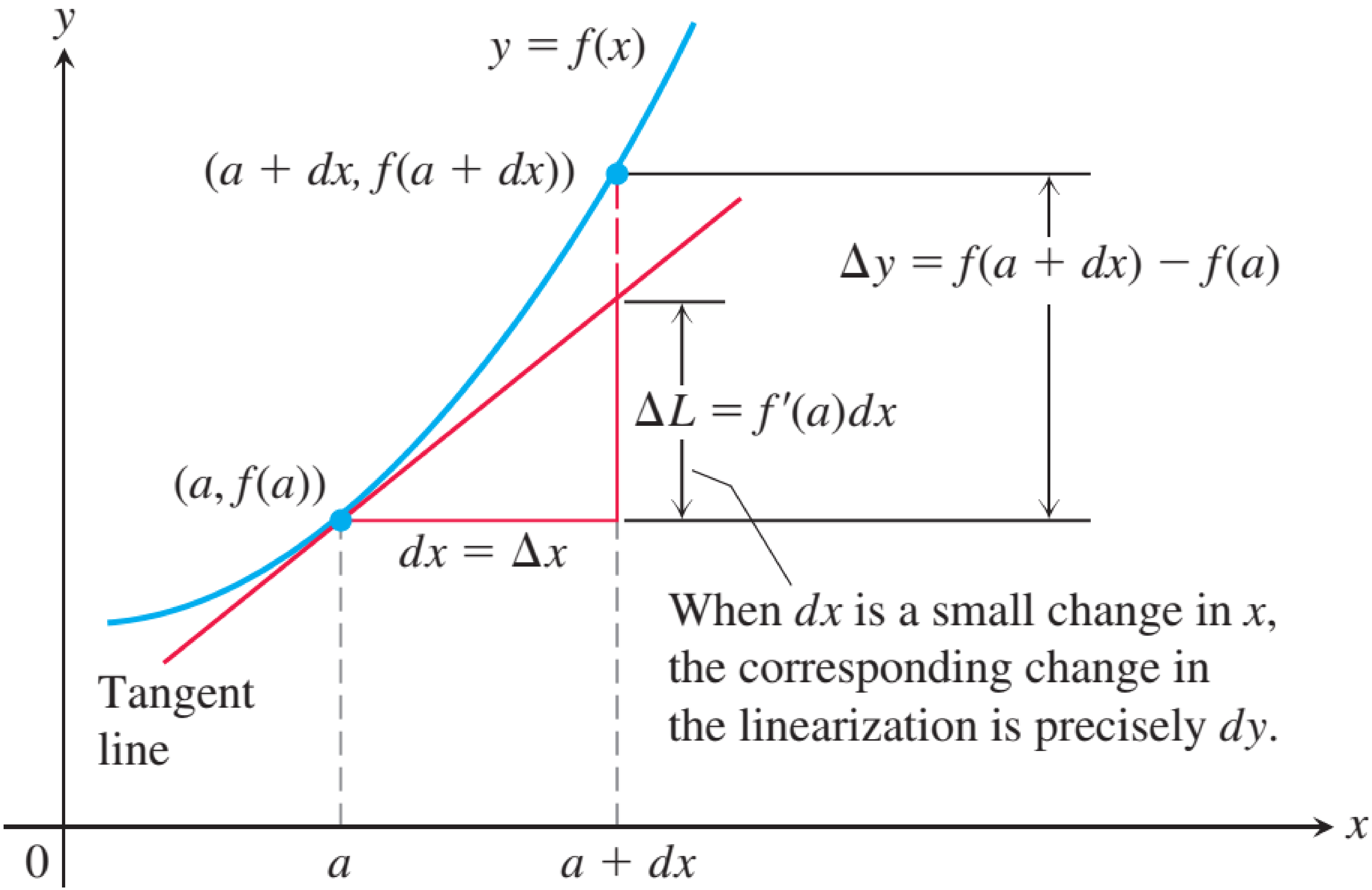
$$\Delta L = L(a + dx) - L(a)$$

$$\Delta L = \underbrace{f(a) + f'(a)\big[(a + dx) - a\big]}_{L(a + dx)} - \underbrace{f(a)}_{L(a)}$$

$$\Delta L = f'(a)\,dx.$$

dy represents the amount that the tangent line rises or falls (the change in the linearization), whereas ∇y represents the amount that the curve $y = f(x)$ rises or falls when x changes by an amount dx.

Figure 1



y
y = f(x)
(a + dx, f(a + dx))
Δy = f(a + dx) − f(a)
ΔL = f′(a)dx
(a, f(a))
dx = Δx
When dx is a small change in x, the corresponding change in the linearization is precisely dy.
Tangent line
x
0
a
a + dx

Problem

Compare the values of Δy and dy if $y = f(x) = x^3 + x^2 - 2x + 1$ and x changes from 2 to 2.05

Solution

(a) We have

$$f(2) = 2^3 + 2^2 - 2(2) + 1 = 9$$

$$f(2.05) = (2.05)^3 + (2.05)^2 - 2(2.05) + 1 = 9.717625$$

$$\Delta y = f(2.05) - f(2) = 0.717625$$

In general,

$$dy = f'(x)\,dx = (3x^2 + 2x - 2)\,dx$$

When $x = 2$ and $dx = \Delta x = 0.05$, this becomes

$$dy = [3(2)^2 + 2(2) - 2]0.05 = 0.7$$

Problem

The radius of a sphere was measured and found to be 21 cm with a possible error in measurement of at most 0.05 cm. What is the maximum error in using this value of the radius to compute the volume of the sphere?

Solution

If the radius of the sphere is r, then its volume is $V = \frac{4}{3}\pi r^3$. If the error in the measured value of r is denoted by $dr = \Delta r$, then the corresponding error in the calculated value of V is ΔV, which can be approximated by the differential

$$dV = 4\pi r^2\,dr$$

When $r = 21$ and $dr = 0.05$, this becomes

$$dV = 4\pi(21)^2\,0.05 \approx 277$$

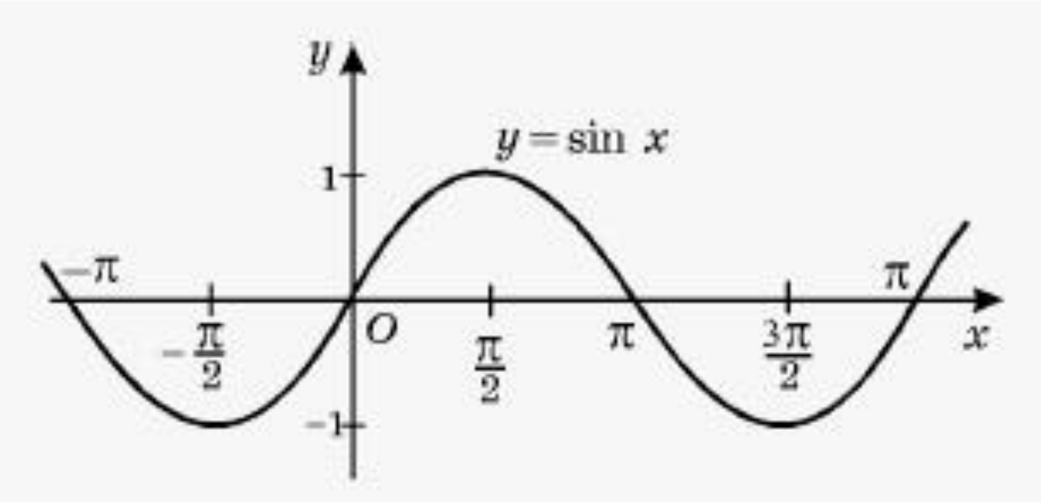
The maximum error in the calculated volume is about 277 cm^3.

Absolute Maximum & Minimum Values of a Function

1 Definition Let c be a number in the domain D of a function f. Then $f(c)$ is the

- **absolute maximum** value of f on D if $f(c) \geqslant f(x)$ for all x in D.
- **absolute minimum** value of f on D if $f(c) \leqslant f(x)$ for all x in D.

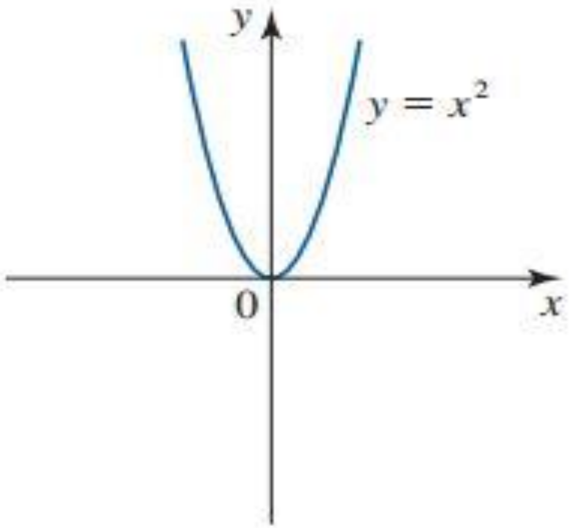
e.g. the absolute maximum and minimum values of the function $y = \sin x$ are $f(\frac{3\pi}{2}) = -1$ and $f(\frac{\pi}{2}) = 1$ respectively.



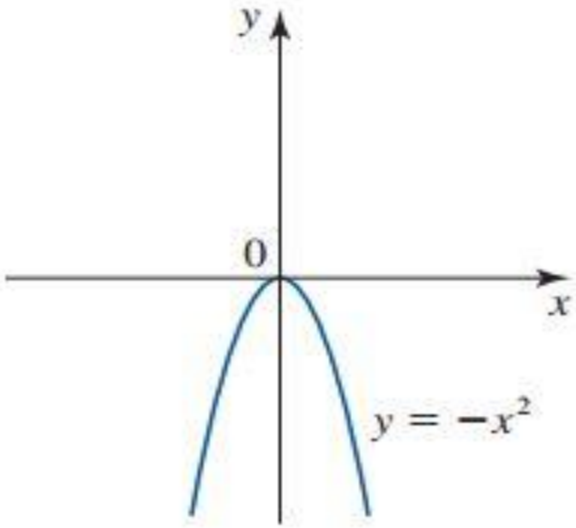
An absolute maximum or minimum is sometimes called global maximum or minimum. The maximum and minimum values of f are called extreme values of f.

Absolute maximum and minimum values are also called Absolute Extrema of the function

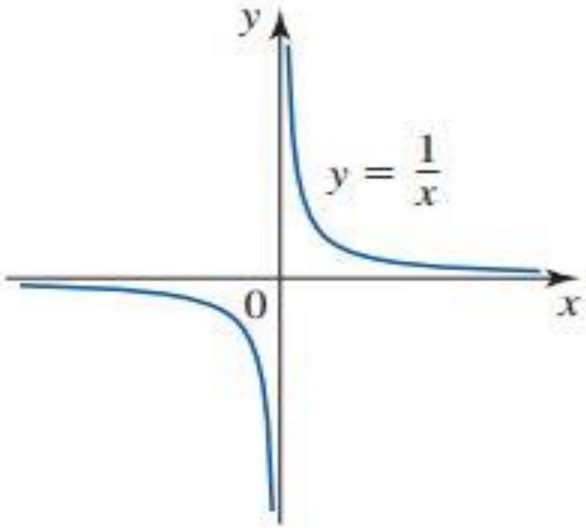
Note: It is not compulsory that a function will always have absolute extrema.



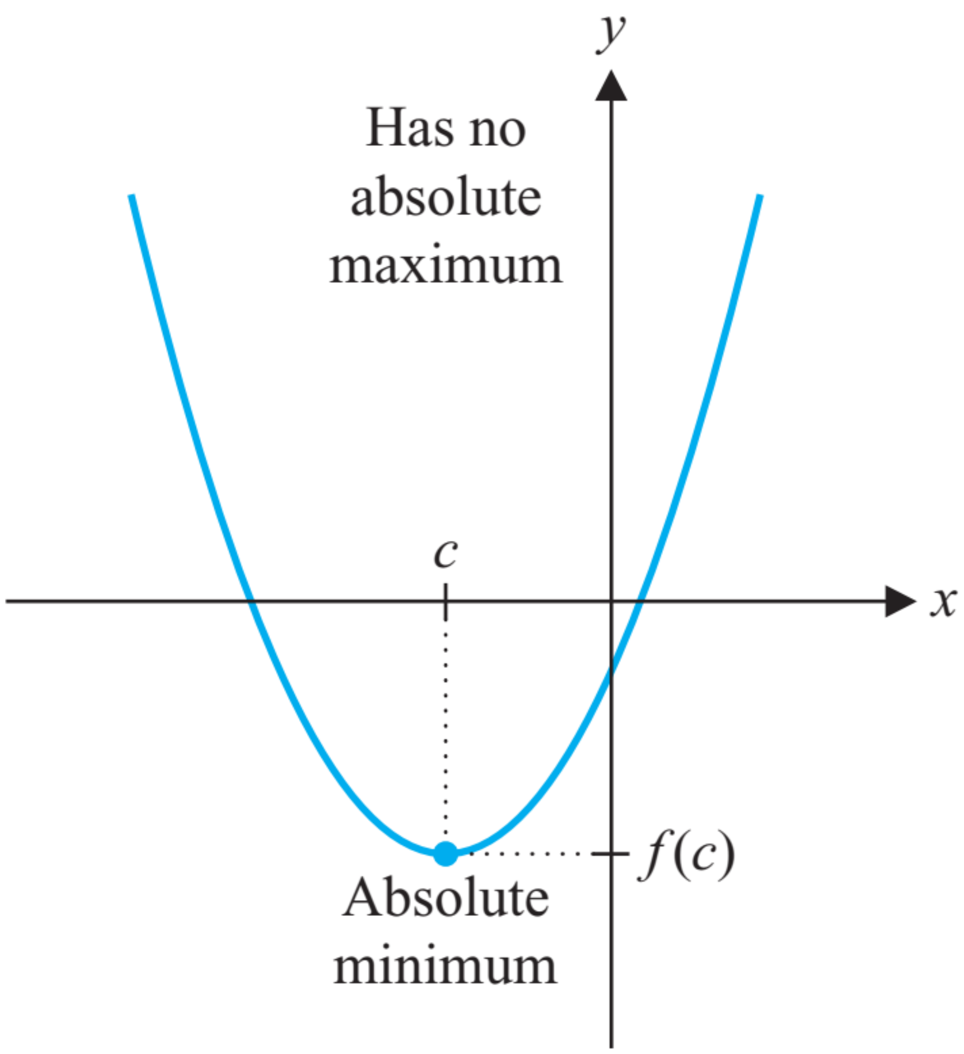
Absolute Minimum
$f(0) = 0$

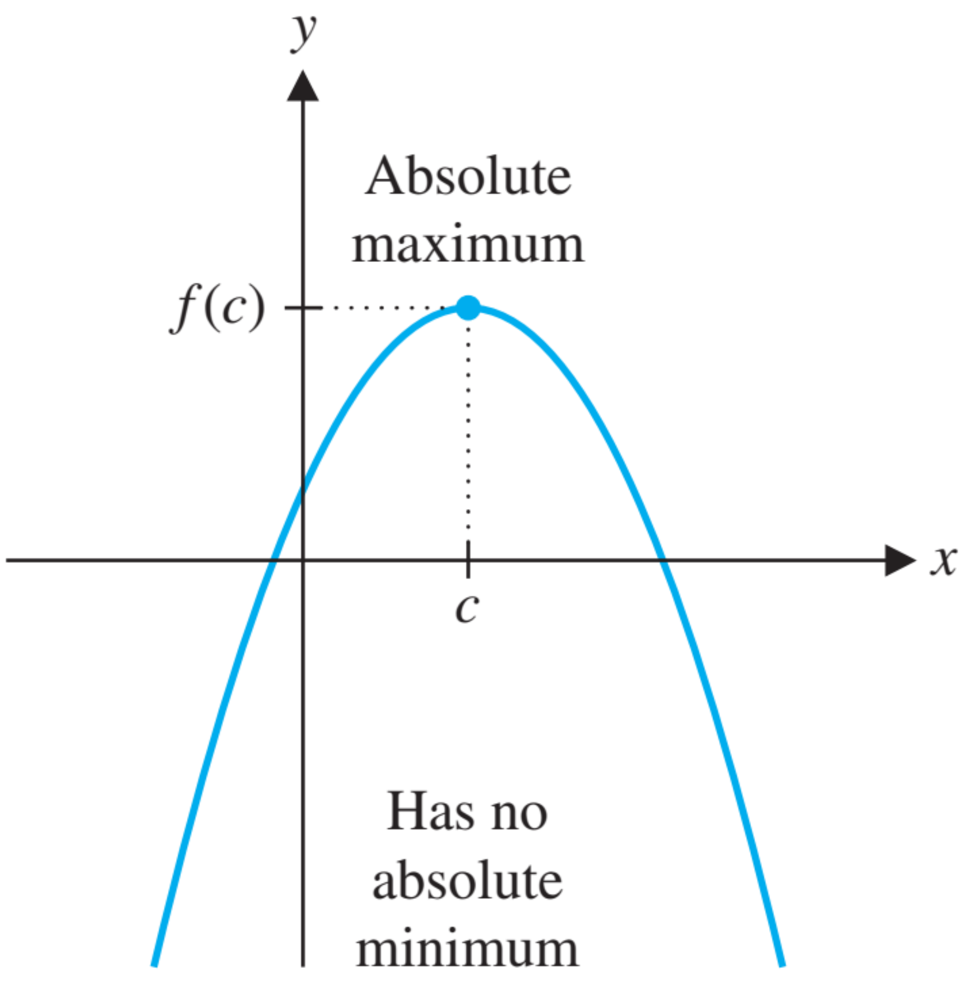


Absolute Maximum
$f(0) = 0$

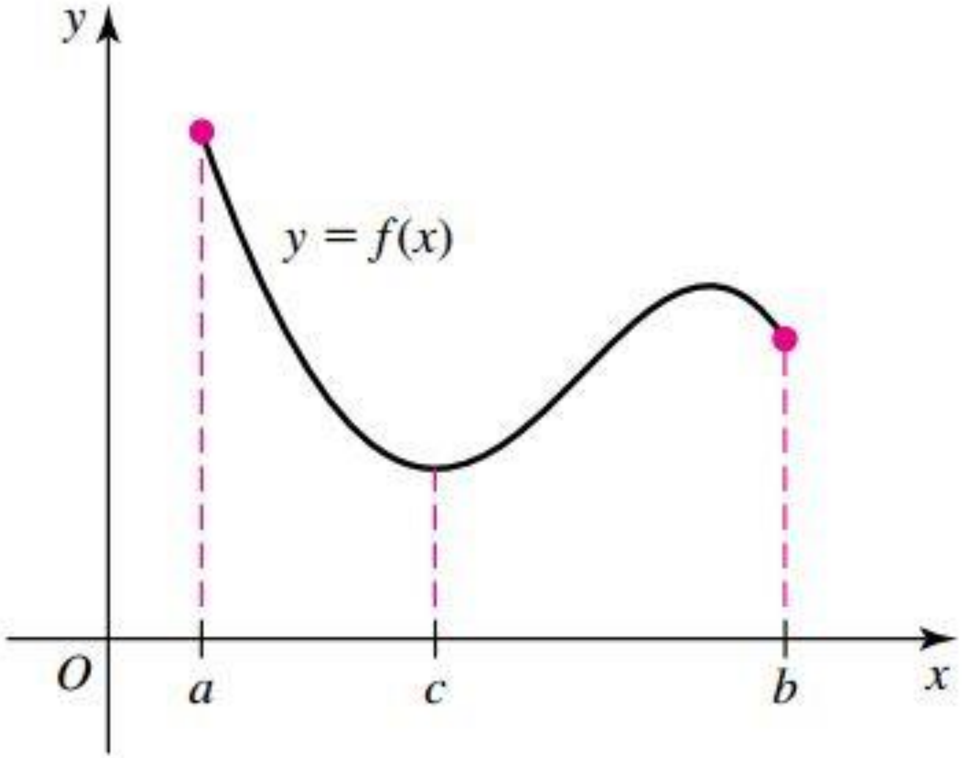


No Absolute Extrema


y
Has no
absolute
maximum
c
x
f(c)
Absolute
minimum

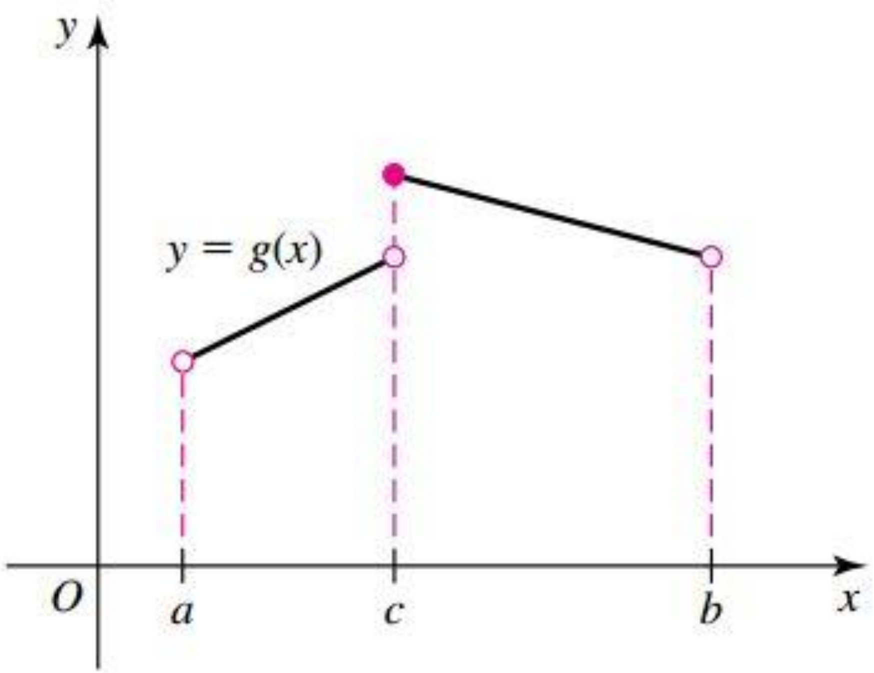

y
Absolute
maximum
f(c)
x
c
Has no
absolute
minimum

$\boxed{3}$ **The Extreme Value Theorem** If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.



(a)

a. The function f is continuous on the closed interval $[a, b]$, so the Extreme Value Theorem guarantees an absolute maximum (which occurs at a) and an absolute minimum (which occurs at c).



(b)

b. The function g does not satisfy the conditions of the Extreme Value Theorem because it is not continuous, and it is defined only on the open interval (a, b). It does not have an absolute minimum value. It does, however, have an absolute maximum at c. Therefore, a function may violate the conditions of the Extreme Value Theorem and still have an absolute maximum or minimum (or both).

6 **Definition** A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Problem

Find the critical numbers of (a) $f(x) = x^3 - 3x^2 + 1$ and (b) $f(x) = x^{3/5}(4 - x)$.

Solution

(a) The derivative of f is $f'(x) = 3x^2 - 6x = 3x(x - 2)$. Since $f'(x)$ exists for all x, the only critical numbers of f occur when $f'(x) = 0$, that is, when $x = 0$ or $x = 2$.

(b) First note that the domain of f is $\mathbb{R}$. The Product Rule gives

$$f'(x) = x^{3/5}(-1) + (4 - x)\left(\tfrac{3}{5}x^{-2/5}\right) = -x^{3/5} + \frac{3(4 - x)}{5x^{2/5}}$$

$$f'(x) = \frac{-5x + 3(4 - x)}{5x^{2/5}} = \frac{12 - 8x}{5x^{2/5}}$$

[The same result could be obtained by first writing $f(x) = 4x^{3/5} - x^{8/5}$.] Therefore $f'(x) = 0$ if $12 - 8x = 0$, that is, $x = \frac{3}{2}$, and $f'(x)$ does not exist when $x = 0$. Thus the critical numbers are $\frac{3}{2}$ and 0.

Since the Extreme Value Theorem guarantees us that there is always an absolute maximum and minimum value of a function that is continuous on a closed interval [a, b]. Here is the method to find them:

The Closed Interval Method To find the *absolute* maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Problem

Find the absolute maximum and minimum values of the function

$$f(x) = x^3 - 3x^2 + 1 \qquad -\tfrac{1}{2} \leqslant x \leqslant 4$$

Solution

Since f is continuous on $\left[-\frac{1}{2}, 4\right]$, we can use the Closed Interval Method.

$$f(x) = x^3 - 3x^2 + 1 \quad \Rightarrow \quad f'(x) = 3x^2 - 6x = 3x(x - 2).$$

Since f' exists for all x, the only critical numbers of f occur when $f'(x) = 0$, that is, when $x = 0$ or $x = -2.$ Notice that each of these critical numbers lies in the interval $\left(-\frac{1}{2}, 4\right)$.

The values of f at these critical numbers are

$$f(0) = 1 \qquad\qquad f(2) = -3$$

The values of f at the endpoints of the interval are

$$f\left(-\tfrac{1}{2}\right) = \tfrac{1}{8} \qquad\qquad f(4) = 17$$

Comparing these four numbers, we see that the absolute maximum value is $f(4) = 17$ and the absolute minimum value is $f(2) = -3$.

Problem

Find the absolute maximum and minimum values of $f(x) = 10x(2 - \ln x)$ on the interval $[1, e^2]$.

Solution

The first derivative is

$$f'(x) = 10(2 - \ln x) - 10x\left(\frac{1}{x}\right) = 10(1 - \ln x).$$

The only critical point in the domain $[1, e^2]$ is the point $x = e$, where $\ln x = 1$. The values of f at this one critical point and at the endpoints are

$$\begin{aligned} &\text{Critical point value:} && f(e) = 10e \\ &\text{Endpoint values:} && f(1) = 10(2 - \ln 1) = 20 \\ & && f(e^2) = 10e^2(2 - 2\ln e) = 0. \end{aligned}$$

We can see from this list that the function's absolute maximum value is $10e \approx 27.2$; it occurs at the critical interior point $x = e$. The absolute minimum value is 0 and occurs at the right endpoint $x = e^2$.

Practice Problems

Find the absolute maximum and absolute minimum values of f on the given interval.

(I) $f(x) = x^3 - 6x^2 + 5, \quad [-3, 5]$

(II) $f(t) = t - \sqrt[3]{t}, \quad [-1, 4]$

(III) $f(x) = x^{-2} \ln x, \quad \left[\frac{1}{2}, 4\right]$

Local Maximum & Minimum Values of a Function

Let c be a number in the domain of a function f.

(i) $f(c)$ is a **local maximum** of f if there exists an open interval (a, b) containing c such that $f(x) \leq f(c)$ for all x in (a, b).

(ii) $f(c)$ is a **local minimum** of f if there exists an open interval (a, b) containing c such that $f(x) \geq f(c)$ for all x in (a, b).

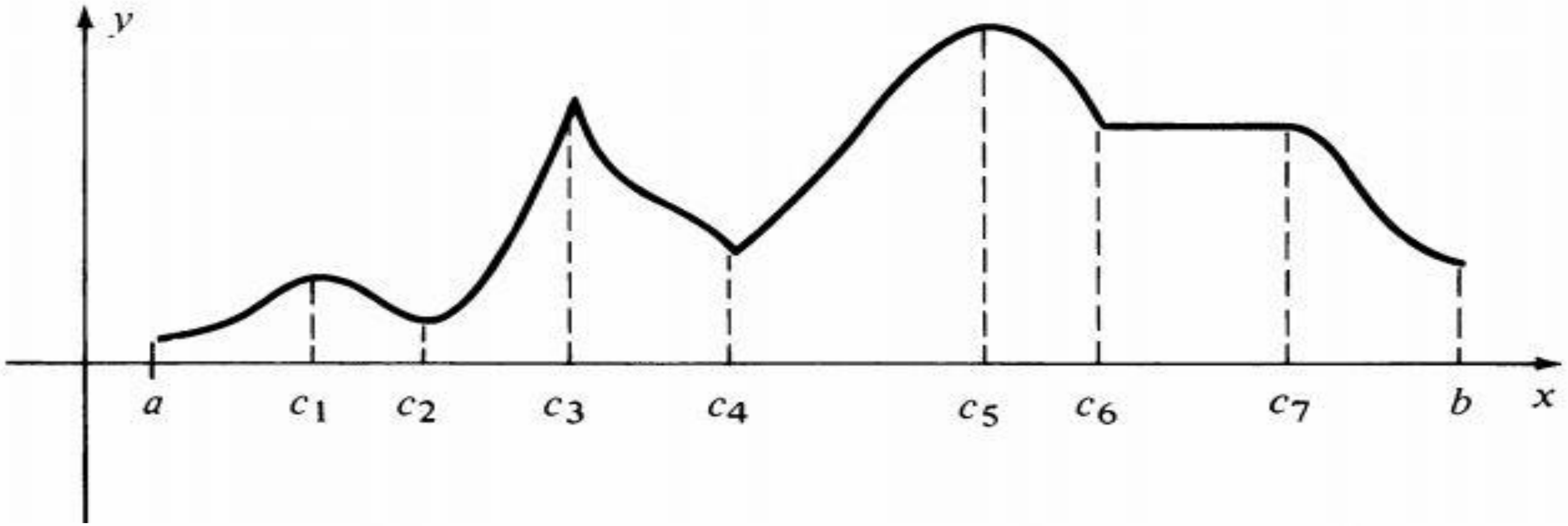
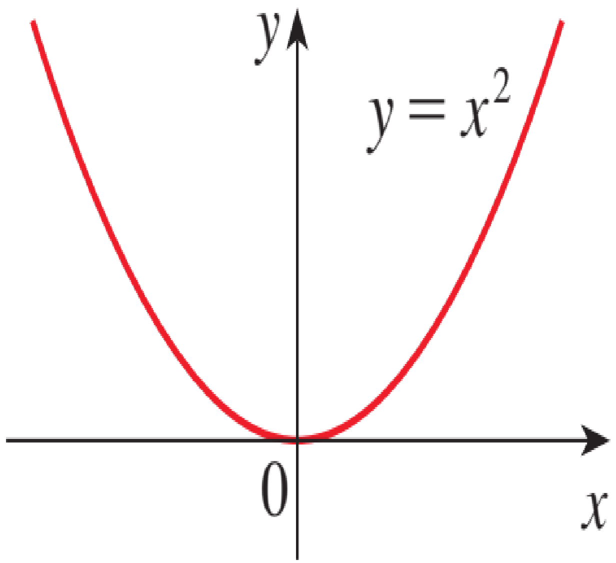


Figure 4.1

For the function whose graph is sketched in Figure 4.1, local maxima occur at c_1, c_3, and c_5 whereas local minima occur at c_2 and c_4. The functional values that correspond to numbers in the open interval (c_6, c_7) are *both* local maxima and local minima. (Why?) The local extrema may not include the absolute minimum or maximum values of f. For example, with reference to Figure 4.1, $f(a)$ is the minimum value of f on $[a, b]$, although it is not a local minimum since there is no *open* interval I contained in $[a, b]$ such that $f(a)$ is the least value of f on I. The number $f(c_5)$ is both a local maximum and the absolute maximum for f on $[a, b]$.

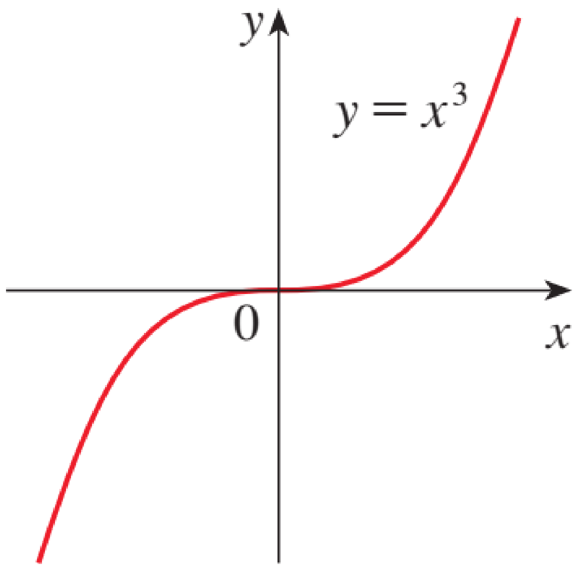
4 **Fermat's Theorem** If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

In the figure we see that the parabola $y = x^2$ has a local minimum value at $x = 0$ and here the first order derivative is zero.



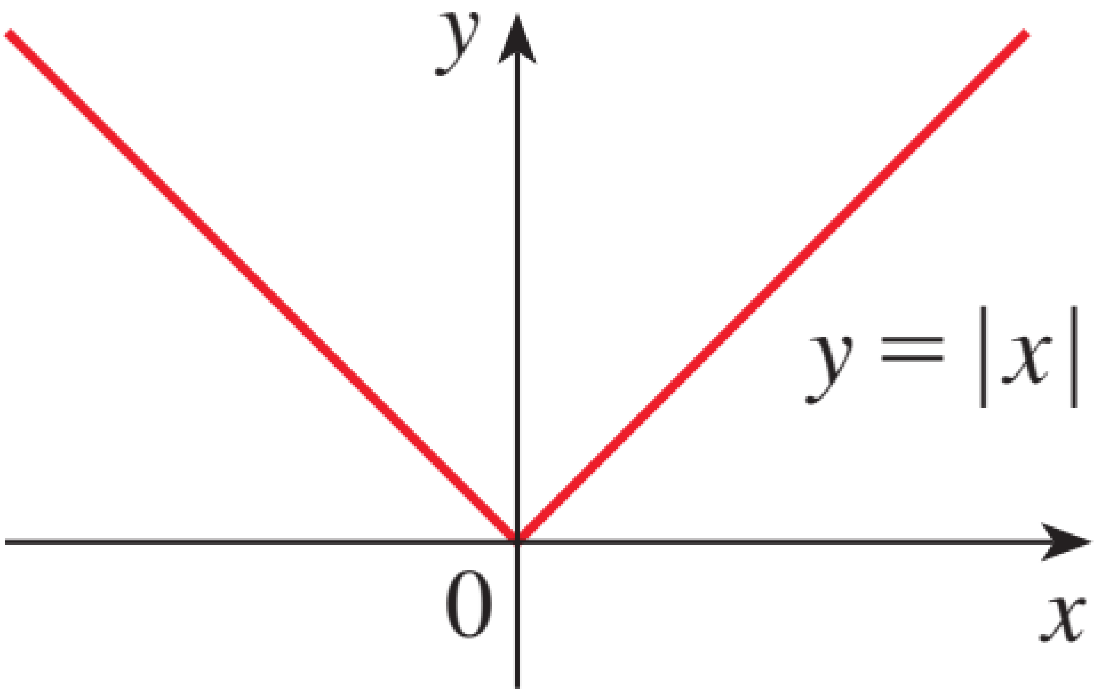
The converse of Fermat's Theorem is not true in general.

(It means that if the first order derivative is zero at a point then it is not compulsory that it is a point where f has local maximum or minimum.)



If $f(x) = x^3$, then $f'(0) = 0$, but f has no maximum or minimum.

An example that illustrate the idea that it is also possible that local maximum or minimum value can occurs at a point where the derivative does not exist.



If $f(x) = |x|$, then $f(0) = 0$ is a minimum value, but $f'(0)$ does not exist.

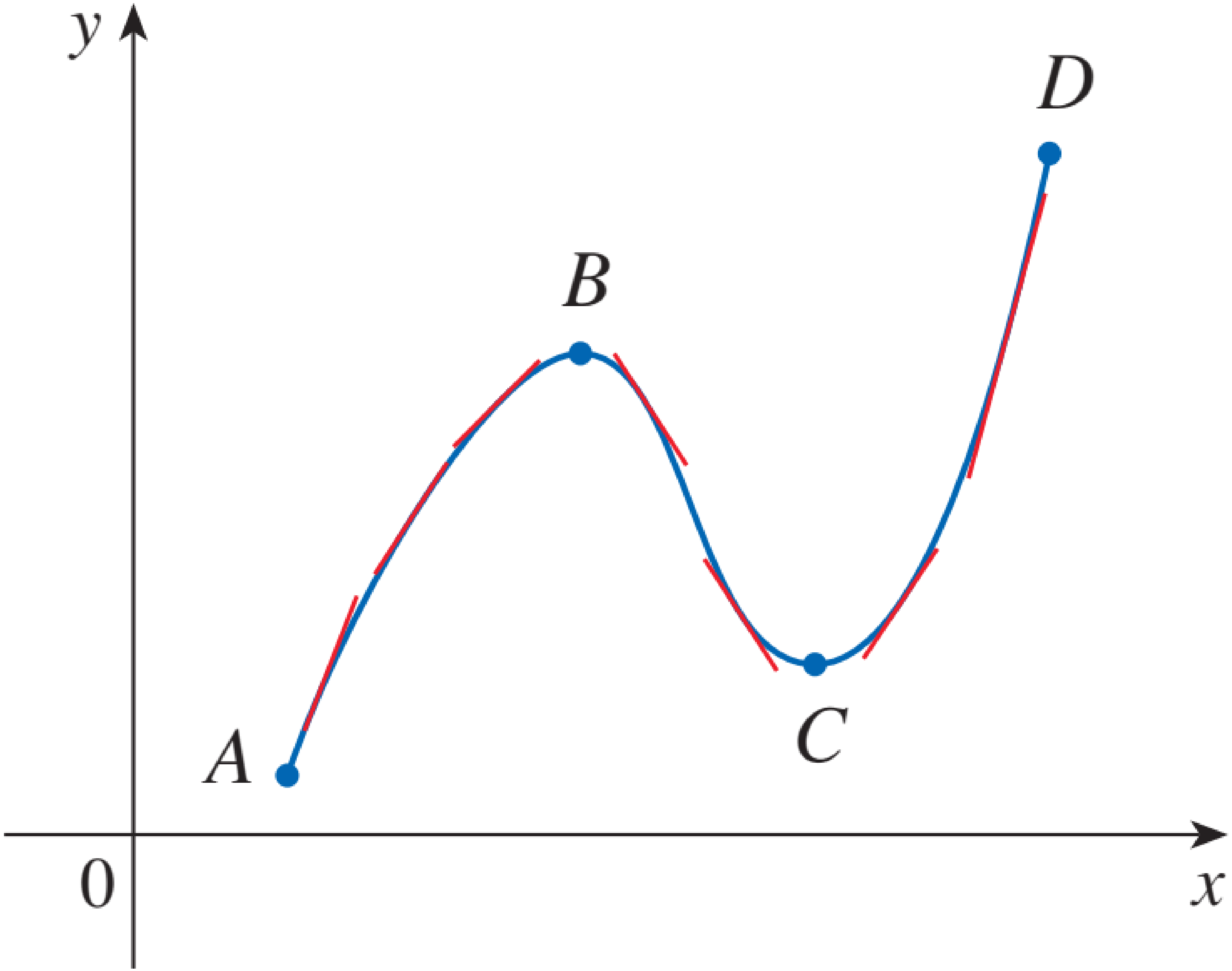
Conclusion :

Since the Fermat's theorem tell us that if there is a local extrema at a point c and the derivative at c exist then it will be zero. Also we saw that local extrema can occur at those points where derivative does not exist. This suggest us that local extrema can only occur at only those point where derivative is zero or does not exist and these are critical points of the function. Thus critical points of a function are the candidates for local extrema of a function.

Increasing/Decreasing Test

(a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

(b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

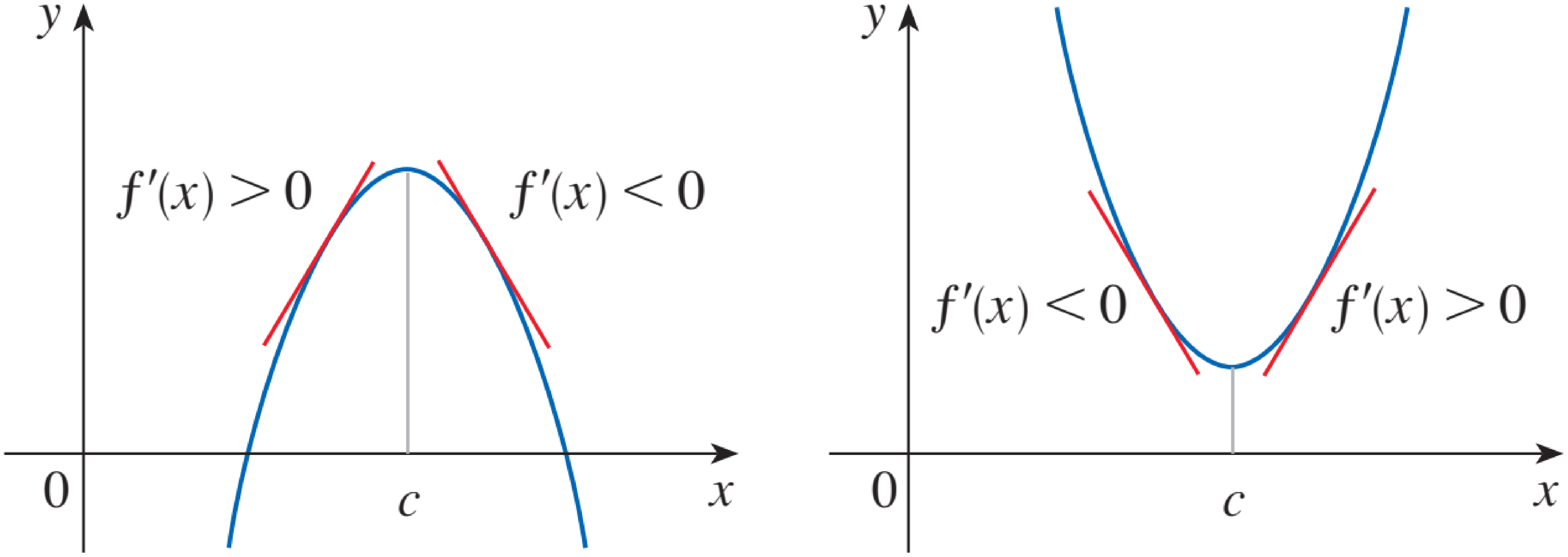

y
D
B
A
C
0
x

The First Derivative Test Suppose that c is a critical number of a continuous function f.

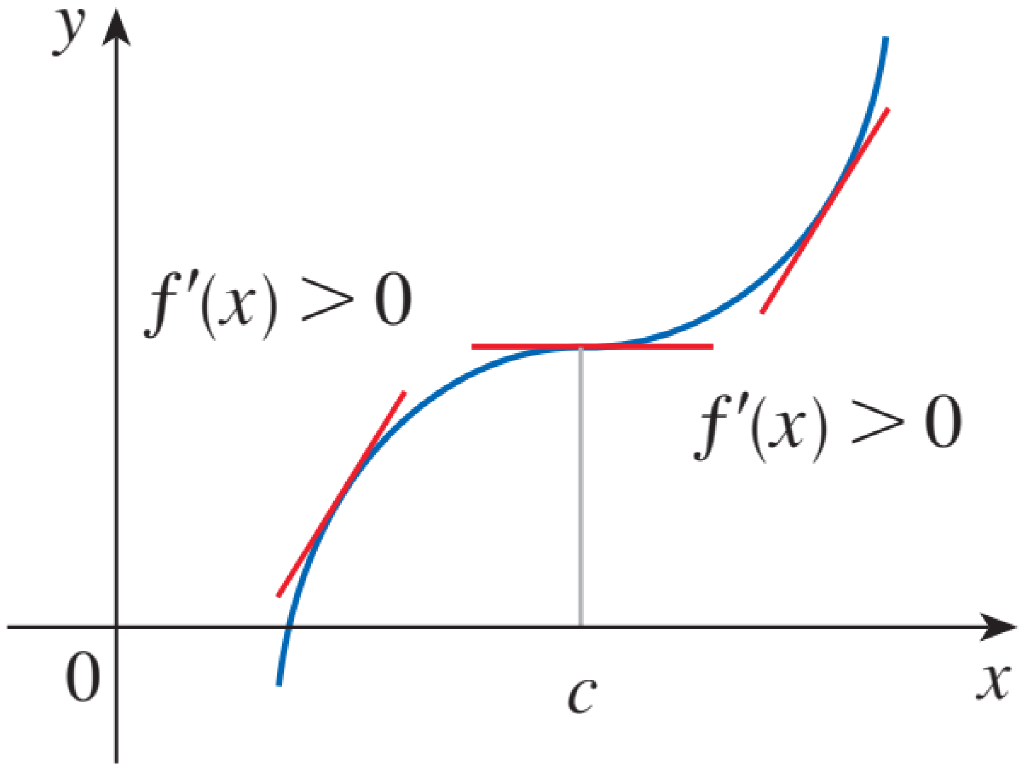
(a) If f' changes from positive to negative at c, then f has a local maximum at c.

(b) If f' changes from negative to positive at c, then f has a local minimum at c.

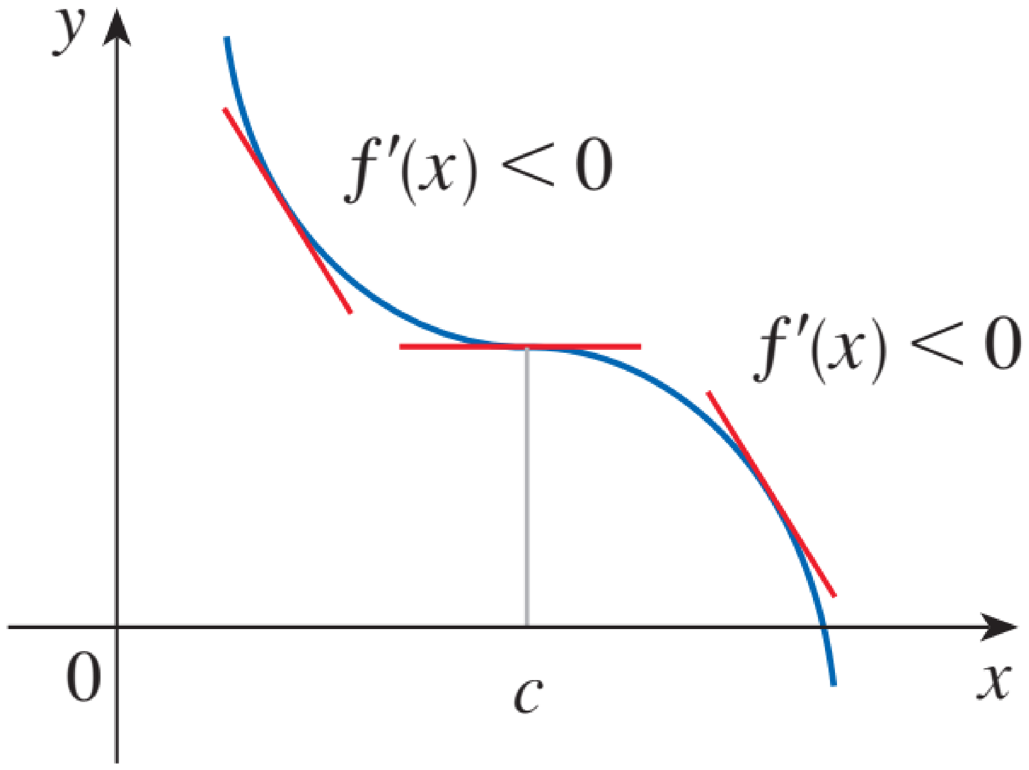
(c) If f' is positive to the left and right of c, or negative to the left and right of c, then f has no local maximum or minimum at c.



(a) Local maximum at c (b) Local minimum at c



(c) No maximum or minimum at c



(d) No maximum or minimum at c

Problem

Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the open intervals on which f is increasing and on which f is decreasing.

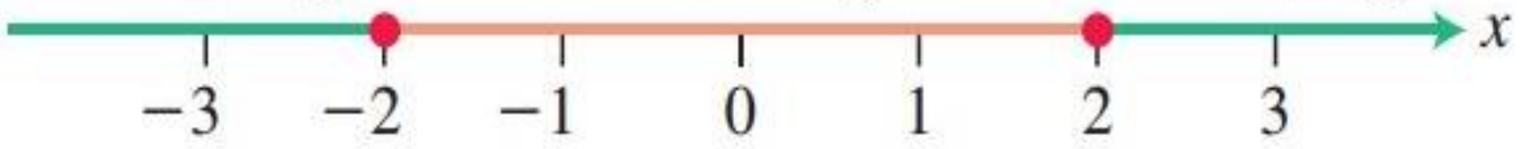
Solution

$$f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 3(x+2)(x-2)$$

is zero at $x = -2$ and $x = 2$. These critical points subdivide the domain of f to create nonoverlapping open intervals $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$ on which f' is either positive or negative.

Table on next slide…

Interval	$-\infty < x < -2$	$-2 < x < 2$	$2 < x < \infty$
f' evaluated	$f'(-3) = 15$	$f'(0) = -12$	$f'(3) = 15$
Sign of f'	$+$	$-$	$+$
Behavior of f	increasing	decreasing	increasing



Problem

Using the First Derivative Test Consider the function

$$f(x) = 3x^4 - 4x^3 - 6x^2 + 12x + 1.$$

a. Find the intervals on which f is increasing and decreasing.

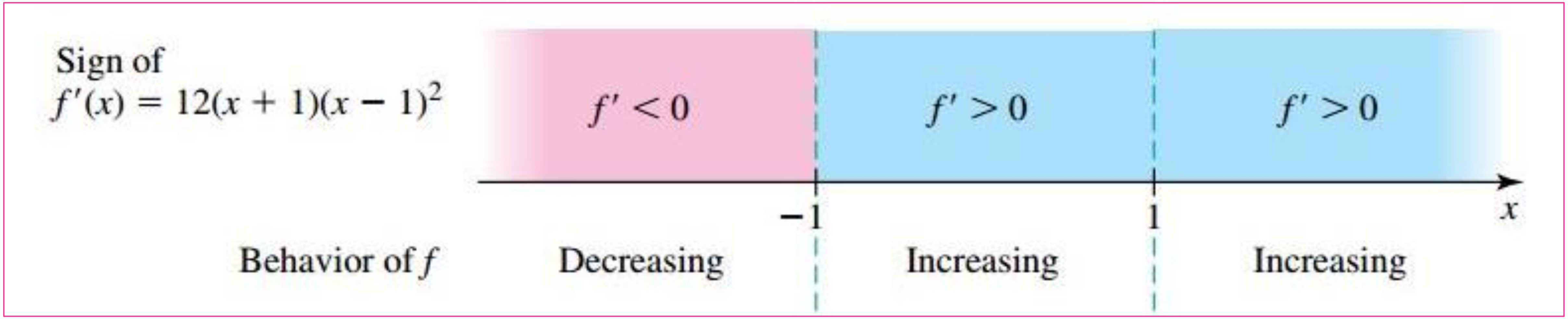
b. Identify the local extrema of f.

Solution

a. Differentiating f, we find that

$$\begin{aligned} f'(x) &= 12x^3 - 12x^2 - 12x + 12 \\ &= 12(x^3 - x^2 - x + 1) \\ &= 12(x + 1)(x - 1)^2. \end{aligned}$$

Solving $f'(x) = 0$ gives the critical points $x = -1$ and $x = 1$. The critical points determine the intervals $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$ on which f' does not change sign.


Sign of
$f'(x) = 12(x+1)(x-1)^2$
$f' < 0$
$f' > 0$
$f' > 0$
-1
1
x
Behavior of f
Decreasing
Increasing
Increasing

Problem Find the local maxima and minima of f if $f(x) = x^{1/3}(8 - x)$.

Solution

By the Product Rule,

$$f'(x) = x^{1/3}(-1) + (8 - x)\tfrac{1}{3}x^{-2/3}$$
$$= \frac{-3x + (8 - x)}{3x^{2/3}} = \frac{4(2 - x)}{3x^{2/3}}$$

and hence the critical numbers of f are 0 and 2.

Interval	k	Test Value $f'(k)$	Sign of $f'(x)$	Variation of f
$(-\infty, 0)$	-1	4	$+$	increasing on $(-\infty, 0]$
$(0, 2)$	1	$\frac{4}{3}$	$+$	increasing on $[0, 2]$
$(2, \infty)$	8	-2	$-$	decreasing on $[2, \infty)$

By the First Derivative Test, f has a local maximum at 2 since the sign of $f'(x)$ changes from $+$ to $-$ as x increases through 2. This local maximum is $f(2) = 2^{1/3}(8 - 2) = 6\sqrt[3]{2} \approx 7.6$. The function does not have an extremum at 0 since the sign of $f'(x)$ does not change as x increases through 0.

First Derivative Test for Absolute Extreme Values Suppose that c is a critical number of a continuous function f defined on an interval.

(a) If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the absolute maximum value of f.

(b) If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then $f(c)$ is the absolute minimum value of f.

What Does f'' Say about f?

Figure 6 shows the graphs of two increasing functions on (a, b). Both graphs join point A to point B but they look different because they bend in different directions. How can we distinguish between these two types of behavior?

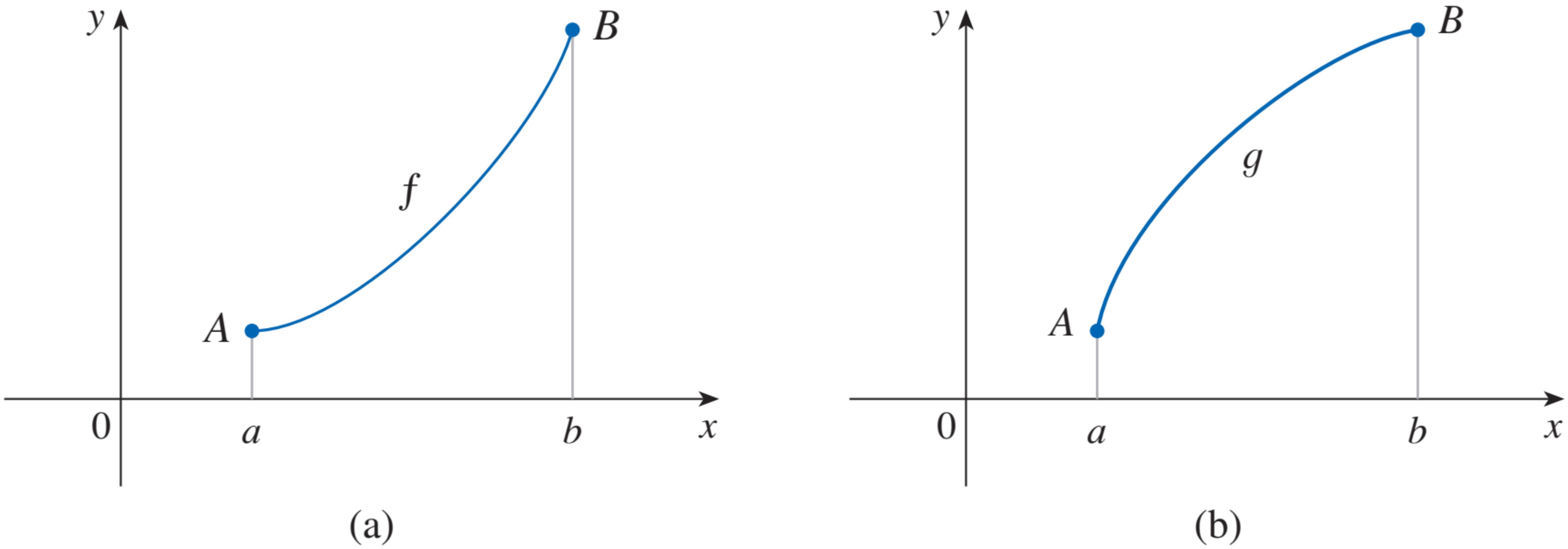
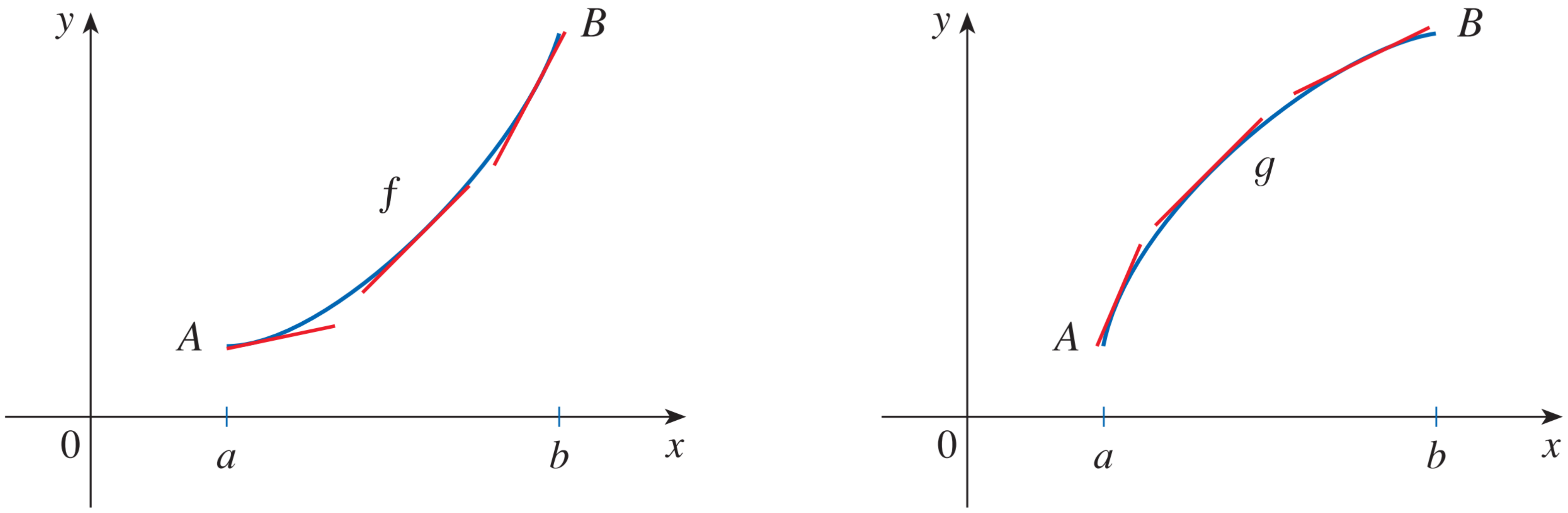


FIGURE 6

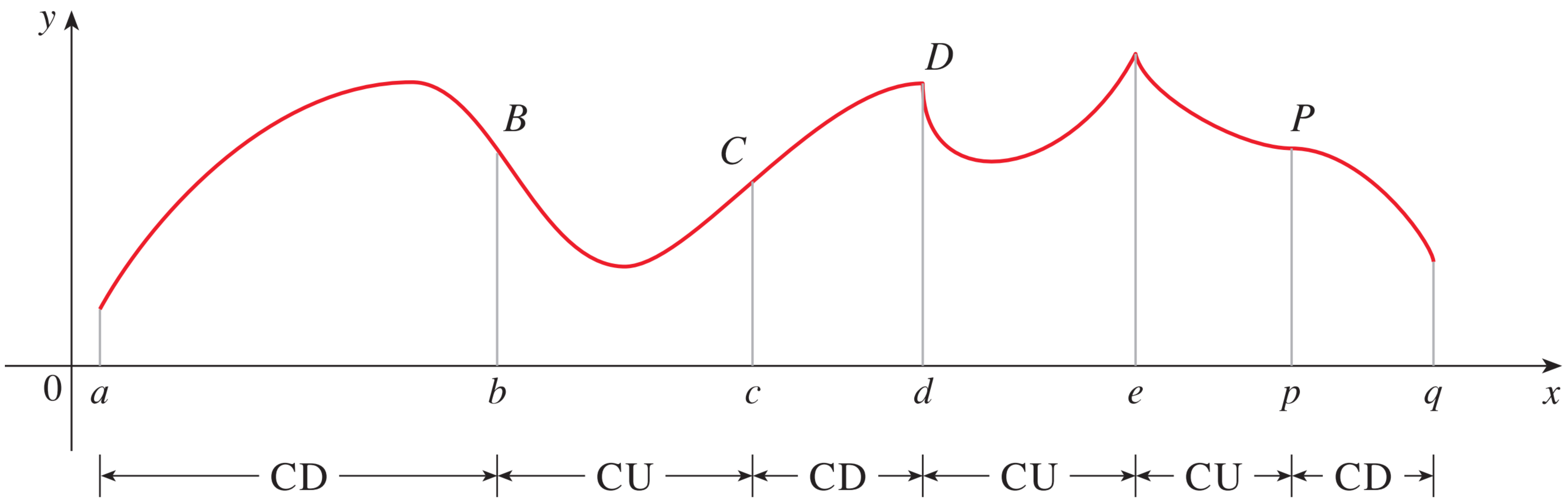
In Figure 7 tangents to these curves have been drawn at several points. In part (a) the curve lies above the tangents and f is called *concave upward* on (a, b). In part (b) the curve lies below the tangents and g is called *concave downward* on (a, b).



(a) Concave upward (b) Concave downward

FIGURE 7

Definition If the graph of f lies above all of its tangents on an interval I, then f is called **concave upward** on I. If the graph of f lies below all of its tangents on I, then f is called **concave downward** on I.


y
D
B
C
P
0
a
b
c
d
e
p
q
x
CD
CU
CD
CU
CU
CD

Concavity Test

(a) If $f''(x) > 0$ on an interval I, then the graph of f is concave upward on I.

(b) If $f''(x) < 0$ on an interval I, then the graph of f is concave downward on I.

Definition A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

Determining the Intervals of Concavity of a Function

1. Find all values of x for which $f''(x) = 0$ or $f''(x)$ does not exist. Use these values of x to partition the domain of f into open intervals.
2. Select a test point c in each interval found in step 1 and determine the sign of $f''(c)$ in that interval.
3. If $f''(c) > 0$, the graph of f is concave upward on that interval.
4. If $f''(c) < 0$, the graph of f is concave downward on that interval.

Problem

For the function $f(x) = x^3 - 6x^2 + 9x + 30,$ determine all intervals where f is concave up and all intervals where f is concave down. List all inflection points for f.

Solution

To determine concavity, we need to find the second derivative $f''(x)$. The first derivative is $f'(x) = 3x^2 - 12x + 9,$ so the second derivative is $f''(x) = 6x - 12.$ If the function changes concavity, it occurs either when $f''(x) = 0$ or $f''(x)$ is undefined. Since f'' is defined for all real numbers $x,$ we need only find where $f''(x) = 0.$ Solving the equation $6x - 12 = 0,$ we see that $x = 2$ is the only place where f could change concavity. We now test points over the intervals $(-\infty, 2)$ and $(2, \infty)$ to determine the concavity of f. The points $x = 0$ and $x = 3$ are test points for these intervals.

Interval	**Test Point**	**Sign of $f''(x) = 6x - 12$ at Test Point**	**Conclusion**
$(-\infty, 2)$	$x = 0$	$-$	f is concave down
$(2, \infty)$	$x = 3$	$+$	f is concave up.

We conclude that f is concave down over the interval $(-\infty, 2)$ and concave up over the interval $(2, \infty)$. Since f changes concavity at $x = 2$, the point $(2, f(2)) = (2, 32)$ is an inflection point.

The Second Derivative Test

Another application of the second derivative is the following test for identifying local maximum and minimum values. It is a consequence of the Concavity Test, and it serves as an alternative to the First Derivative Test.

The Second Derivative Test Suppose f'' is continuous near c.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

Note: If $f'(c) = 0$ and $f''(c) = 0$, then the second derivative test is inconclusive. In this case we have only one choice for investigating local extrema that is by First derivative Test.

Problem

Use the second derivative to find the location of all local extrema for $f(x) = x^5 - 5x^3$.

Solution

To apply the second derivative test, we first need to find critical points c where $f'(c) = 0$. The derivative is $f'(x) = 5x^4 - 15x^2$. Therefore, $f'(x) = 5x^4 - 15x^2 = 5x^2(x^2 - 3) = 0$ when $x = 0, \pm\sqrt{3}$.

To determine whether f has a local extrema at any of these points, we need to evaluate the sign of f'' at these points. The second derivative is

$$f''(x) = 20x^3 - 30x = 10x(2x^2 - 3).$$

In the following table, we evaluate the second derivative at each of the critical points and use the second derivative test to determine whether f has a local maximum or local minimum at any of these points.

x	$f''(x)$	**Conclusion**
$-\sqrt{3}$	$-30\sqrt{3}$	Local maximum
0	0	Second derivative test is inconclusive
$\sqrt{3}$	$30\sqrt{3}$	Local minimum

By the second derivative test, we conclude that f has a local maximum at $x = -\sqrt{3}$ and f has a local minimum at $x = \sqrt{3}$. The second derivative test is inconclusive at $x = 0$. To determine whether f has a local extrema at $x = 0$, we apply the first derivative test. To evaluate the sign of $f'(x) = 5x^2\left(x^2 - 3\right)$ for $x \in \left(-\sqrt{3},\, 0\right)$ and $x \in \left(0,\, \sqrt{3}\right)$, let $x = -1$ and $x = 1$ be the two test points. Since $f'(-1) < 0$ and $f'(1) < 0$, we conclude that f is decreasing on both intervals and, therefore, f does not have a local extrema at $x = 0$.

Practice Problem 1

(a) Find the intervals on which f is increasing or decreasing.
(b) Find the local maximum and minimum values of f.
(c) Find the intervals of concavity and the inflection points.

(I) $f(x) = x^4 - 2x^2 + 3$

(II) $f(x) = \dfrac{x}{x^2 + 1}$

(III) $f(x) = x^2 - x - \ln x$

(IV) $f(x) = \cos^2 x - 2 \sin x, \quad 0 \leqslant x \leqslant 2\pi$

Practice Problem 2

For what values of the numbers a and b does the function

$$f(x) = axe^{bx^2}$$

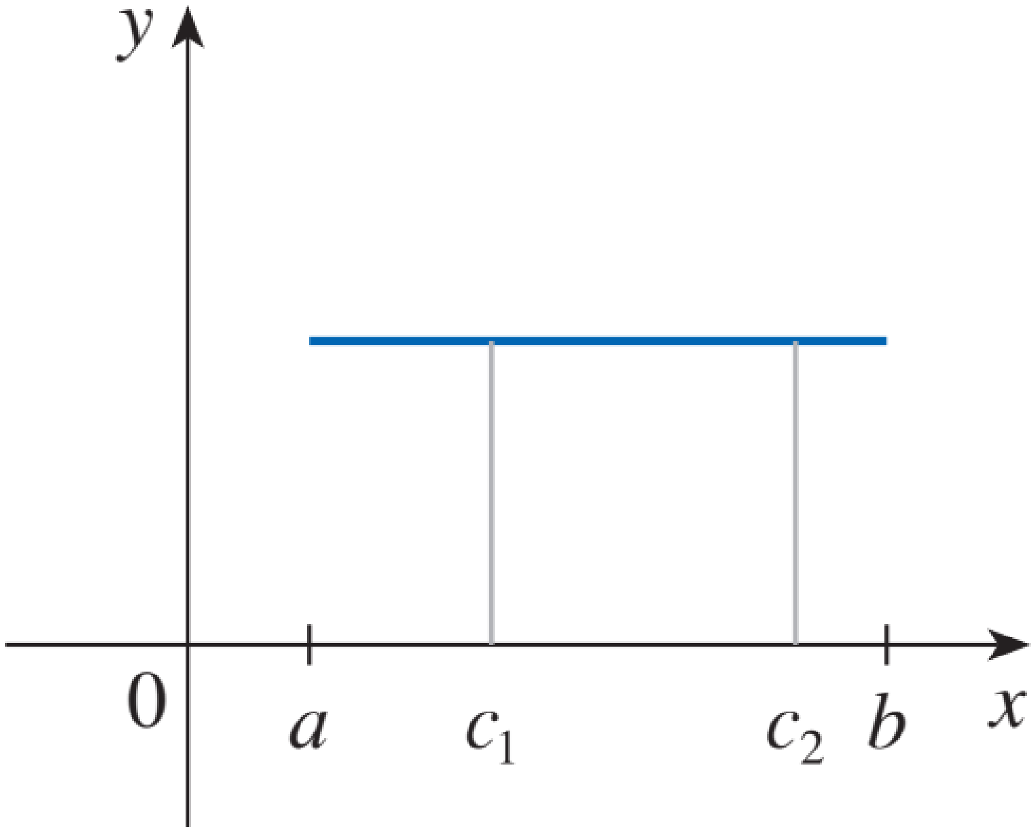
have the maximum value $f(2) = 1$?

Rolle's Theorem Let f be a function that satisfies the following three hypotheses:

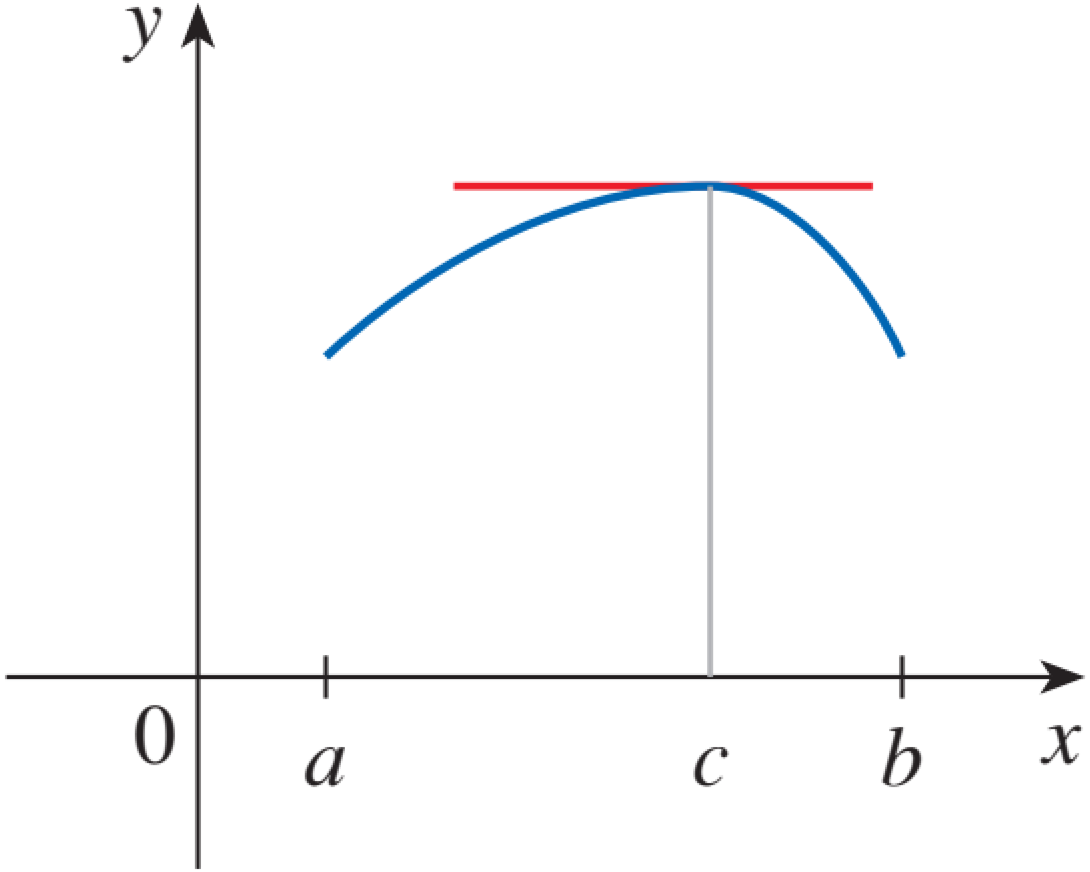
1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).
3. $f(a) = f(b)$

Then there is a number c in (a, b) such that $f'(c) = 0$.

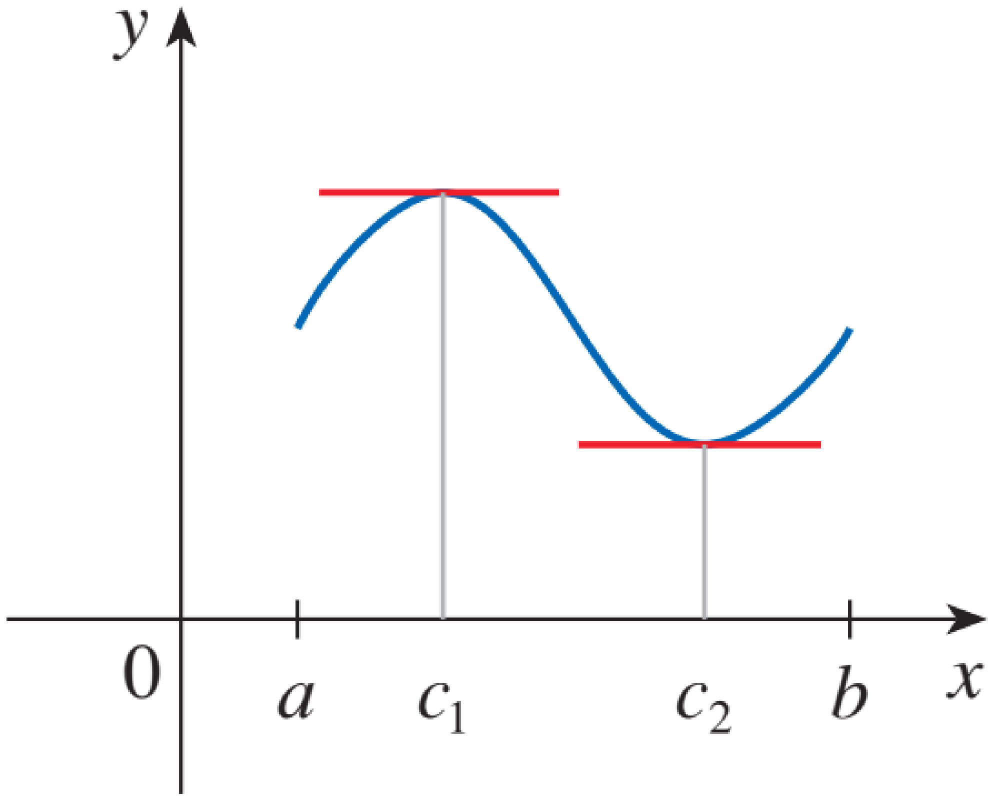
Rolle's Theorem tell us that if a function f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) and $f(a) = f(b)$ i.e. function value at a and b are the same, then there will exist at least one point c between a and b such that the tangent line at $(c, f(c))$ is horizontal.



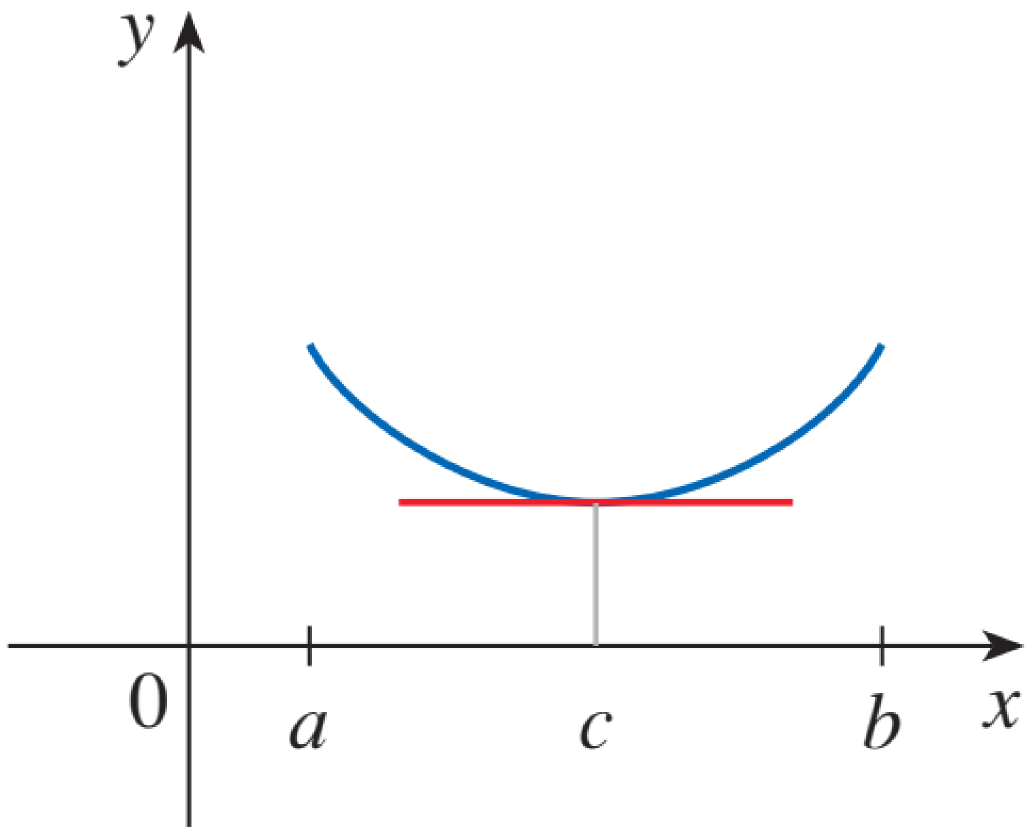
(a)



(b)



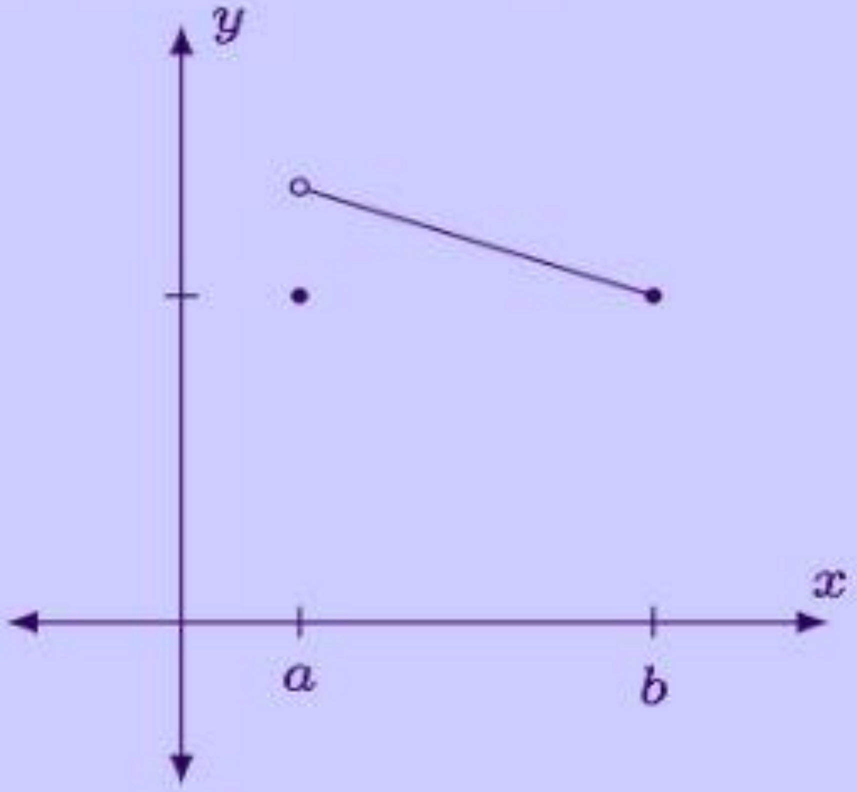
(c)

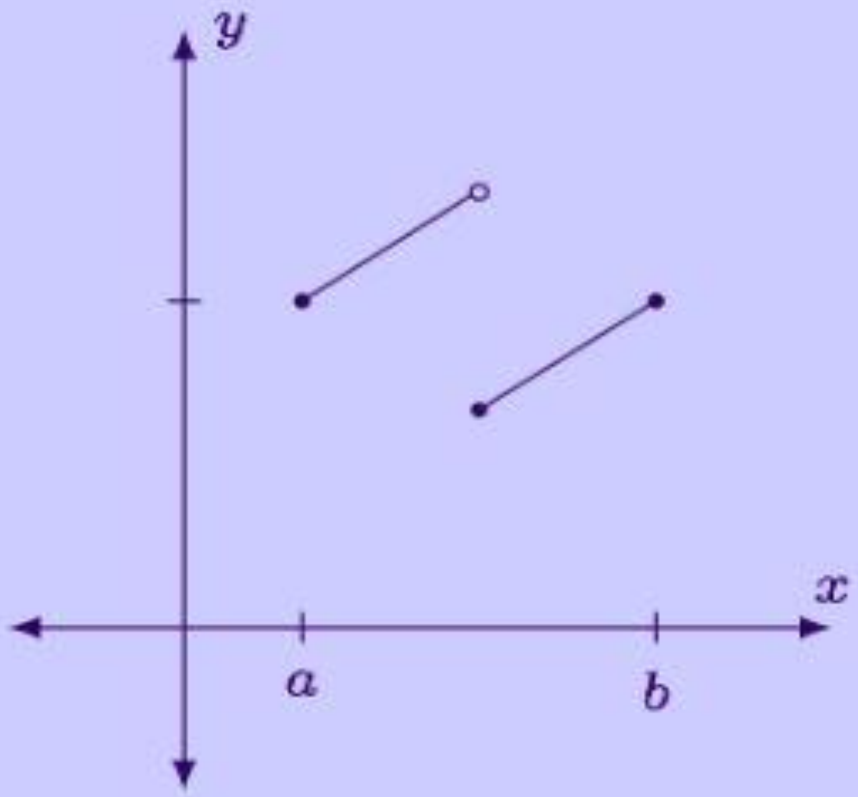


(d)

Why Is Continuity Necessary?

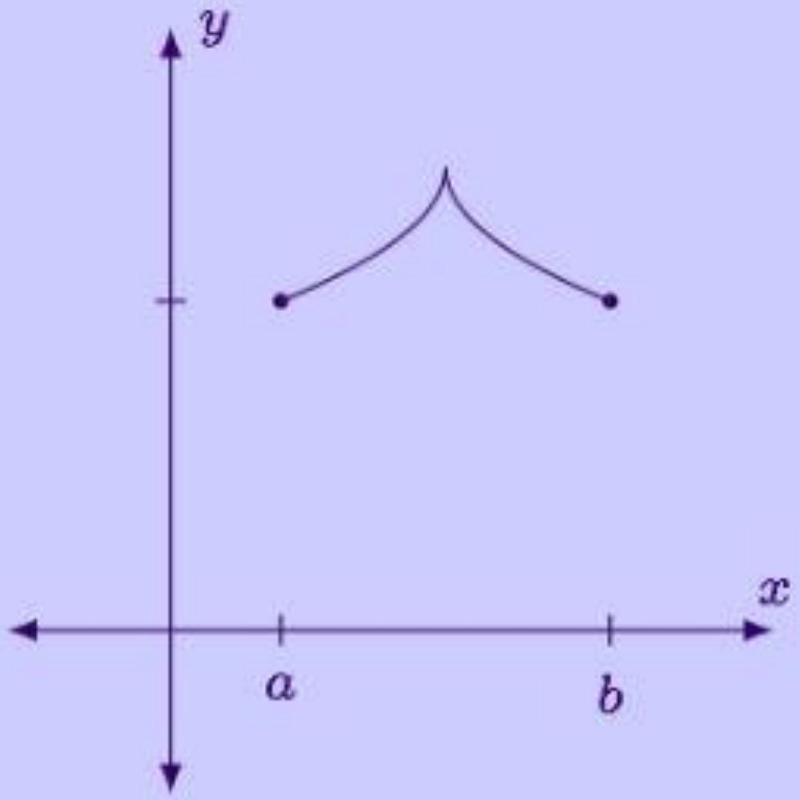
Functions that aren't continuous on $[a, b]$ might not have a point that has a horizontal tangent line. The graphs below are examples of such functions.

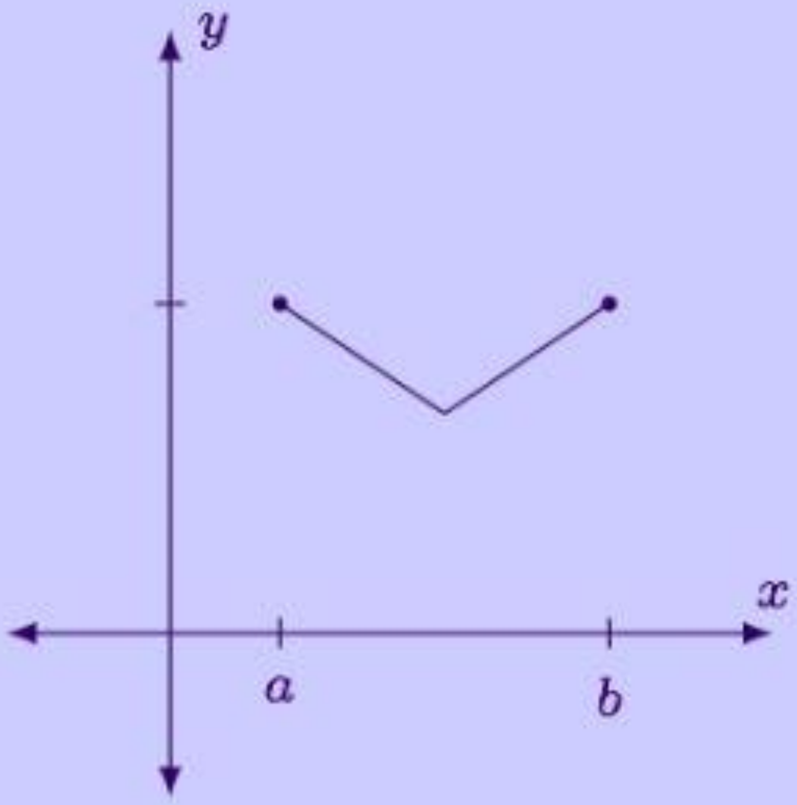




Why is Differentiability Necessary?

Functions that are continuous but not differentiable everywhere on (a, b) will either have a corner or a cusp somewhere in the inteval. When this happens, they might not have a horizontal tangent line, as shown in the examples below.





Indeterminate Forms and l'Hospital's Rule

L'Hospital's Rule Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \qquad \text{and} \qquad \lim_{x \to a} g(x) = 0$$

or that

$$\lim_{x \to a} f(x) = \pm\infty \qquad \text{and} \qquad \lim_{x \to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.) Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

NOTE 1 L'Hospital's Rule says that the limit of a quotient of functions is equal to the limit of the quotient of their derivatives, provided that the given conditions are satisfied. It is especially important to verify the conditions regarding the limits of f and g before using l'Hospital's Rule.

NOTE 2 L'Hospital's Rule is also valid for one-sided limits and for limits at infinity or negative infinity; that is, "$x \to a$" can be replaced by any of the symbols $x \to a^+$, $x \to a^-$, $x \to \infty$, or $x \to -\infty$.

Notice that when using l'Hospital's Rule we differentiate the numerator and denominator *separately*. We do *not* use the Quotient Rule.

Problem

$$\text{Find } \lim_{x \to 1} \frac{\ln x}{x - 1}.$$

Solution

Since

$$\lim_{x \to 1} \ln x = \ln 1 = 0 \qquad \text{and} \qquad \lim_{x \to 1} (x - 1) = 0$$

the limit is an indeterminate form of type $\frac{0}{0}$, so we can apply l'Hospital's Rule:

$$\lim_{x \to 1} \frac{\ln x}{x - 1} = \lim_{x \to 1} \frac{\dfrac{d}{dx}(\ln x)}{\dfrac{d}{dx}(x - 1)} = \lim_{x \to 1} \frac{1/x}{1}$$

$$= \lim_{x \to 1} \frac{1}{x} = 1$$

Problem

Calculate $\lim\limits_{x \to \infty} \dfrac{e^x}{x^2}$.

Solution

We have $\lim_{x \to \infty} e^x = \infty$ and $\lim_{x \to \infty} x^2 = \infty$, so the limit is an indeterminate form of type $\frac{\infty}{\infty}$, and l'Hospital's Rule gives

$$\lim_{x \to \infty} \frac{e^x}{x^2} = \lim_{x \to \infty} \frac{\dfrac{d}{dx}(e^x)}{\dfrac{d}{dx}(x^2)} = \lim_{x \to \infty} \frac{e^x}{2x}$$

Since $e^x \to \infty$ and $2x \to \infty$ as $x \to \infty$, the limit on the right side is also indeterminate.

A second application of l'Hospital's Rule gives

$$\lim_{x \to \infty} \frac{e^x}{x^2} = \lim_{x \to \infty} \frac{e^x}{2x} = \lim_{x \to \infty} \frac{e^x}{2} = \infty$$

Problem Use L'Hospital's Rule to calculate the following limit

$$\lim_{x \to 0} \frac{x - \sin x}{x^3}$$

Solution

$$\lim_{x \to 0} \frac{x - \sin x}{x^3} \qquad \frac{0}{0}\text{; apply l'Hôpital's Rule.}$$

$$= \lim_{x \to 0} \frac{1 - \cos x}{3x^2} \qquad \text{Still } \frac{0}{0}\text{; apply l'Hôpital's Rule again.}$$

$$\lim_{x\to 0}\frac{x - \sin x}{x^3} = \lim_{x\to 0}\frac{\sin x}{6x}$$

Still $\frac{0}{0}$; apply l'Hôpital's Rule again.

$$\lim_{x\to 0}\frac{x - \sin x}{x^3} = \lim_{x\to 0}\frac{\cos x}{6} = \frac{1}{6}$$

Indeterminate Products (Type $0.\infty$)

If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$ (or $-\infty$), then it isn't clear what the value of $\lim_{x\to a} [f(x)\,g(x)]$, if any, will be. There is a struggle between f and g. If f wins, the answer will be 0; if g wins, the answer will be ∞ (or $-\infty$). Or there may be a compromise where the answer is a finite nonzero number.

This kind of limit is called an **indeterminate form of type $0 \cdot \infty$**. We can deal with it by writing the product fg as a quotient:

$$fg = \frac{f}{1/g} \qquad \text{or} \qquad fg = \frac{g}{1/f}$$

This converts the given limit into an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ so that we can use l'Hospital's Rule.

Problem Evaluate $\lim_{x \to 0^+} x \ln x$.

Solution

The given limit is indeterminate because, as $x \to 0^+$, the first factor (x) approaches 0 while the second factor ($\ln x$) approaches $-\infty$. Writing $x = 1/(1/x)$, we have $1/x \to \infty$ as $x \to 0^+$, so l'Hospital's Rule gives

$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/x} = \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} (-x) = 0$$

Indeterminate Difference (Type $\infty - \infty$)

If $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$, then the limit

$$\lim_{x \to a} [f(x) - g(x)]$$

is called an **indeterminate form of type $\infty - \infty$**. Again there is a contest between f and g. Will the answer be ∞ (f wins) or will it be $-\infty$ (g wins) or will they compromise on a finite number? To find out, we try to convert the difference into a quotient (for instance, by using a common denominator, or rationalization, or factoring out a common factor) so that we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Problem

$$\text{Compute } \lim_{x\to 1^+}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right).$$

Solution

First notice that $1/(\ln x) \to \infty$ and $1/(x-1) \to \infty$ as $x \to 1^+$, so the limit is indeterminate of type $\infty - \infty$. Here we can start with a common denominator:

$$\lim_{x\to 1^+}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)=\lim_{x\to 1^+}\frac{x-1-\ln x}{(x-1)\ln x}$$

Both numerator and denominator have a limit of 0, so l'Hospital's Rule applies, giving

$$\lim_{x\to 1^+} \frac{x - 1 - \ln x}{(x - 1)\ln x} = \lim_{x\to 1^+} \frac{1 - \dfrac{1}{x}}{(x - 1)\cdot \dfrac{1}{x} + \ln x} = \lim_{x\to 1^+} \frac{x - 1}{x - 1 + x\ln x}$$

Again we have an indeterminate limit of type $\frac{0}{0}$, so we apply l'Hospital's Rule a second time:

$$\lim_{x\to 1^+} \frac{x - 1}{x - 1 + x\ln x} = \lim_{x\to 1^+} \frac{1}{1 + x\cdot \dfrac{1}{x} + \ln x} = \lim_{x\to 1^+} \frac{1}{2 + \ln x} = \frac{1}{2}$$

Indeterminate Powers (Type $0^0, \infty^0, 1^\infty$)

Several indeterminate forms arise from the limit

$$\lim_{x \to a} [f(x)]^{g(x)}$$

1. $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$ type 0^0
2. $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = 0$ type ∞^0
3. $\lim_{x \to a} f(x) = 1$ and $\lim_{x \to a} g(x) = \pm\infty$ type 1^∞

Each of these three cases can be treated either by taking the natural logarithm:

$$\text{let} \quad y = [f(x)]^{g(x)}, \quad \text{then} \quad \ln y = g(x) \ln f(x)$$

or to write the function as an exponential:

$$[f(x)]^{g(x)} = e^{g(x) \ln f(x)}$$

Problem Calculate $\lim_{x \to 0^+} (1 + \sin 4x)^{\cot x}$.

Solution

First notice that as $x \to 0^+$, we have $1 + \sin 4x \to 1$ and $\cot x \to \infty$, so the given limit is indeterminate (type 1^∞). Let

$$y = (1 + \sin 4x)^{\cot x}$$

Then

$$\ln y = \ln[(1 + \sin 4x)^{\cot x}] = \cot x \ln(1 + \sin 4x) = \frac{\ln(1 + \sin 4x)}{\tan x}$$

so l'Hospital's Rule gives

$$\lim_{x\to0^+} \ln y = \lim_{x\to0^+} \frac{\ln(1 + \sin 4x)}{\tan x} = \lim_{x\to0^+} \frac{\dfrac{4\cos 4x}{1 + \sin 4x}}{\sec^2 x} = 4$$

So far we have computed the limit of ln y, but what we want is the limit of y. To find this we use the fact that $y = e^{\ln y}$:

$$\lim_{x\to0^+} (1 + \sin 4x)^{\cot x} = \lim_{x\to0^+} y = \lim_{x\to0^+} e^{\ln y} = e^4$$

Practice Problems

Compute the limits.

(i) $$\lim_{x \to 0} \frac{\tan x - x}{x^3}$$

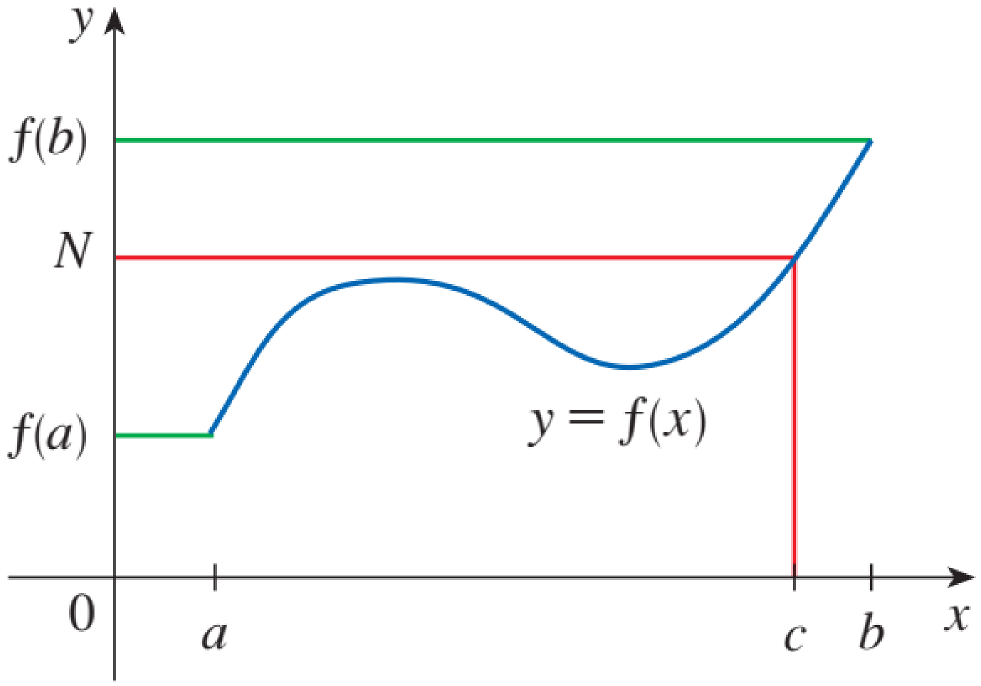
(ii) $$\lim_{x \to \infty} (e^x - x)$$

(iii) $$\lim_{x \to 0^+} x^x$$

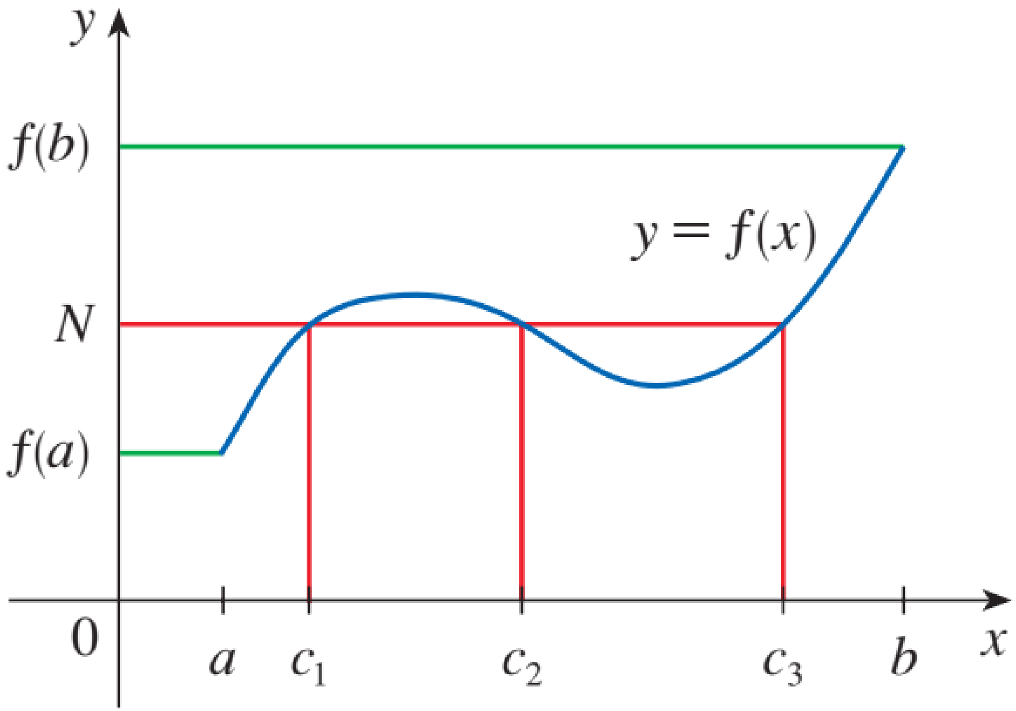
(iv) $$\lim_{x \to 3} \frac{\ln(x/3)}{3 - x}$$

10 The Intermediate Value Theorem Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

The Intermediate Value Theorem states that a continuous function takes on every intermediate value between the function values $f(a)$ and $f(b)$.



(a)



(b)

Root of an Equation & Zero of a Function

Any number r for which $f(r) = 0$ is called a ***root of the equation*** $f(x) = 0$. Also, we say that r is a ***zero of the function*** $f(x)$.

For example, the equation $2x^2 + 5x - 3 = 0$ has two real roots $r_1 = 0.5$ and $r_2 = -3$, whereas the corresponding function $f(x) = 2x^2 + 5x - 3 = (2x - 1)(x + 3)$ has two real zeros, $r_1 = 0.5$ and $r_2 = -3$.

THEOREM

Let f be a continuous function on $[a, b]$, satisfying $f(a) f(b) < 0$. Then f has a root between a and b, that is, there exists a number r satisfying $a < r < b$ and $f(r) = 0$.

Problem Show that there is a solution of the equation

$$4x^3 - 6x^2 + 3x - 2 = 0$$

between 1 and 2.

Solution

Let $f(x) = 4x^3 - 6x^2 + 3x - 2$. We are looking for a solution of the given equation, that is, a number c between 1 and 2 such that $f(c) = 0$.

$$f(1) = 4 - 6 + 3 - 2 = -1 < 0$$

$$f(2) = 32 - 24 + 6 - 2 = 12 > 0$$

Since $f(1)f(2) < 0$ this implies that there is a root or solution of the given equation in the interval $[1,2]$.

Integration

Antiderivatives

Definition A function F is called an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

Consider the derivative formula $\dfrac{d}{dx}(x) = 1$. It implies that an antiderivative of $f(x) = 1$ is $F(x) = x$ because $F'(x) = f(x)$. Using the same logic, we can write

$$\frac{d}{dx}(x^2) = 2x \quad \Rightarrow \quad \text{an antiderivative of } f(x) = 2x \text{ is } F(x) = x^2$$

$$\frac{d}{dx}(\sin x) = \cos x \quad \Rightarrow \quad \text{an antiderivative of } f(x) = \cos x \text{ is } F(x) = \sin x.$$

$\boxed{1}$ **Theorem** If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

Indefinite Integral Notation For convenience, let us introduce a notation for an antiderivative of a function. If $F'(x) = f(x)$, we shall represent the most general antiderivative of f by

$$\int f(x)\,dx = F(x) + C.$$

The symbol $\int$ was introduced by Leibniz and is called an **integral sign**. The notation $\int f(x)\,dx$ is called the **indefinite integral** of $f(x)$ with respect to x. The function $f(x)$ is called the **integrand**. The process of finding an antiderivative is called **antidifferentiation** or **integration**. The number C is called a **constant of integration**. Just as $\frac{d}{dx}(\ )$ denotes the operation of differentiation of $(\ )$ *with respect to* x, the symbolism $\int (\ )\,dx$ denotes the operation of integration of $(\ )$ *with respect to* x.

Differentiation and integration are fundamentally inverse operations. If $\int f(x)\,dx = F(x) + C$, then F is an antiderivative of f, that is, $F'(x) = f(x)$ and so

$$\int F'(x)\,dx = F(x) + C.$$

$$\frac{d}{dx}\int f(x)\,dx = \frac{d}{dx}(F(x) + C) = F'(x) = f(x)$$

- *An antiderivative of the derivative of a function is that function plus a constant.*
- *The derivative of an antiderivative of a function is that function.*

DIFFERENTIATION FORMULA	INTEGRATION FORMULA
1. $\dfrac{d}{dx}[x] = 1$	$\displaystyle\int dx = x + C$
2. $\dfrac{d}{dx}\left[\dfrac{x^{r+1}}{r+1}\right] = x^r \ (r \neq -1)$	$\displaystyle\int x^r\, dx = \dfrac{x^{r+1}}{r+1} + C \ (r \neq -1)$
3. $\dfrac{d}{dx}[\sin x] = \cos x$	$\displaystyle\int \cos x\, dx = \sin x + C$
4. $\dfrac{d}{dx}[-\cos x] = \sin x$	$\displaystyle\int \sin x\, dx = -\cos x + C$
5. $\dfrac{d}{dx}[\tan x] = \sec^2 x$	$\displaystyle\int \sec^2 x\, dx = \tan x + C$
6. $\dfrac{d}{dx}[-\cot x] = \csc^2 x$	$\displaystyle\int \csc^2 x\, dx = -\cot x + C$
7. $\dfrac{d}{dx}[\sec x] = \sec x \tan x$	$\displaystyle\int \sec x \tan x\, dx = \sec x + C$

DIFFERENTIATION FORMULA	INTEGRATION FORMULA
8. $\frac{d}{dx}[-\csc x] = \csc x \cot x$	$\int \csc x \cot x \, dx = -\csc x + C$
9. $\frac{d}{dx}[e^x] = e^x$	$\int e^x \, dx = e^x + C$
10. $\frac{d}{dx}\left[\frac{b^x}{\ln b}\right] = b^x \; (0 < b,\ b \neq 1)$	$\int b^x \, dx = \frac{b^x}{\ln b} + C \; (0 < b,\ b \neq 1)$
11. $\frac{d}{dx}[\ln \lvert x \rvert] = \frac{1}{x}$	$\int \frac{1}{x} \, dx = \ln \lvert x \rvert + C$
12. $\frac{d}{dx}[\tan^{-1} x] = \frac{1}{1 + x^2}$	$\int \frac{1}{1 + x^2} \, dx = \tan^{-1} x + C$
13. $\frac{d}{dx}[\sin^{-1} x] = \frac{1}{\sqrt{1 - x^2}}$	$\int \frac{1}{\sqrt{1 - x^2}} \, dx = \sin^{-1} x + C$

$\boxed{4}$ **The Substitution Rule** If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du$$

Find $\int x^3 \cos(x^4 + 2)\, dx$.

Find $\int \sqrt{1 + x^2}\, x^5\, dx$.

Evaluate $\int \frac{e^x}{\sqrt{1 - e^{2x}}}\, dx$.

Evaluate $\int \sqrt{2x + 1}\, dx$.

Calculate $\int \tan x\, dx$.

Evaluate $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$.

Find $\int \frac{x}{\sqrt{1 - 4x^2}}\, dx$.

Find $\int \frac{x}{\sqrt{x + 1}}\, dx$.

Evaluate $\int t^4 \sqrt[3]{3 - 5t^5}\, dt$.

Important Formulas

$$\int \tan u \, du = \ln |\sec u| + C$$

$$\int \cot u \, du = \ln |\sin u| + C$$

$$\int \sec u \, du = \ln |\sec u + \tan u| + C$$

$$\int \csc u \, du = \ln |\csc u - \cot u| + C$$

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a} + C, \quad a > 0$$

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$$

$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a} + C$$

$$\int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + C, n \neq -1$$

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + C$$

Integration By Parts

The Product Rule states that if f and g are differentiable functions, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

In the notation for indefinite integrals this equation becomes

$$\int [f(x)g'(x) + g(x)f'(x)]\,dx = f(x)g(x)$$

or

$$\int f(x)g'(x)\,dx + \int g(x)f'(x)\,dx = f(x)g(x)$$

We can rearrange this equation as

$$\boxed{\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx}$$

is called the **formula for integration by parts**.

It is perhaps easier to remember in the following notation.

Let $u = f(x)$ and $v = g(x)$. Then the differentials are

$du = f'(x)\,dx$ and $dv = g'(x)\,dx$, so, by the Substitution Rule,

the formula for integration by parts becomes

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Guidelines For Integration By Parts

The main goal in integration by parts is to choose u and dv to obtain a new integral that is easier to evaluate than the original. In general, there are no hard and fast rules for doing this; it is mainly a matter of experience that comes from lots of practice. A strategy that often works is to choose u and dv so that u becomes "simpler" when differentiated, while leaving a dv that can be readily integrated to obtain v.

There is another useful strategy for choosing u and dv that can be applied when the integrand is a product of two functions from *different* categories in the list :

Logarithmic, Inverse trigonometric, Algebraic, Trigonometric, Exponential

In this case you will often be successful if you take u to be the function whose category occurs earlier in the list and take dv to be the rest of the integrand. The acronym LIATE will help you to remember the order. The method does not work all the time, but it works often enough to be useful.

The LIATE method is discussed in the article "A Technique for Integration by Parts," *American Mathematical Monthly*, Vol. 90, 1983, pp. 210–211, by Herbert Kasube.

Problem

Find $\int x \sin x \, dx$.

Solution

Let

$$u = x \qquad dv = \sin x \, dx$$

$$du = dx \qquad v = -\cos x$$

$$\int x \sin x \, dx$$

$$= \int \overbrace{x}^{u} \, \overbrace{\sin x \, dx}^{dv} = \overbrace{x}^{u} \, \overbrace{(-\cos x)}^{v} - \int \overbrace{(-\cos x)}^{v} \, \overbrace{dx}^{du}$$

$$= -x \cos x + \int \cos x \, dx$$

$$= -x \cos x + \sin x + C$$

Problem Evaluate $\int \ln x \, dx$.

Solution Here we don't have much choice for u and dv. Let

$$u = \ln x \qquad dv = dx$$

Then

$$du = \frac{1}{x}\,dx \qquad v = x$$

Integrating by parts, we get

$$\int \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x} \, dx$$

$$= x \ln x - \int dx$$

$$= x \ln x - x + C$$

Problem 1

Evaluate $\int e^x \sin x \, dx$.

Problem 2

Find $\int t^2 e^t \, dt$.

A Tabular Method For Repeated Integration By Parts

Integrals of the form

$$\int p(x)\,f(x)\,dx$$

where $p(x)$ is a polynomial, can sometimes be evaluated using repeated integration by parts.

Tabular Integration by Parts

Step 1. Differentiate $p(x)$ repeatedly until you obtain 0, and list the results in the first column.

Step 2. Integrate $f(x)$ repeatedly and list the results in the second column.

Step 3. Draw an arrow from each entry in the first column to the entry that is one row down in the second column.

Step 4. Label the arrows with alternating $+$ and $-$ signs, starting with a $+$.

Step 5. For each arrow, form the product of the expressions at its tip and tail and then multiply that product by $+1$ or -1 in accordance with the sign on the arrow. Add the results to obtain the value of the integral.

Problem

Evaluate $\int (x^2 - x)\cos x \, dx$.

Solution

REPEATED DIFFERENTIATION		REPEATED INTEGRATION
$x^2 - x$	$+$	$\cos x$
$2x - 1$	$-$	$\sin x$
2	$+$	$-\cos x$
0		$-\sin x$

$$\int (x^2 - x) \cos x \, dx$$

$$= (x^2 - x) \sin x + (2x - 1) \cos x - 2 \sin x + C$$

$$= (x^2 - x - 2) \sin x + (2x - 1) \cos x + C$$

Integration by parts can be used to derive ***reduction formulas*** for integrals. These are formulas that express an integral involving a power of a function in terms of an integral that involves a *lower* power of that function.

Problem Prove the reduction formula

$$\int \sin^n x \, dx = -\frac{1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} \int \sin^{n-2} x \, dx$$

where $n \geqslant 2$ is an integer.

Solution Let

$$u = \sin^{n-1} x \qquad dv = \sin x \, dx$$

$$du = (n-1)\sin^{n-2}x \cos x\, dx \qquad\qquad v = -\cos x$$

and integration by parts gives

$$\int \sin^n x\, dx = -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x \cos^2 x\, dx$$

Since $\cos^2 x = 1 - \sin^2 x$, we have

$$\int \sin^n x\,dx = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^n x\,dx$$

We solve this equation for the desired integral by taking the last term on the right side to the left side. Thus we have

$$n\int \sin^n x\,dx = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x\,dx$$

or

$$\boxed{\int \sin^n x\,dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\,dx}$$

★ Prove the reduction formula

$$\int \sec^n x\,dx = \frac{1}{n-1}\sec^{n-2}x\tan x + \frac{n-2}{n-1}\int \sec^{n-2}x\,dx$$

where $n \geq 2$.

Proof

Let $I = \int \sec^n x\,dx = \int \sec^{n-2}x \cdot \sec^2 x\,dx$

Take $u = \sec^{n-2}x$ & $dv = \sec^2 x\,dx$

$\Rightarrow du = (n-2)\sec^{n-3}x \cdot \sec x\tan x\,dx$

$\Rightarrow \boxed{du = (n-2)\sec^{n-2}x\tan x\,dx}$ & $\boxed{V = \tan x}$

So $I = \sec^{n-2}x \tan x - (n-2)\int \sec^{n-2}x \tan^2 x\, dx$

$$= \sec^{n-2}x \tan x - (n-2)\int \sec^{n-2}x\,(\sec^2 x - 1)\, dx$$

$$= \sec^{n-2}x \tan x - (n-2)\int \sec^{n}x\, dx + (n-2)\int \sec^{n-2}x\, dx$$

$$\Rightarrow I = \sec^{n-2}x \tan x - (n-2)I + (n-2)\int \sec^{n-2}x\, dx$$

$$\Rightarrow (1+n-2)I = \sec^{n-2}x \tan x + (n-2)\int \sec^{n-2}x\, dx$$

$$\Rightarrow (n-1)I = \sec^{n-2}x \tan x + (n-2)\int \sec^{n-2}x\, dx$$

$$\Rightarrow I = \frac{1}{n-1}\sec^{n-2}x \tan x + \frac{n-2}{n-1}\int \sec^{n-2}x\, dx$$

Thus

$$\boxed{\int \sec^{n}x\, dx = \frac{1}{n-1}\sec^{n-2}x \tan x + \frac{n-2}{n-1}\int \sec^{n-2}x\, dx}$$

Proved

$$\int \cot^5 x\,dx = \int \cot^3 x \cdot \cot^2 x\,dx = \int \cot^3 x \cdot (\csc^2 x - 1)\,dx$$

$$= \int \cot^3 x \cdot \csc^2 x\,dx - \int \cot^3 x\,dx$$

$$= -\frac{\cot^4 x}{4} - \int \cot x \cdot \cot^2 x\,dx$$

$$= -\frac{\cot x}{4} - \int \cot x \cdot (\csc^2 x - 1)\,dx$$

$$= \frac{-\cot x}{4} - \int \cot x \csc^2 x\,dx + \int \cot x\,dx$$

$$\Rightarrow \boxed{\int \cot^5 x\,dx = \frac{-\cot^4 x}{4} + \frac{\cot^2 x}{2} + \ln|\sin x| + C} \text{ Ans}$$

Practice Problem 1

Prove the reduction formula

$$\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx$$

where $n \geqslant 2$ is an integer.

Practice Problem 2 Evaluate the integral.

$$\int (1 + x^2)\, e^{3x}\, dx$$

$$\int \frac{z}{10^z}\, dz$$

$$\int \ln \sqrt{x}\, dx$$

$$\int \csc^5 x\, dx$$

Hint: Write the power as 3+2.

$$\int t^4 \ln t\, dt$$

$$\int \cos^5 x\, dx$$

Hint: Reduction Formula

Strategy for Evaluating $\int \sin^m x \cos^n x \, dx$

(a) If the power of cosine is odd ($n = 2k + 1$), save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of sine:

$$\int \sin^m x \cos^{2k+1} x \, dx = \int \sin^m x \, (\cos^2 x)^k \cos x \, dx$$

$$= \int \sin^m x \, (1 - \sin^2 x)^k \cos x \, dx$$

Then substitute $u = \sin x$.

(b) If the power of sine is odd ($m = 2k + 1$), save one sine factor and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of cosine:

$$\int \sin^{2k+1}x \cos^n x\, dx = \int (\sin^2 x)^k \cos^n x \, \sin x \, dx$$

$$= \int (1 - \cos^2 x)^k \cos^n x \, \sin x \, dx$$

Then substitute $u = \cos x$.

(c) If the powers of both sine and cosine are even, use the half-angle identities

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x) \qquad\qquad \cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$

It is sometimes helpful to use the identity

$$\sin x \cos x = \tfrac{1}{2}\sin 2x$$

Problem

Evaluate $$\int \sin^4 x \cos^5 x \, dx$$

Solution

$$\int \sin^4 x \cos^5 x \, dx = \int \sin^4 x \cos^4 x \cos x \, dx$$

$$\int \sin^4 x \cos^5 x \, dx = \int \sin^4 x \cos^4 x \cos x \, dx$$

$$= \int \sin^4 x (1 - \sin^2 x)^2 \cos x \, dx$$

$$= \tfrac{1}{5} \sin^5 x - \tfrac{2}{7} \sin^7 x + \tfrac{1}{9} \sin^9 x + C$$

Problem

Evaluate $\displaystyle\int \sin^5 x \cos^2 x \, dx.$

Solution

$$\sin^5 x \cos^2 x = (\sin^2 x)^2 \cos^2 x \sin x = (1 - \cos^2 x)^2 \cos^2 x \sin x$$

Substituting $u = \cos x$, we have $du = -\sin x \, dx$ and so

$$\int \sin^5 x \cos^2 x \, dx = \int (\sin^2 x)^2 \cos^2 x \sin x \, dx$$

$$= \int (1 - \cos^2 x)^2 \cos^2 x \sin x \, dx$$

$$= \int (1 - u^2)^2 u^2 (-du) = -\int (u^2 - 2u^4 + u^6) \, du$$

$$= -\left(\frac{u^3}{3} - 2 \frac{u^5}{5} + \frac{u^7}{7} \right) + C$$

$$\int \sin^5 x \cos^2 x \, dx =$$

$$= -\tfrac{1}{3}\cos^3 x + \tfrac{2}{5}\cos^5 x - \tfrac{1}{7}\cos^7 x + C$$

$$\int \sin^{11} x \cos^{8} x \, dx$$

$$= \int \sin^{10} x \, \underbrace{\sin x} \, \cos^{8} x \, dx$$

$$\boxed{\sin^2 x = 1 - \cos^2 x}$$

$$= \int (\sin^2 x)^5 \cos^8 x \cdot \sin x \, dx$$

$$= \int (1 - \cos^2 x)^5 \cdot \cos^8 x \cdot \sin x \, dx$$

Let $u = \cos x$ $\quad du = -\sin x \, dx$

$$\int \sin^{11} x \cos^{8} x \, dx = -\int (1 - u^2)^5 \cdot u^8 \cdot du$$

$$(1-u^2)^5 = \left(1 + (-u^2)\right)^5$$

$$= 1^5 + 5\cdot 1^4(-u^2) + \frac{5\cdot 4}{2!}\cdot 1^3\cdot(-u^2)^2$$

$$+ \frac{5\cdot 4\cdot 3}{3!}\, 1^2 (-u^2)^3 + \frac{5\cdot 4\cdot 3\cdot 2}{4!}\, 1 (-u^2)^4$$

$$+ (-u^2)^5$$

$$\Rightarrow \boxed{(1-u^2)^5 = 1 - 5u^2 + 10u^4 - 10u^6 + 5u^8 - u^{10}}$$

By ^using^ Binomial Theorem

$$(x+y)^n = x^n + n x^{n-1} y + \frac{n(n-1)}{2!} x^{n-2} y^2 + \dots\ y^n$$

$$\int \sin^{11} x \cos^8 x\, dx$$

$$= -\int (1 - 5u^2 + 10u^4 - 10u^6 + 5u^8 - u^{10}) \cdot u^8 du$$

$$= -\int (u^8 - 5u^{10} + 10u^{12} - 10u^{14} + 5u^{16} - u^{18}) du$$

$$= -\left[\frac{u^9}{9} - 5\frac{u^{11}}{11} + 10\frac{u^{13}}{13} - 10\frac{u^{15}}{15} + 5\frac{u^{17}}{17} - \frac{u^{19}}{19} \right] + C$$

$$\int \sin^{11}x \cos^{8}x \, dx =$$

$$-\left[\frac{\cos^9 x}{9} - \frac{5}{11}\cos^{11}x + \frac{10}{13}\cos^{13}x - \frac{10}{15}\cos^{15}x + \frac{5}{17}\cos^{17}x - \frac{1}{19}\cos^{19}x \right] + C$$

Problem

Evaluate $\displaystyle\int \sin^4 x \cos^4 x \, dx$

Solution

$$\int \sin^4 x \cos^4 x \, dx = \int (\sin^2 x)^2 (\cos^2 x)^2 \, dx$$

$$\int \sin^4 x \cos^4 x \, dx$$

$$= \int \left(\tfrac{1}{2}[1 - \cos 2x]\right)^2 \left(\tfrac{1}{2}[1 + \cos 2x]\right)^2 \, dx$$

$$= \frac{1}{16} \int (1 - \cos^2 2x)^2 \, dx = \frac{1}{16} \int \sin^4 2x \, dx$$

$$= \frac{1}{32} \int \sin^4 u \, du$$

$$\boxed{\begin{aligned} u &= 2x \\ du &= 2\,dx \text{ or } dx = \tfrac{1}{2}\,du \end{aligned}}$$

$$= \frac{1}{32}\left(\frac{3}{8}u - \frac{1}{4}\sin 2u + \frac{1}{32}\sin 4u\right) + C$$

$$= \frac{3}{128}x - \frac{1}{128}\sin 4x + \frac{1}{1024}\sin 8x + C$$

Strategy for Evaluating $\int \tan^m x \sec^n x \, dx$

(a) If the power of secant is even ($n = 2k$, $k \geqslant 2$), save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to express the remaining factors in terms of $\tan x$:

$$\int \tan^m x \sec^{2k} x \, dx = \int \tan^m x \, (\sec^2 x)^{k-1} \sec^2 x \, dx$$

$$= \int \tan^m x \, (1 + \tan^2 x)^{k-1} \sec^2 x \, dx$$

Then substitute $u = \tan x$.

(b) If the power of tangent is odd ($m = 2k + 1$), save a factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to express the remaining factors in terms of $\sec x$:

$$\int \tan^{2k+1}x \, \sec^n x \, dx = \int (\tan^2 x)^k \, \sec^{n-1}x \, \sec x \, \tan x \, dx$$

$$= \int (\sec^2 x - 1)^k \, \sec^{n-1}x \, \sec x \, \tan x \, dx$$

Then substitute $u = \sec x$.

For other cases, the guidelines are not as clear-cut. We may need to use identities, integration by parts, and occasionally a little ingenuity.

$\int \tan^m x \sec^n x \, dx$	PROCEDURE	RELEVANT IDENTITIES
$\begin{cases} m \text{ even} \\ n \text{ odd} \end{cases}$	• Use the relevant identities to reduce the integrand to powers of $\sec x$ alone. • Then use the reduction formula for powers of $\sec x$.	$\tan^2 x = \sec^2 x - 1$

Problem Evaluate $\int \tan^6 x \sec^4 x \, dx$.

(1st Case)

Solution

$$\int \tan^6 x \sec^4 x \, dx = \int \tan^6 x \sec^2 x \sec^2 x \, dx$$

$$= \int \tan^6 x \, (1 + \tan^2 x) \sec^2 x \, dx$$

Let $u = \tan x$ then $du = \sec^2 x \, dx$

$$\int \tan^6 x \sec^4 x \, dx = \int u^6 (1 + u^2) \, du = \int (u^6 + u^8) \, du$$

$$\int \tan^6 x \sec^4 x \, dx$$

$$= \frac{u^7}{7} + \frac{u^9}{9} + C = \tfrac{1}{7}\tan^7 x + \tfrac{1}{9}\tan^9 x + C$$

Problem

Find $\int \tan^5\theta \ \sec^7\theta \ d\theta$.

(2nd Case)

Solution

$$\int \tan^5\theta \ \sec^7\theta \ d\theta = \int \tan^4\theta \ \sec^6\theta \ \sec\theta \ \tan\theta \ d\theta$$

$$= \int (\sec^2\theta - 1)^2 \sec^6\theta \ \sec\theta \ \tan\theta \ d\theta$$

Let $u = \sec\theta$, so $du = \sec\theta \ \tan\theta \ d\theta$

$$\int \tan^5\theta \sec^7\theta \, d\theta = \int (u^2 - 1)^2 u^6 \, du$$

$$= \int (u^{10} - 2u^8 + u^6) \, du$$

$$= \frac{u^{11}}{11} - 2\frac{u^9}{9} + \frac{u^7}{7} + C$$

$$= \tfrac{1}{11}\sec^{11}\theta - \tfrac{2}{9}\sec^9\theta + \tfrac{1}{7}\sec^7\theta + C$$

Problem Evaluate

$$\int \tan^2 x \sec x \, dx$$

(3rd case)

Solution

$$\int \tan^2 x \sec x \, dx = \int (\sec^2 x - 1) \sec x \, dx$$

$$= \int \sec^3 x \, dx - \int \sec x \, dx$$

We know that

$$\int \sec^n x \, dx = \frac{1}{n-1} \sec^{n-2} x \tan x + \left(\frac{n-2}{n-1}\right) \int \sec^{n-2} x \, dx$$

Therefore

$$\int \sec^3 x\,dx = \frac{1}{2}\sec x\tan x + \frac{1}{2}\int \sec x\,dx$$

$$= \frac{1}{2}\sec x\tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$$

Thus

$$\int \tan^2 x \sec x\,dx =$$

$$= \frac{1}{2}\sec x\tan x + \frac{1}{2}\ln|\sec x + \tan x| - \ln|\sec x + \tan x| + C$$

$$= \frac{1}{2}\sec x\tan x - \frac{1}{2}\ln|\sec x + \tan x| + C$$ Ans

Strategy for Evaluating $\int \cot^m x \csc^n x \, dx$

If the power of $\csc x$ is even save a factor of $\csc^2 x$ and use $\csc^2 x = 1 + \cot^2 x$ to express the remaining factors in terms of $\cot x$ and then substitute $u = \cot x$.

If the power of $\cot x$ is odd save a factor of $\csc x \cot x$ and use $\cot^2 x = \csc^2 x - 1$ to express the remaining factors in terms of $\csc x$ and then substitute $u = \csc x$.

Evaluate $\int \cot^6 x \csc^4 x\, dx$

Solution

$$\int \cot^6 x \cdot \csc^4 x\, dx = \int \cot^6 x \cdot \csc^2 x \cdot \csc^2 x\, dx$$

$$= \int \cot^6 x \left(1+\cot^2 x\right) \cdot \csc^2 x\, dx$$

Let $u = \cot x$

$\Rightarrow du = -\csc^2 x\, dx$

$\Rightarrow -du = +\csc^2 x\, dx$

So

$$\int \cot^6 x\,(1+\cot^2 x)\cdot \csc^2 x\,dx$$

$$= -\int u^6(1+u^2)\,du = -\int (u^6+u^8)\,du$$

$$= -\left(\frac{u^7}{7}+\frac{u^9}{9}\right)+C = -\left(\frac{\cot^7 x}{7}+\frac{\cot^9 x}{9}\right)+C$$

Thus

$$\int \cot^2 x\cdot \csc^4 x\,dx = -\left(\frac{\cot^7 x}{7}+\frac{\cot^9 x}{9}\right)+C$$

$\boxed{2}$ To evaluate the integrals (a) $\int \sin mx \cos nx \, dx$, (b) $\int \sin mx \sin nx \, dx$, or (c) $\int \cos mx \cos nx \, dx$, use the corresponding identity:

$$\text{(a)} \quad \sin A \cos B = \tfrac{1}{2}[\sin(A - B) + \sin(A + B)]$$

$$\text{(b)} \quad \sin A \sin B = \tfrac{1}{2}[\cos(A - B) - \cos(A + B)]$$

$$\text{(c)} \quad \cos A \cos B = \tfrac{1}{2}[\cos(A - B) + \cos(A + B)]$$

$$
\begin{aligned}
\int \sin 4x \cos 5x \, dx &= \int \tfrac{1}{2}[\sin(-x) + \sin 9x] \, dx \\
&= \tfrac{1}{2} \int (-\sin x + \sin 9x) \, dx \\
&= \tfrac{1}{2}\left(\cos x - \tfrac{1}{9} \cos 9x\right) + C
\end{aligned}
$$

Practice Problems

$$\int \cos^6 y \sin^3 y \, dy$$

$$\int \cos^3(t/2) \sin^2(t/2) \, dt$$

$$\int \csc^5\theta \, \cos^3\theta \, d\theta$$

$$\int \tan^3 x \, \sec^6 x \, dx$$

$$\int \frac{1 - \tan^2 x}{\sec^2 x} \, dx$$

$$\int \sin 2\theta \sin 6\theta \, d\theta$$

Trigonometric Substitution

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a \sin\theta, \quad -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a \tan\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a \sec\theta, \quad 0 \leqslant \theta < \frac{\pi}{2}$ or $\pi \leqslant \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

Problem

$$\text{Evaluate } \int \frac{\sqrt{9 - x^2}}{x^2}\, dx.$$

Solution

Let $x = 3 \sin \theta$, where $-\pi/2 \leqslant \theta \leqslant \pi/2$.

Then $dx = 3 \cos \theta \, d\theta$ and

$$\sqrt{9 - x^2} = \sqrt{9 - 9\sin^2\theta} = \sqrt{9\cos^2\theta} = 3\left|\cos\theta\right| = 3\cos\theta$$

(Note that $\cos\theta \geqslant 0$ because $-\pi/2 \leqslant \theta \leqslant \pi/2$.)

$$\int \frac{\sqrt{9 - x^2}}{x^2}\,dx = \int \frac{3\cos\theta}{9\sin^2\theta}\,3\cos\theta\,d\theta$$

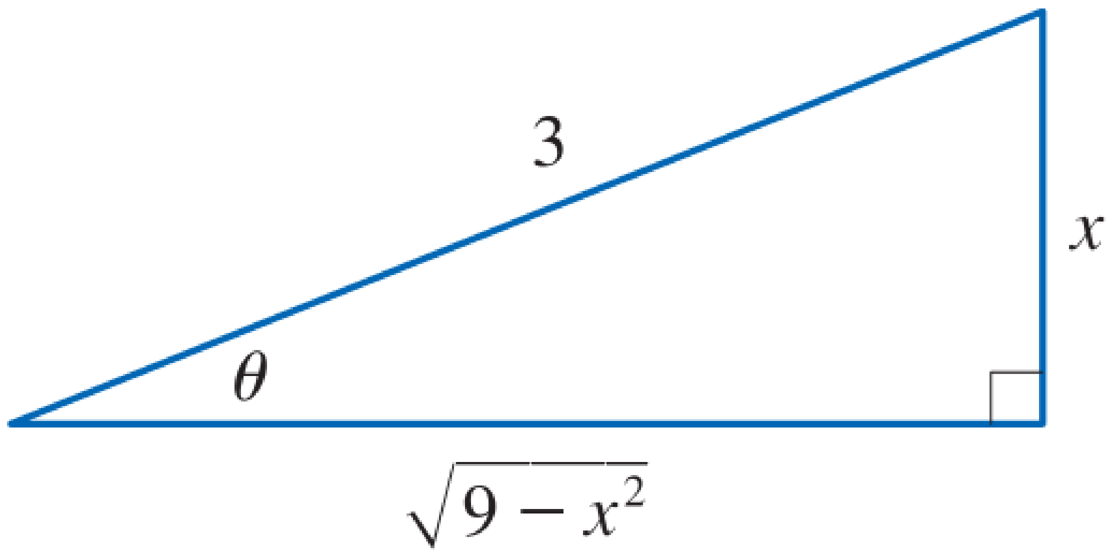
$$= \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta = \int \cot^2\theta\,d\theta$$

$$\int \frac{\sqrt{9 - x^2}}{x^2}\,dx = \int (\csc^2\theta - 1)\,d\theta$$

$$= -\cot\theta - \theta + C$$

Because this is an indefinite integral, we must return to the original variable x. This can be done either by using trigonometric identities to express $\cot\theta$ in terms of $\sin\theta = x/3$ or by drawing a diagram, as in Figure 1, where θ is interpreted as an angle of a right triangle. Since $\sin\theta = x/3$, we label the opposite side and the hypotenuse as having lengths x and 3. Then the Pythagorean Theorem gives the length of the adjacent side as $\sqrt{9 - x^2}$, so we can simply read the value of $\cot\theta$ from the figure:

Figure 1



$$\cot \theta = \frac{\sqrt{9 - x^2}}{x}$$

$$\int \frac{\sqrt{9 - x^2}}{x^2}\, dx = -\cot\theta - \theta + C = -\frac{\sqrt{9 - x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C$$

Problem

Find $\displaystyle\int \frac{1}{x^2\sqrt{x^2+4}}\,dx.$

Solution

Let $x = 2\tan\theta$, $-\pi/2 < \theta < \pi/2$. Then $dx = 2\sec^2\theta\, d\theta$ and

$$\sqrt{x^2+4} = \sqrt{4(\tan^2\theta + 1)} = \sqrt{4\sec^2\theta} = 2|\sec\theta| = 2\sec\theta$$

So we have

$$\int \frac{dx}{x^2\sqrt{x^2+4}} = \int \frac{2\sec^2\theta\, d\theta}{4\tan^2\theta \cdot 2\sec\theta} = \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\, d\theta$$

To evaluate this trigonometric integral we put everything in terms of $\sin\theta$ and $\cos\theta$:

$$\frac{\sec\theta}{\tan^2\theta} = \frac{1}{\cos\theta} \cdot \frac{\cos^2\theta}{\sin^2\theta} = \frac{\cos\theta}{\sin^2\theta}$$

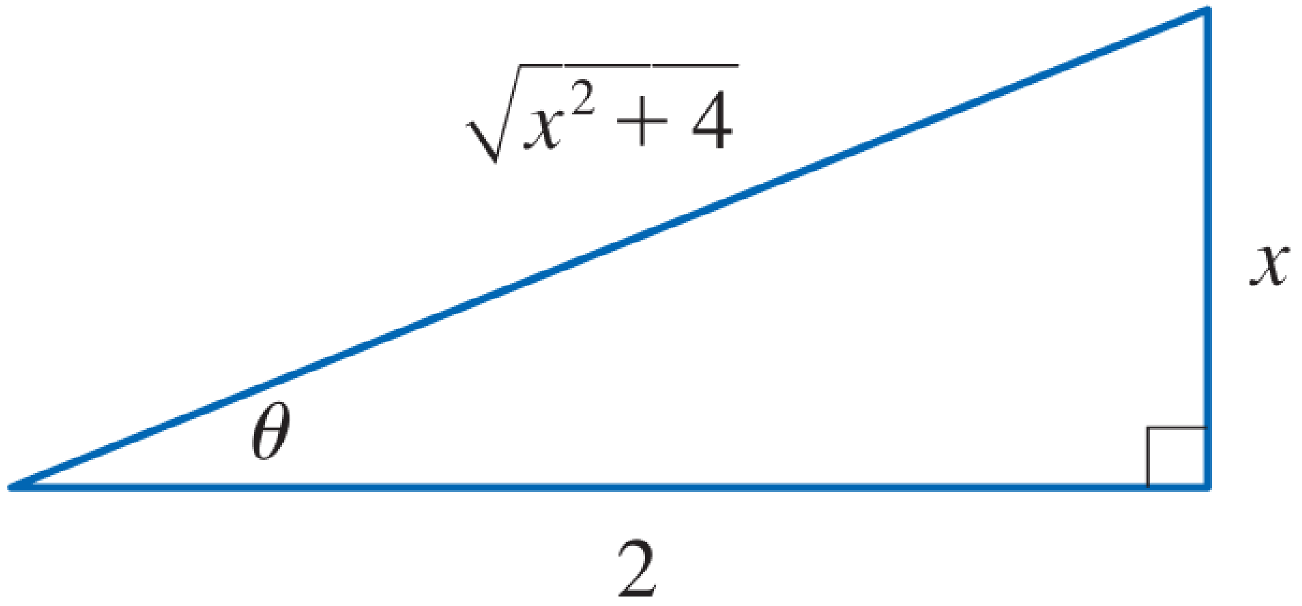
Therefore, making the substitution $u = \sin\theta$, we have

$$\int \frac{dx}{x^2\sqrt{x^2 + 4}} = \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \frac{1}{4}\int \frac{du}{u^2}$$

$$= \frac{1}{4}\left(-\frac{1}{u}\right) + C = -\frac{1}{4\sin\theta} + C$$

$$= -\frac{\csc\theta}{4} + C$$

We use Figure 2 to determine that $\csc \theta = \sqrt{x^2 + 4}/x$ and so



$$\tan \theta = \frac{x}{2}$$

Figure 2

$$\int \frac{dx}{x^2\sqrt{x^2 + 4}} = -\frac{\sqrt{x^2 + 4}}{4x} + C$$

$\boxed{\text{Ans.}}$

Practice Problems

Evaluate the integral.

$$\int \frac{x^2}{\sqrt{9 - x^2}}\, dx$$

$$\int \frac{dx}{\sqrt{x^2 + 2x + 5}}$$

$$\int x\sqrt{1 - x^4}\, dx$$

Integration of Rational Functions by Partial Fraction

To illustrate the method, observe that by taking the fractions $2/(x - 1)$ and $1/(x + 2)$ to a common denominator we obtain

$$\frac{2}{x-1} - \frac{1}{x+2} = \frac{2(x+2) - (x-1)}{(x-1)(x+2)} = \frac{x+5}{x^2+x-2}$$

If we now reverse the procedure, we see how to integrate the function on the right side of this equation:

$$\int \frac{x+5}{x^2+x-2}\,dx = \int \left(\frac{2}{x-1} - \frac{1}{x+2}\right) dx$$

$$= 2\ln|x-1| - \ln|x+2| + C$$

To see how the method of partial fractions works in general, let's consider a rational function

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials. It's possible to express f as a sum of simpler fractions provided that the degree of P is less than the degree of Q. Such a rational function is called *proper*.

If f is *improper*, that is, $\deg(P) \geqslant \deg(Q)$, then we must take the preliminary step of dividing Q into P (by long division) until a remainder $R(x)$ is obtained such that $\deg(R) < \deg(Q)$. The result is

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

where S and R are also polynomials.

Problem

$$\text{Find} \int \frac{x^3 + x}{x - 1}\, dx.$$

Solution

Since the degree of the numerator is greater than the degree of the denominator, we first perform the long division.

$$
\begin{array}{r|l}
 & \quad x^2 + x + 2 \\
\cline{2-2}
x - 1 & x^3 \qquad\quad + x \\
 & \underline{x^3 - x^2} \\
 & \qquad\quad x^2 + x \\
 & \qquad\quad \underline{x^2 - x} \\
 & \qquad\qquad 2x \\
 & \qquad\qquad \underline{2x - 2} \\
 & \qquad\qquad\quad 2
\end{array}
$$

$$\int \frac{x^3 + x}{x - 1}\,dx = \int \left(x^2 + x + 2 + \frac{2}{x - 1}\right) dx$$

$$= \frac{x^3}{3} + \frac{x^2}{2} + 2x + 2\ln|x - 1| + C$$

CASE I: The denominator $Q(x)$ is a product of distinct linear factors

This means that we can write

$$Q(x) = (a_1x + b_1)(a_2x + b_2)\cdots(a_kx + b_k)$$

where no factor is repeated (and no factor is a constant multiple of another). In this case the partial fraction theorem states that there exist constants $A_1, A_2, \ldots, A_k$ such that

$$\boxed{2} \qquad \frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_k}{a_kx + b_k}$$

Problem

$$\text{Evaluate} \int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x}\, dx.$$

Solution Since the degree of the numerator is less than the degree of the denominator, we don't need to divide. We factor the denominator as

$$2x^3 + 3x^2 - 2x = x(2x^2 + 3x - 2) = x(2x - 1)(x + 2)$$

Since the denominator has three distinct linear factors, the partial fraction decomposition of the integrand has the form

$$\frac{x^2 + 2x - 1}{x(2x - 1)(x + 2)} = \frac{A}{x} + \frac{B}{2x - 1} + \frac{C}{x + 2}$$

To determine the values of A, B, and C, we multiply both sides of this equation by the least common denominator, $x(2x - 1)(x + 2)$, obtaining

$$\boxed{4} \qquad x^2 + 2x - 1 = A(2x - 1)(x + 2) + Bx(x + 2) + Cx(2x - 1)$$

Expanding the right side of Equation 4 and writing it in the standard form for polynomials, we get

5

$$x^2 + 2x - 1 = (2A + B + 2C)x^2 + (3A + 2B - C)x - 2A$$

$$\begin{aligned} 2A + B + 2C &= 1 \\ 3A + 2B - C &= 2 \\ -2A &= -1 \end{aligned}$$

Solving, we get $A = \frac{1}{2}$, $B = \frac{1}{5}$, and $C = -\frac{1}{10}$, and so

$$\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x}\, dx = \int \left(\frac{1}{2}\frac{1}{x} + \frac{1}{5}\frac{1}{2x - 1} - \frac{1}{10}\frac{1}{x + 2} \right) dx$$

$$= \tfrac{1}{2}\ln|x| + \tfrac{1}{10}\ln|2x - 1| - \tfrac{1}{10}\ln|x + 2| + K$$

Practice Problem

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x - a}{x + a} \right| + C$$

CASE II: The denominator $Q(x)$ is a product of linear factors, some of which are repeated

Suppose the first linear factor $(a_1x + b_1)$ is repeated r times; that is, $(a_1x + b_1)^r$ occurs in the factorization of $Q(x)$. Then instead of the single term $A_1/(a_1x + b_1)$ we will use

$$\frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_1x + b_1)^2} + \cdots + \frac{A_r}{(a_1x + b_1)^r}$$

By way of illustration, we could write

$$\frac{x^3 - x + 1}{x^2(x-1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3}$$

Problem

$$\text{Find} \int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\, dx.$$

Solution

The first step is to divide. The result of long division is

$$\frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} = x + 1 + \frac{4x}{x^3 - x^2 - x + 1}$$

The second step is to factor the denominator $Q(x) = x^3 - x^2 - x + 1$. Since $Q(1) = 0$, we know that $x - 1$ is a factor and we obtain

$$x^3 - x^2 - x + 1 = (x - 1)(x^2 - 1) = (x - 1)(x - 1)(x + 1)$$

$$= (x - 1)^2(x + 1)$$

Since the linear factor $x - 1$ occurs twice, the partial fraction decomposition is

$$\frac{4x}{(x - 1)^2(x + 1)} = \frac{A}{x - 1} + \frac{B}{(x - 1)^2} + \frac{C}{x + 1}$$

Multiplying by the least common denominator, $(x-1)^2(x+1)$, we get

$$\boxed{8} \qquad \begin{aligned} 4x &= A(x-1)(x+1) + B(x+1) + C(x-1)^2 \\ &= (A+C)x^2 + (B-2C)x + (-A+B+C) \end{aligned}$$

Now we equate coefficients:

$$\begin{aligned} A \qquad\quad + \quad C &= 0 \\ B - 2C &= 4 \\ -A + B + \quad C &= 0 \end{aligned}$$

Solving, we obtain $A = 1$, $B = 2$, and $C = -1$, so

$$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\,dx = \int \left[x + 1 + \frac{1}{x - 1} + \frac{2}{(x - 1)^2} - \frac{1}{x + 1}\right] dx$$

$$= \frac{x^2}{2} + x + \ln|x - 1| - \frac{2}{x - 1} - \ln|x + 1| + K$$

$$= \frac{x^2}{2} + x - \frac{2}{x - 1} + \ln\left|\frac{x - 1}{x + 1}\right| + K$$

CASE III: $Q(x)$ contains irreducible quadratic factors, none of which is repeated

If $Q(x)$ has the factor $ax^2 + bx + c$, where $b^2 - 4ac < 0$, then, the expression for $R(x)/Q(x)$ will have a term of the form

$$\frac{Ax + B}{ax^2 + bx + c}$$

For instance, the function $f(x) = x/[(x - 2)(x^2 + 1)(x^2 + 4)]$

has a partial fraction decomposition of the form

$$\frac{x}{(x-2)(x^2+1)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{x^2+4}$$

CASE IV: $Q(x)$ contains a repeated irreducible quadratic factors

If $Q(x)$ has the factor $(ax^2 + bx + c)^r$, where $b^2 - 4ac < 0$, then instead of the single partial fraction, the sum

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_rx + B_r}{(ax^2 + bx + c)^r}$$

occurs in the partial fraction decomposition of $R(x)/Q(x)$.

Problem

Write out the form of the partial fraction decomposition of the function

$$\frac{x^3 + x^2 + 1}{x(x-1)(x^2+x+1)(x^2+1)^3}$$

Solution

$$\frac{x^3 + x^2 + 1}{x(x-1)(x^2+x+1)(x^2+1)^3}$$

$$= \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+x+1} + \frac{Ex+F}{x^2+1} + \frac{Gx+H}{(x^2+1)^2} + \frac{Ix+J}{(x^2+1)^3}$$

Rationalizing Substitutions

Some nonrational functions can be changed into rational functions by means of appropriate substitutions. In particular, when an integrand contains an expression of the form $\sqrt[n]{g(x)}$, then the substitution $u = \sqrt[n]{g(x)}$ may be effective.

Problem

Evaluate $\int \frac{\sqrt{x+4}}{x}\,dx$.

Solution

Let $u = \sqrt{x+4}$. Then $u^2 = x + 4$, so $x = u^2 - 4$ and $dx = 2u\,du$.

Therefore

$$\int \frac{\sqrt{x+4}}{x}\,dx = \int \frac{u}{u^2-4}\,2u\,du = 2\int \frac{u^2}{u^2-4}\,du = 2\int \left(1 + \frac{4}{u^2-4}\right) du$$

So $$\int \frac{\sqrt{x+4}}{x}\,dx = 2\int du + 8\int \frac{du}{u^2-4}$$

$$= 2u + 8\cdot\frac{1}{2\cdot 2}\ln\left|\frac{u-2}{u+2}\right| + C$$

$$= 2\sqrt{x+4} + 2\ln\left|\frac{\sqrt{x+4}-2}{\sqrt{x+4}+2}\right| + C$$

Practice Problems

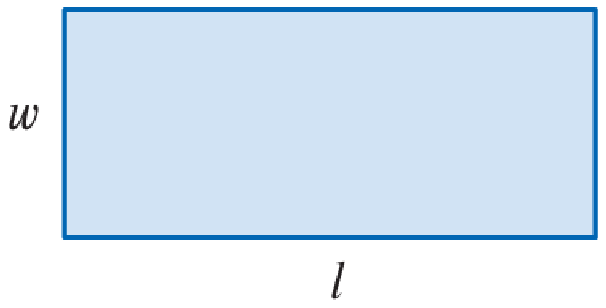
Evaluate the integral.

$$\int \frac{5}{(x-1)(x+4)}\,dx \qquad \int \frac{10}{(x-1)(x^2+9)}\,dx$$

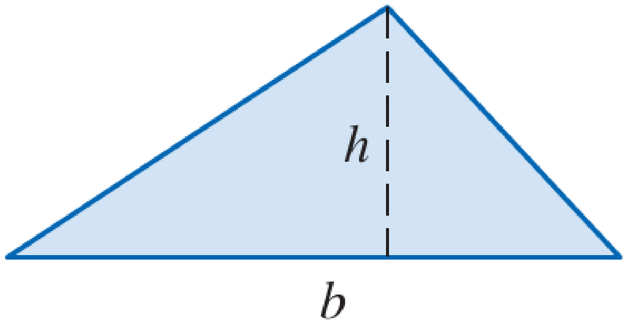
$$\int \frac{3x^2 - x + 8}{x^3 + 4x}\,dx$$

Definite Integral

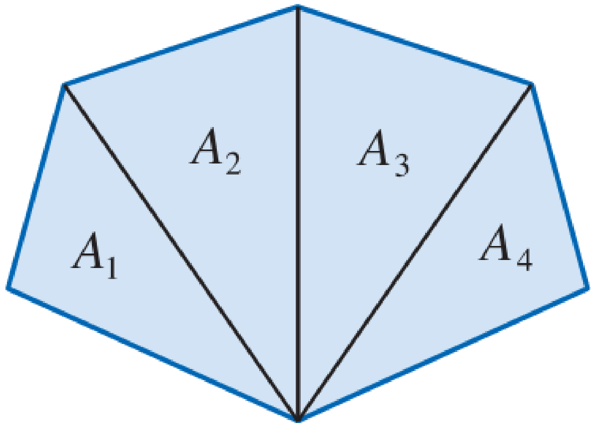
Area of Some Figures with Straight Sides



$A = lw$

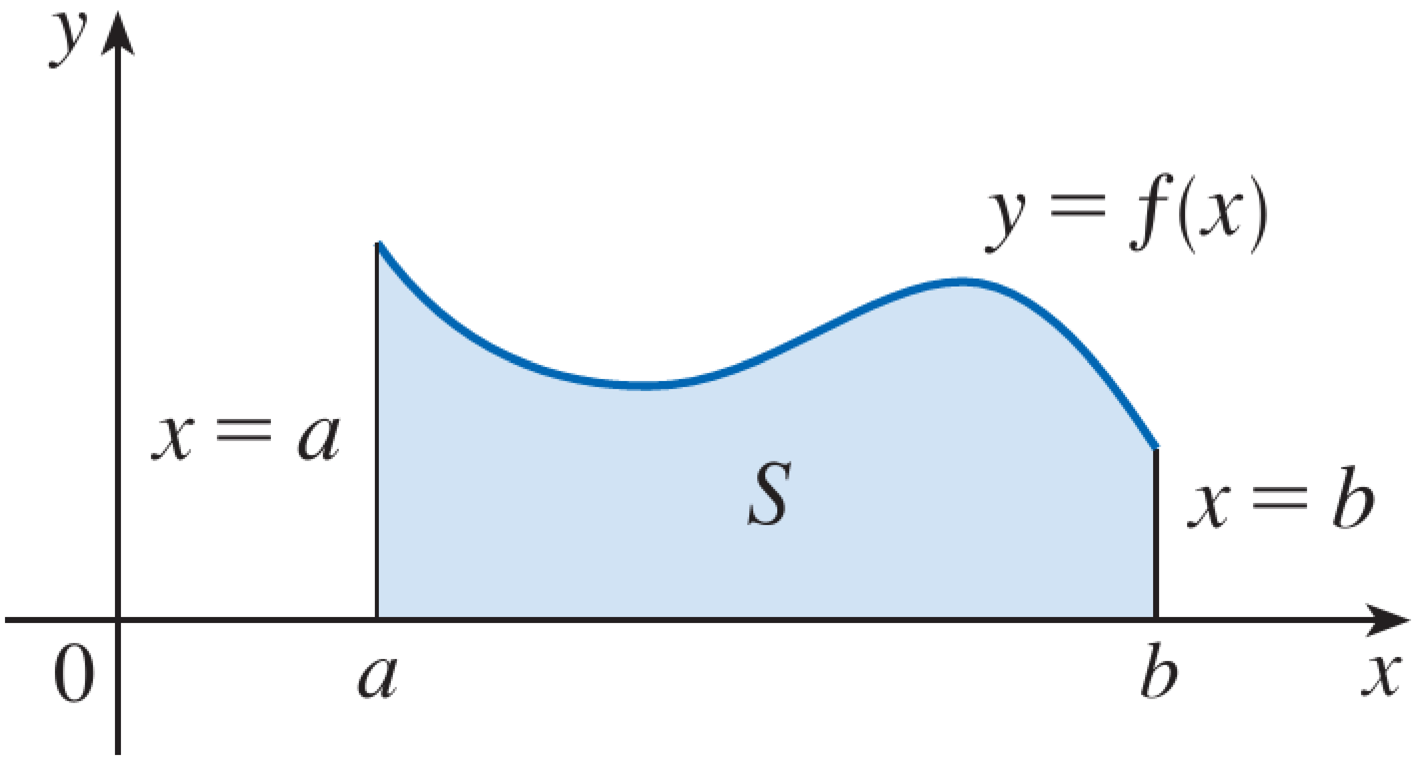


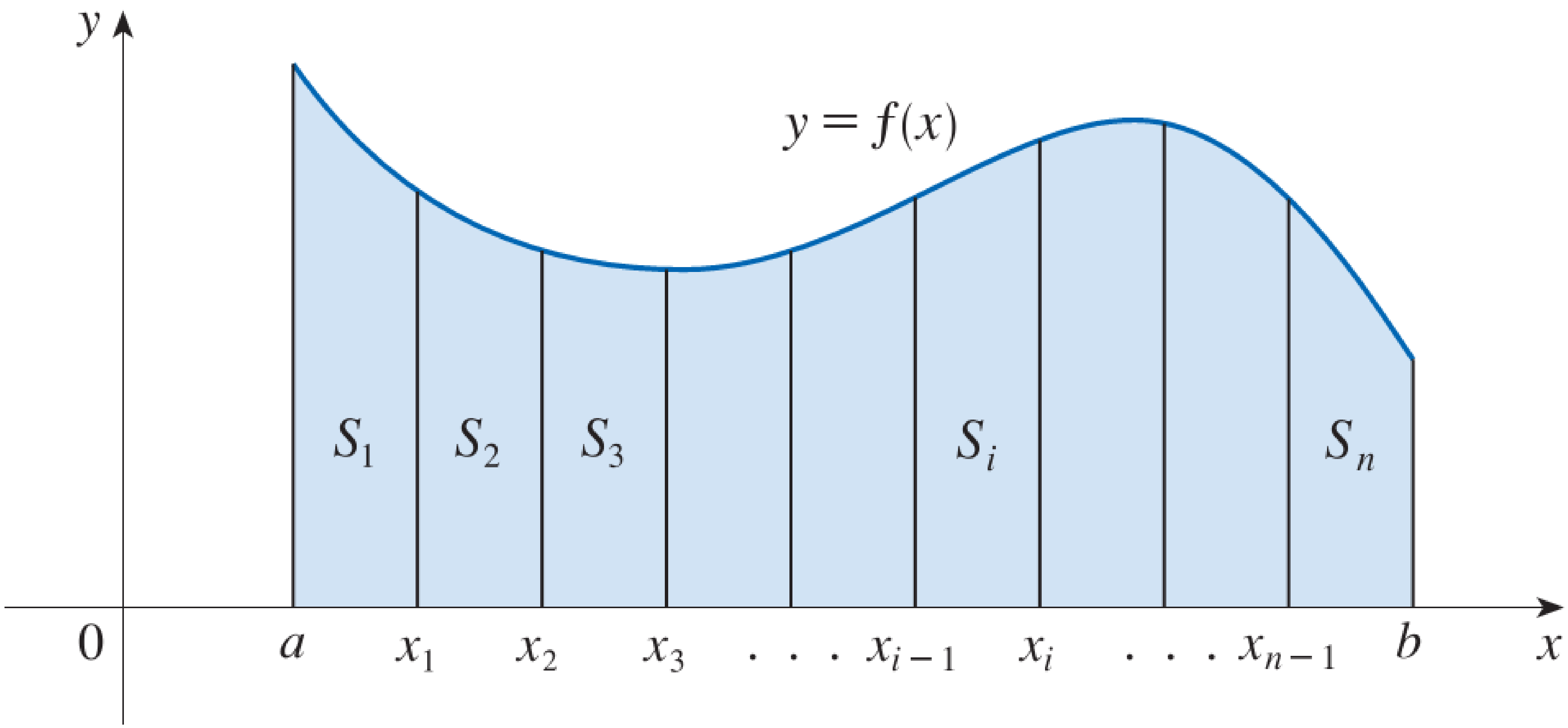
$A = \frac{1}{2} bh$



$A = A_1 + A_2 + A_3 + A_4$

The Area Problem




y
y = f(x)
S1
S2
S3
Si
Sn
0
a
x1
x2
x3
. . . xi − 1
xi
. . . xn − 1
b
x

The width of the interval $[a, b]$ is $b - a$, so the width of each of the n strips is

$$\Delta x = \frac{b - a}{n}$$

These strips divide the interval $[a, b]$ into n subintervals

$$[x_0, x_1], \quad [x_1, x_2], \quad [x_2, x_3], \quad \ldots, \quad [x_{n-1}, x_n]$$

where $x_0 = a$ and $x_n = b$.

The right endpoints of the subintervals are

$$x_1 = a + \Delta x,$$

$$x_2 = a + 2\,\Delta x,$$

$$x_3 = a + 3\,\Delta x,$$

$$\vdots$$

and, in general, $x_i = a + i\,\Delta x$. Now let's approximate the ith strip S_i by a rectangle with width Δx and height $f(x_i)$, which is the value of f at the right endpoint

Then the area of the ith rectangle is $f(x_i)\,\Delta x$. What we think of intuitively as the area of S is approximated by the sum of the areas of these rectangles, which is

$$R_n = f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x$$

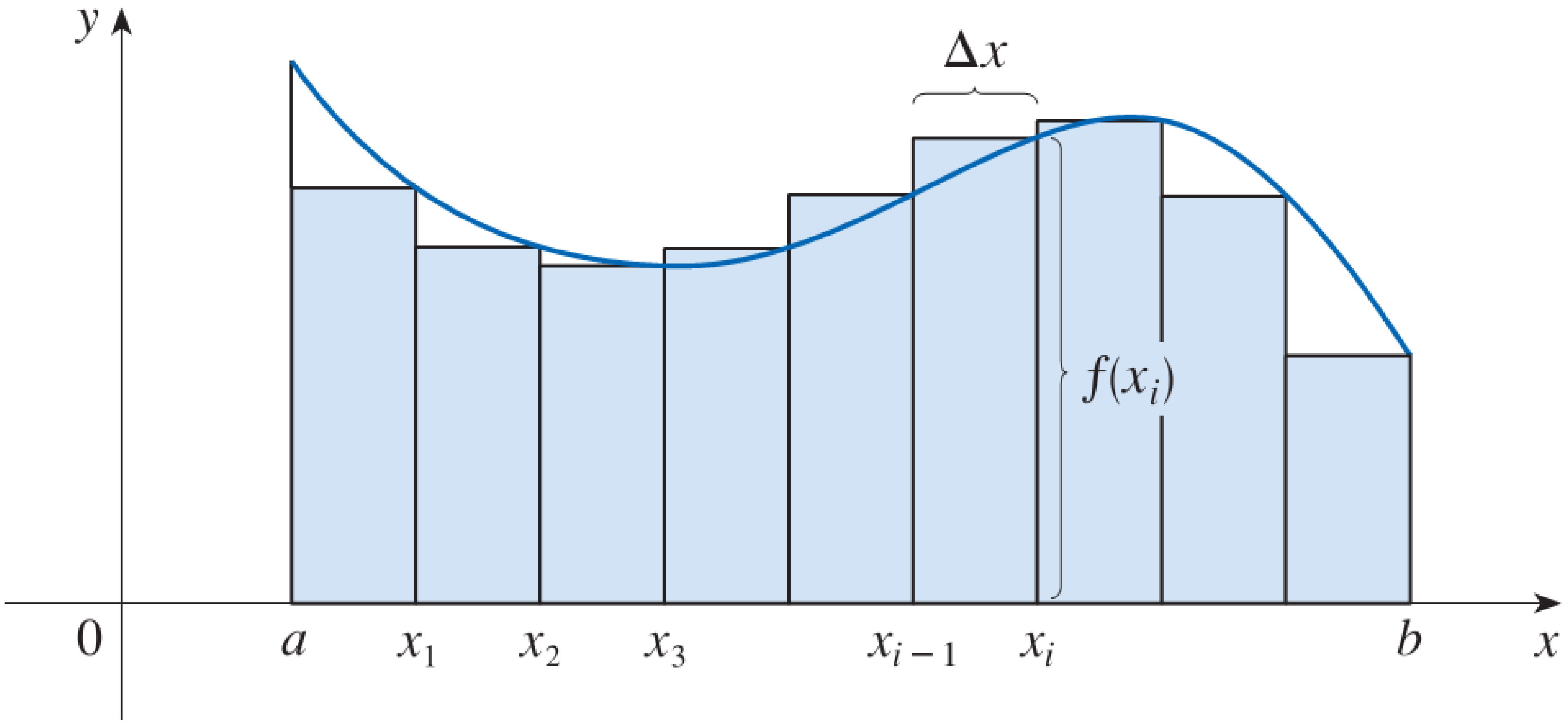
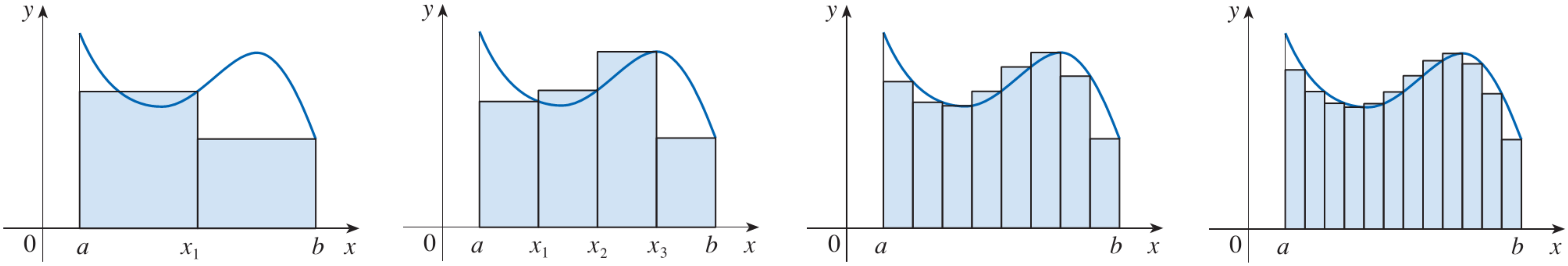


Figure 12 shows this approximation for $n = 2, 4, 8$, and 12. Notice that this approximation appears to become better and better as the number of strips increases, that is, as $n \to \infty$. Therefore we define the area A of the region S in the following way.



(a) $n = 2$ (b) $n = 4$ (c) $n = 8$ (d) $n = 12$

FIGURE 12

2 **Definition** The **area** A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} [f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x]$$

2 **Definition of a Definite Integral** If f is a function defined for $a \leqslant x \leqslant b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0 \, (= a), x_1, x_2, \ldots, x_n \, (= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any **sample points** in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the **definite integral of f from a to b** is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is **integrable** on $[a, b]$.

- It is an elongated S and was chosen because an integral is a limit of sums
- a and b are called the limits of integration; a is the lower limit and b is the upper limit.

The sum

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

that occurs in Definition 2 is called a **Riemann sum**

NOTE 4 Although we have defined $\int_a^b f(x)\,dx$ by dividing $[a, b]$ into subintervals of equal width, there are situations in which it is advantageous to work with subintervals of unequal width.

If the subinterval widths are $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$, we have to ensure that all these widths approach 0 in the limiting process. This happens if the largest width, $\max \Delta x_i$, approaches 0. So in this case the definition of a definite integral becomes

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$$

$\boxed{3}$ **Theorem** If f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a, b]$; that is, the definite integral $\int_a^b f(x)\,dx$ exists.

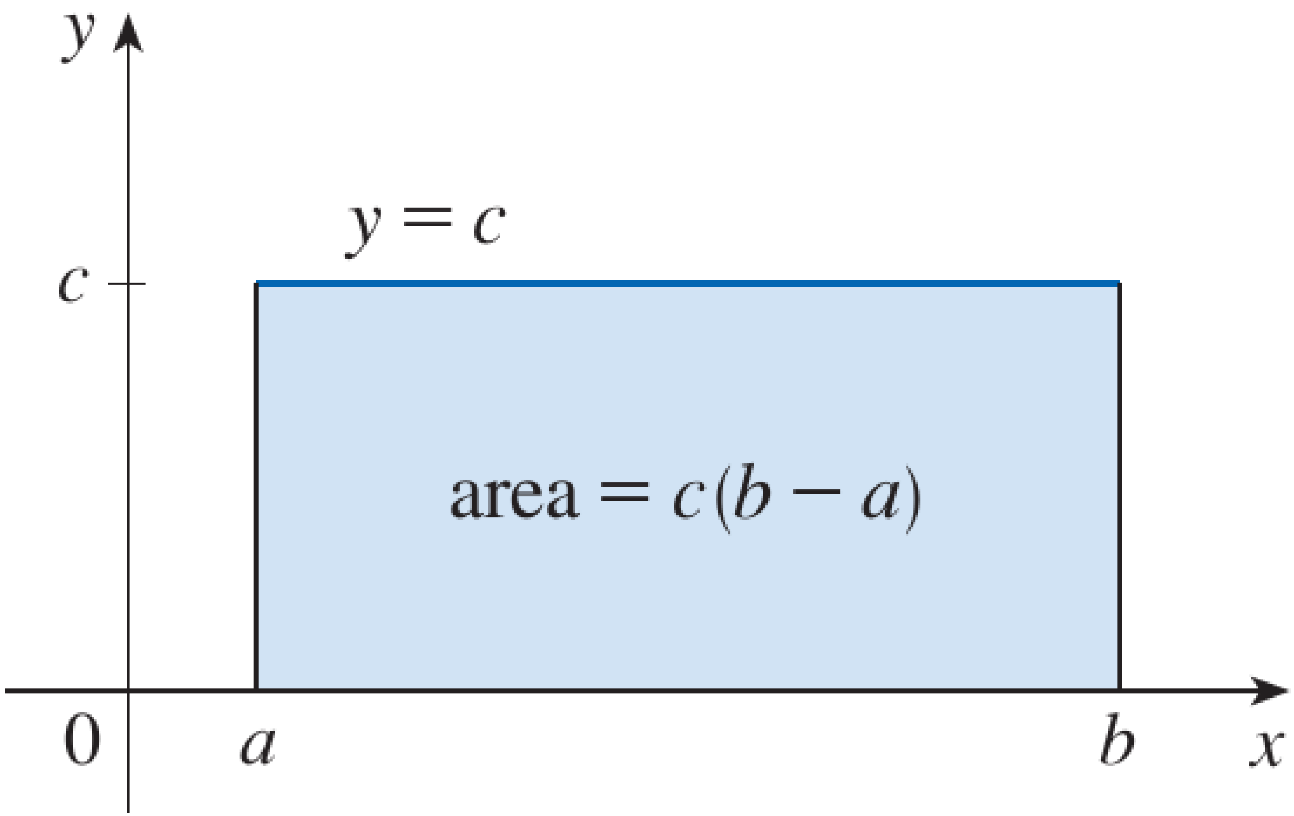
Properties of the Integral

1. $\displaystyle\int_a^b c\,dx = c(b - a),$ where c is any constant

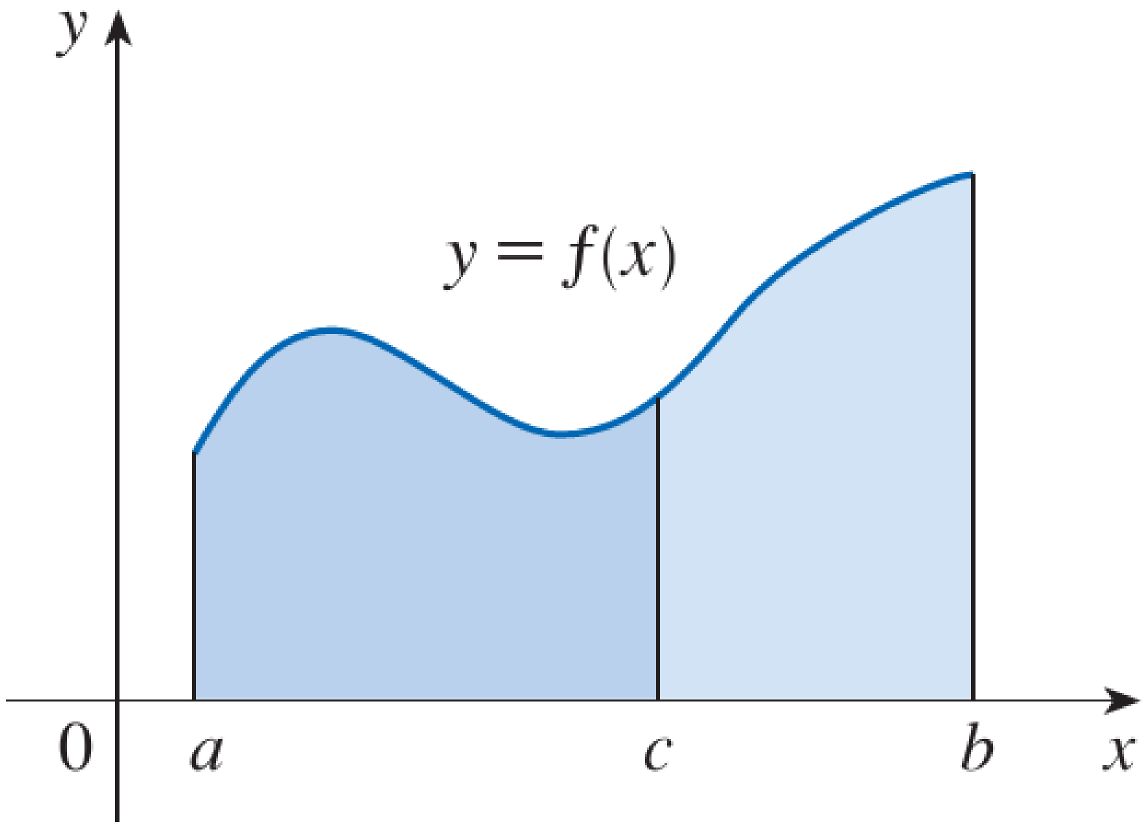
2. $\displaystyle\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

3. $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx,$ where c is any constant

4. $\displaystyle\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$


y
y = c
c
area = c(b − a)
0
a
b
x

5. $$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$



Comparison Properties of the Integral

6. If $f(x) \geqslant 0$ for $a \leqslant x \leqslant b$, then $\int_a^b f(x)\,dx \geqslant 0$.

7. If $f(x) \geqslant g(x)$ for $a \leqslant x \leqslant b$, then $\int_a^b f(x)\,dx \geqslant \int_a^b g(x)\,dx$.

8. If $m \leqslant f(x) \leqslant M$ for $a \leqslant x \leqslant b$, then

$$m(b - a) \leqslant \int_a^b f(x)\,dx \leqslant M(b - a)$$

The Fundamental Theorem of Calculus, Part 2 If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function F such that $F' = f$.

Problem

Evaluate the integral $\int_1^3 e^x\,dx$.

Solution

$$\int_1^3 e^x\,dx = F(3) - F(1) = e^3 - e$$

The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\, dt \qquad a \leqslant x \leqslant b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

Problem

Find the derivative of the function $g(x) = \int_0^x \sqrt{1 + t^2}\, dt$.

Solution

Since $f(t) = \sqrt{1 + t^2}$ is continuous, Part 1 of the Fundamental Theorem of Calculus gives

$$g'(x) = \sqrt{1 + x^2}$$

Problem

Use the Fundamental Theorem to find dy/dx if

(a) $y = \int_a^x (t^3 + 1)\, dt$

(b) $y = \int_x^5 3t \sin t\, dt$

(c) $y = \int_1^{x^2} \cos t\, dt$

(d) $y = \int_{1+3x^2}^4 \frac{1}{2 + e^t}\, dt$

Solution

(a) $$\frac{dy}{dx} = \frac{d}{dx}\int_a^x (t^3 + 1)\,dt = x^3 + 1$$

(b) $$\frac{dy}{dx} = \frac{d}{dx}\int_x^5 3t\sin t\,dt = \frac{d}{dx}\left(-\int_5^x 3t\sin t\,dt\right)$$

$$= -\frac{d}{dx}\int_5^x 3t\sin t\,dt$$

$$= -3x\sin x$$

(c) The upper limit of integration is not x but x^2. This makes y a composition of the two functions

$$y = \int_1^u \cos t\, dt \qquad \text{and} \qquad u = x^2.$$

We must therefore apply the Chain Rule to find dy/dx:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \left(\frac{d}{du} \int_1^u \cos t\, dt \right) \cdot \frac{du}{dx} \qquad = \cos u \cdot \frac{du}{dx}$$

$$= \cos(x^2) \cdot 2x$$

$$= 2x \cos x^2$$

Problem

An Integral with Variable Upper and Lower Limits

If $F(x) = \int_{2x}^{x^2} \sqrt{t^2 + 1}\, dt$, compute $F'(x)$.

Solution

$$F(x) = \int_{2x}^{0} \sqrt{t^2 + 1}\, dt + \int_{0}^{x^2} \sqrt{t^2 + 1}\, dt = -\int_{0}^{2x} \sqrt{t^2 + 1}\, dt + \int_{0}^{x^2} \sqrt{t^2 + 1}\, dt,$$

where we have also switched the limits of integration in the first integral. Using the chain rule

$$F'(x) = -\sqrt{(2x)^2 + 1}\,\frac{d}{dx}(2x) + \sqrt{(x^2)^2 + 1}\,\frac{d}{dx}(x^2)$$

$$= -2\sqrt{4x^2 + 1} + 2x\sqrt{x^4 + 1}.$$

If f is a continuous function and g and h are differentiable functions, then

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(s)\,ds = \frac{d}{dx}\Big[F\,(h(x)) - F\,(g(x))\Big]$$

$$= F'\,(h(x))\,h'(x) - F'\,(g(x))\,g'(x)$$

$$= f\,(h(x))\,h'(x) - f\,(g(x))\,g'(x).$$

For example

$$\frac{d}{dx}\int_{x^2}^{x^3}(\sin t+e^t)\,dt=\left[\sin(x^3)+e^{x^3}\right]\frac{d}{dx}(x^3)-\left[\sin(x^2)+e^{x^2}\right]\frac{d}{dx}(x^2)$$

$$=\left[\sin x^3+e^{x^3}\right](3x^2)-\left[\sin(x^2)+e^{x^2}\right](2x)$$

Similarly

$$\frac{d}{du}\int_{u}^{\sin u}e^{t^2}\,dt=e^{(\sin u)^2}\cdot\frac{d}{du}\sin u-e^{u^2}\cdot\frac{d}{du}(u)$$

$$=e^{\sin^2 u}\cdot\cos u-e^{u^2}\cdot 1$$

$$=\cos u\, e^{\sin^2 u}-e^{u^2}$$

Problem

Find the linearization of

$$f(x) = 2 - \int_{2}^{x+1} \frac{9}{1+t}\,dt$$

at $x = 1$.

Solution We know that Linearization of a function at a number $x=a$ is given by

$$L(x) = f(a) + f'(a)(x-a)$$

Here $a=1$ and $f(x) = 2 - \int_2^{x+1} \frac{9}{1+t}\,dt$

So

$$L(x) = f(1) + f'(1)(x-1) \quad \text{———} \; \star$$

Now

$$f(1) = 2 - \int_2^{1+1} \frac{9}{1+t}\,dt = 2 - \int_2^{2} \frac{9}{1+t}\,dt = 2-0 = 2$$

Also $f'(x) = 0 - \dfrac{d}{dx}\displaystyle\int_2^{x+1} \dfrac{9}{1+t}\,dt = \dfrac{9}{1+x+1}\cdot(1+0)$

$$\Rightarrow f'(x) = \frac{9}{x+2}$$

$$f'(1) = \frac{9}{1+2} = \frac{9}{3} = 3$$

FTOC Part 1 combine with chain rule was used.

Substituting values in eq* , we get

$$L(x) = 2 + 3(x-1) = 3x - 3 + 2$$

$$\Rightarrow \boxed{L(x) = 3x - 1}$$ Ans

Practice Problem 1

Use Part 1 of the Fundamental Theorem of Calculus to find the derivative of the function.

$$g(x) = \int_0^x \sqrt{t + t^3}\, dt$$

$$g(w) = \int_0^w \sin(1 + t^3)\, dt$$

Practice Problem 2

Find the derivative of the function.

$$g(x) = \int_{2x}^{3x} \frac{u^2 - 1}{u^2 + 1} \, du$$

$$g(x) = \int_{1-2x}^{1+2x} t \sin t \, dt$$

$$F(x) = \int_{x}^{x^2} e^{t^2} \, dt$$

Practice Problem 3

Find the linearization of

$$g(x) = 3 + \int_1^{x^2} \sec(t - 1)\, dt$$

at $x = -1$.

$\boxed{7}$ **Integrals of Symmetric Functions** Suppose f is continuous on $[-a, a]$.

(a) If f is even $[f(-x) = f(x)]$, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

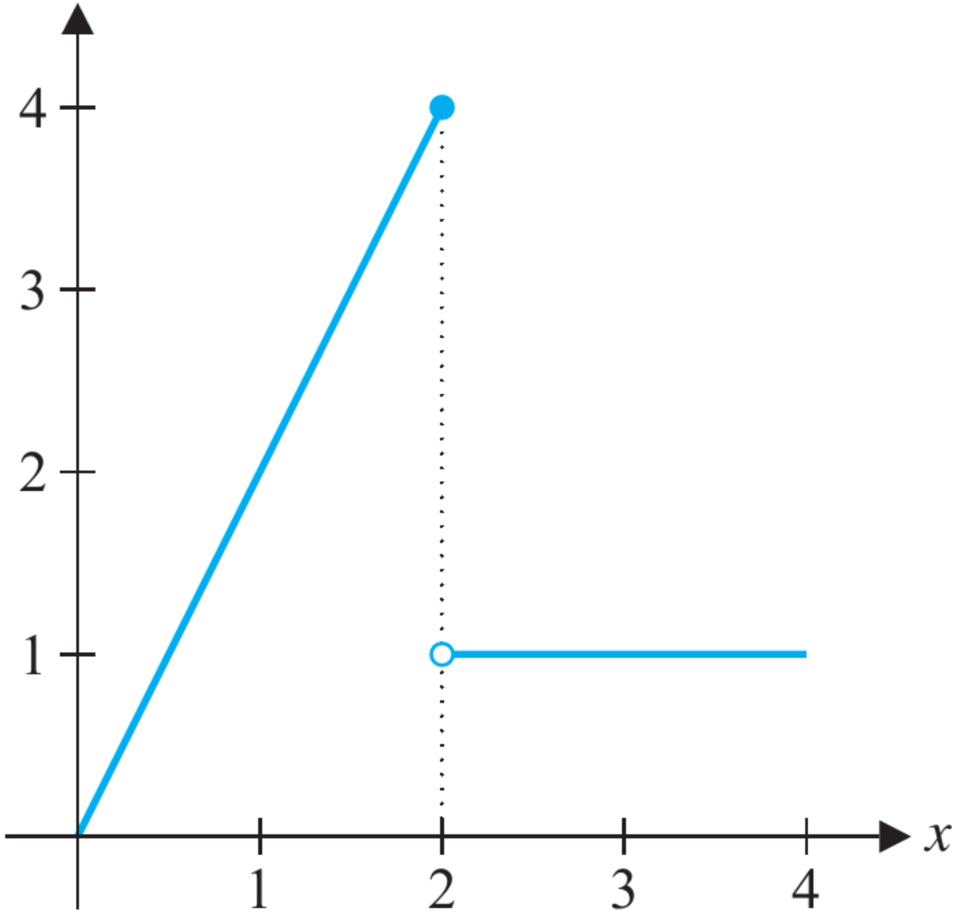
(b) If f is odd $[f(-x) = -f(x)]$, then $\int_{-a}^{a} f(x)\,dx = 0$.

Problem

Evaluate $\int_0^3 f(x)\,dx$, where $f(x)$ is defined by

$$f(x) = \begin{cases} 2x, & \text{if } x \le 2 \\ 1, & \text{if } x > 2 \end{cases}.$$

Solution



Graph of f

$$\int_0^3 f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^3 f(x)\,dx$$

$$= \int_0^2 2x \quad dx + \int_2^3 1 \quad dx$$

$$= 4 + 1 = 5$$

Problem

Evaluate $\int_0^3 |x-2|dx$

Solution

We know that

$$|x-2| = \begin{cases} x-2 & \text{if } x-2 \geq 0 \\ -(x-2) & \text{if } x-2 < 0 \end{cases}$$

OR

$$|x-2| = \begin{cases} x-2 & \text{if } x \geq 2 \\ -(x-2) & \text{if } x < 2 \end{cases}$$

So $\int_0^3 |x-2|\,dx$

$$= \int_0^2 |x-2|\,dx + \int_2^3 |x-2|\,dx$$

$$= -\int_0^2 (x-2)\,dx + \int_2^3 (x-2)\,dx$$

$$= -\left(\frac{x^2}{2}-2x\right)\Bigg|_0^2 + \left(\frac{x^2}{2}-2x\right)\Bigg|_2^3$$

$$= -\left[\left(\frac{2^2}{2}-2(2)\right)-0\right] + \left[\left(\frac{3^2}{2}-2(3)\right)-\left(\frac{2^2}{2}-2(2)\right)\right]$$

$$= -\left[2-4\right] + \left[\frac{9}{2}-6\right] - \left[2-4\right]$$

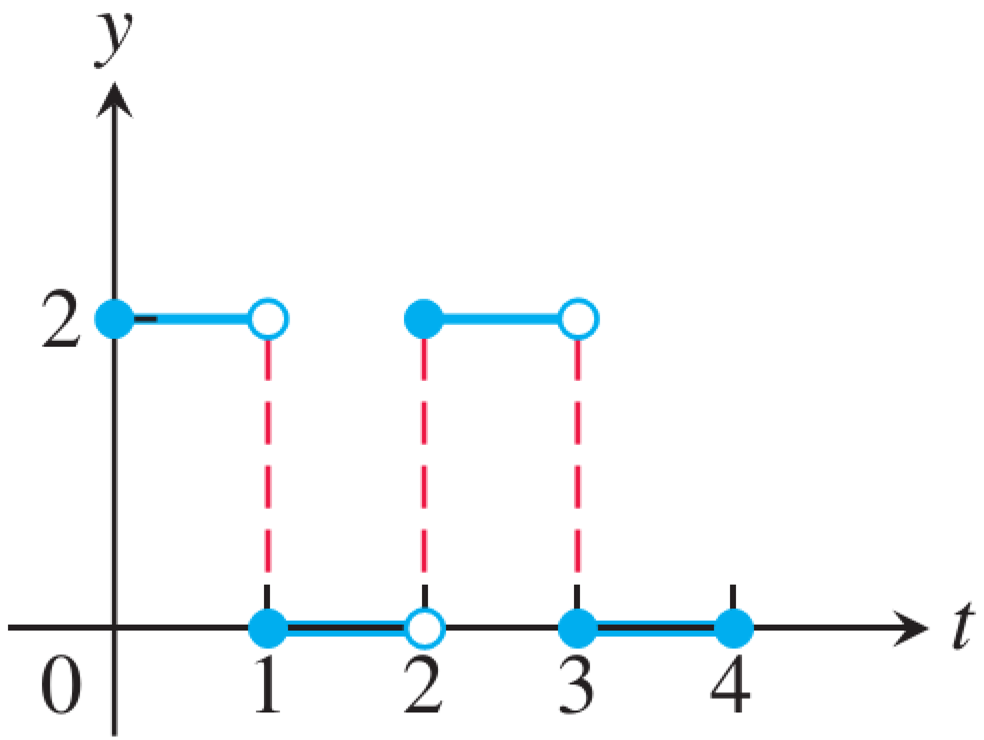
$$= 2-\frac{3}{2}+2 = 4-\frac{3}{2} = \frac{5}{2}$$

Thus

$$\boxed{\int_0^3 |x-2|\,dx = 5/2}$$ Ans

Problem

Evaluate $\int_0^4 f(x)dx$, where f is shown in the figure.



$$f(x)=\begin{cases} 2 & \text{if } 0\le x<1 \\ 0 & \text{if } 1\le x<2 \\ 2 & \text{if } 2\le x<3 \\ 0 & \text{if } 3\le x\le 4 \end{cases}$$

$$\int_0^4 f(x)\,dx = \int_0^1 2\,dx + \int_1^2 0\,dx + \int_2^3 2\,dx + \int_3^4 0\,dx$$

$$= 2x\Big|_0^1 + 0 + 2x\Big|_2^3 + 0$$

$$\Rightarrow \int_0^4 f(x)\,dx = 2(1-0) + 0 + 2(3-2) + 0$$

$$= 2 + 2 = 4$$

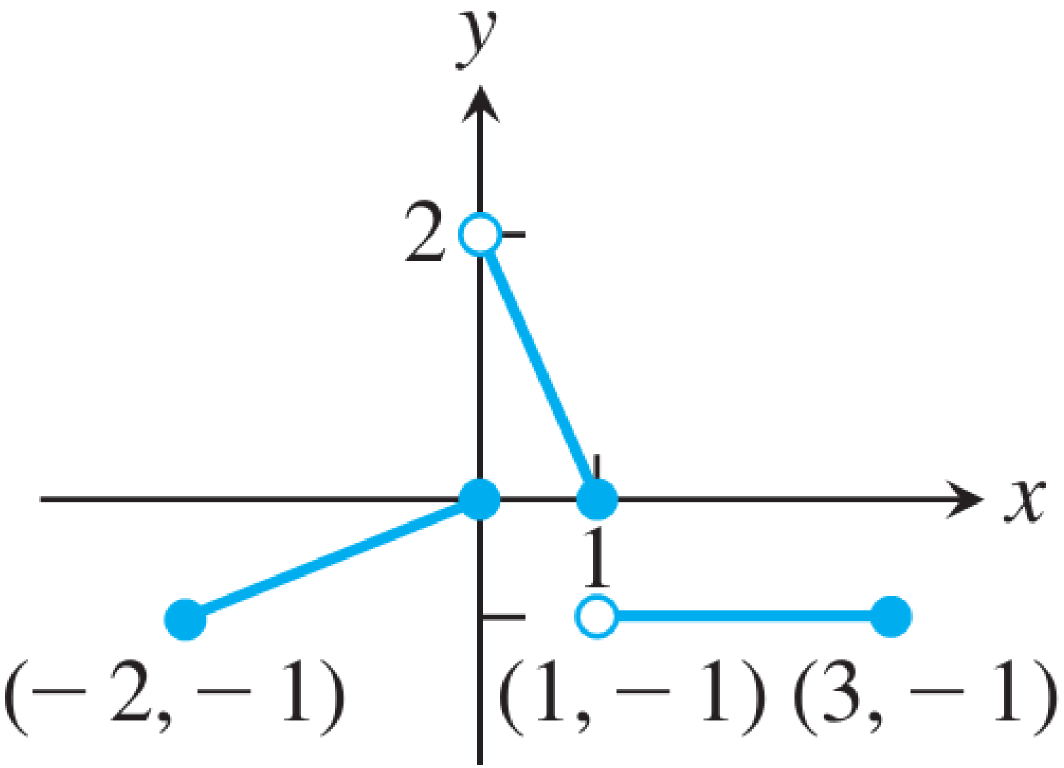
Thus

$$\boxed{\int_0^4 f(x)\,dx = 4}$$

Ans

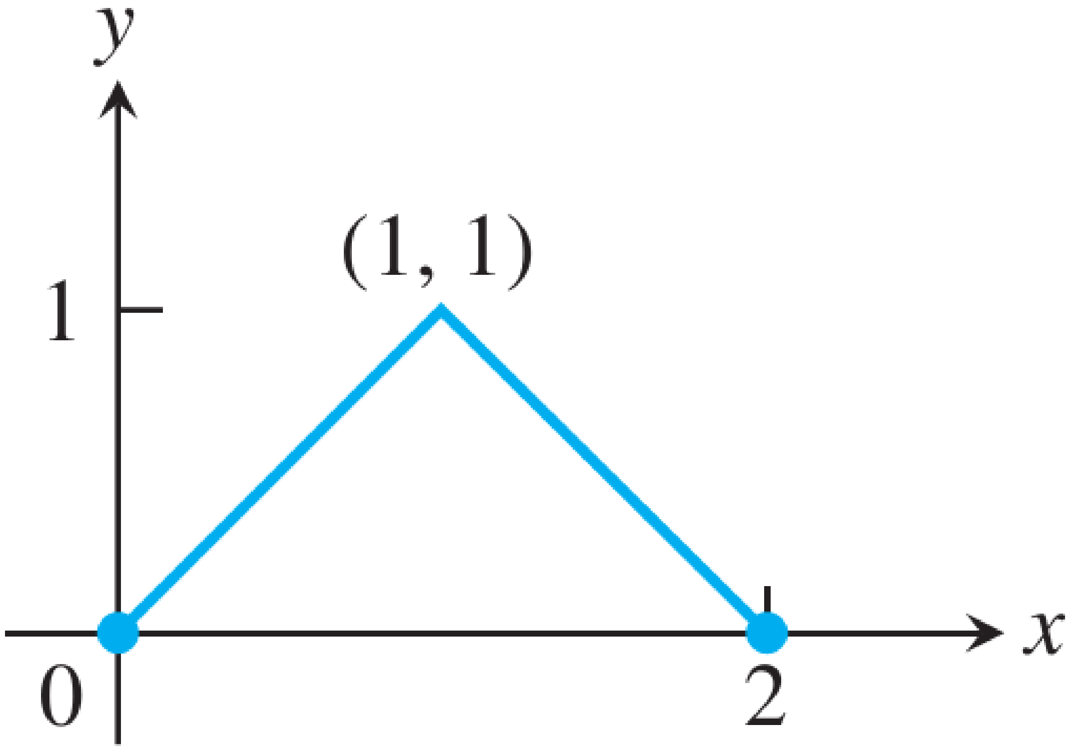
Practice Problem

Evaluate $\int_{-2}^{3} f(x)dx$, where $f(x)$ is shown in below figure.



Practice Problem

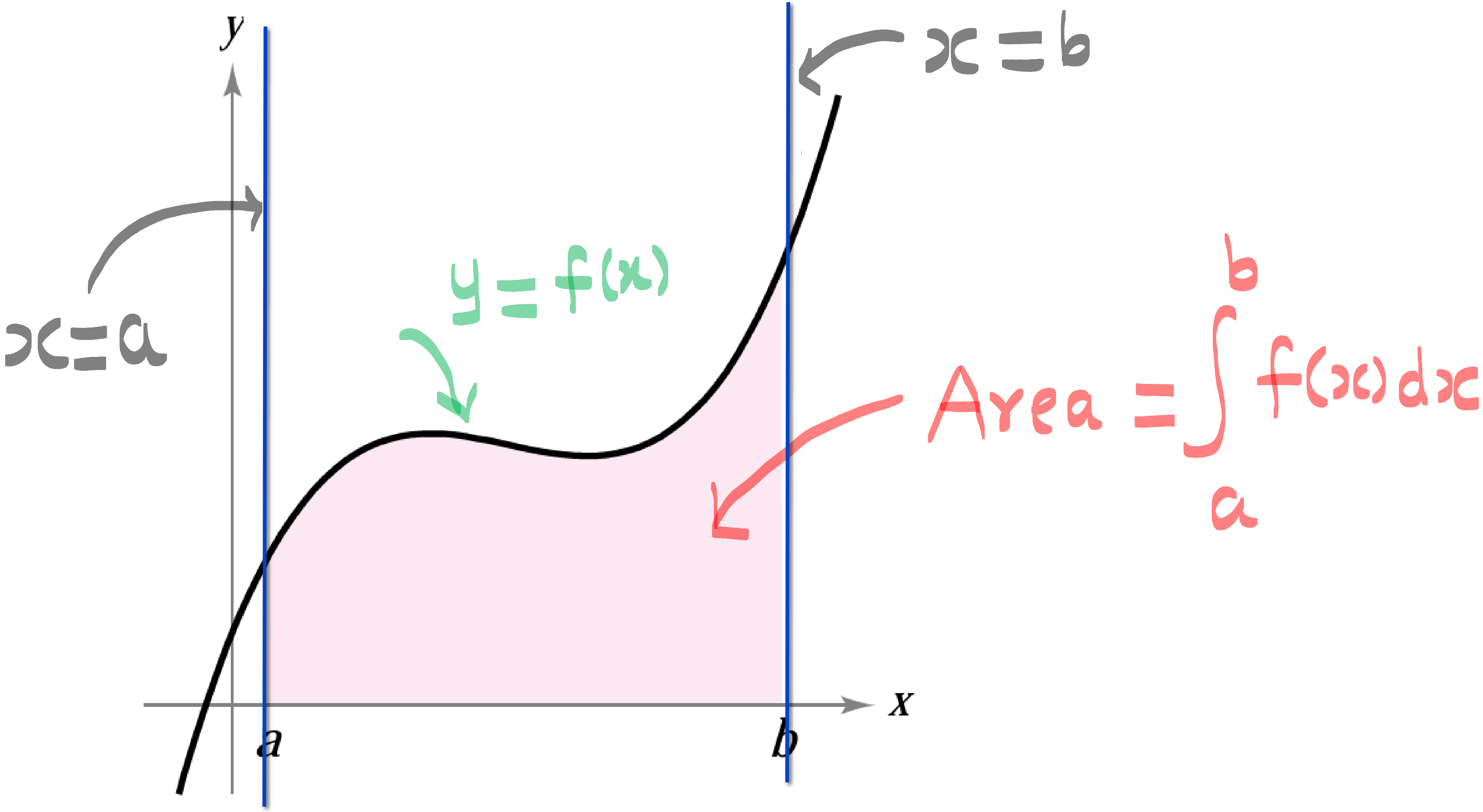
Evaluate $\int_0^2 f(x)dx$, where f is shown in the figure.



Theorem The Definite Integral as the Area of a Region

If f is continuous and nonnegative on the closed interval $[a, b]$, then the area of the region bounded by the graph of f, the x −axis, and the vertical lines $x = a$ and $x = b$ is

$$\text{Area} = \int_a^b f(x)dx$$



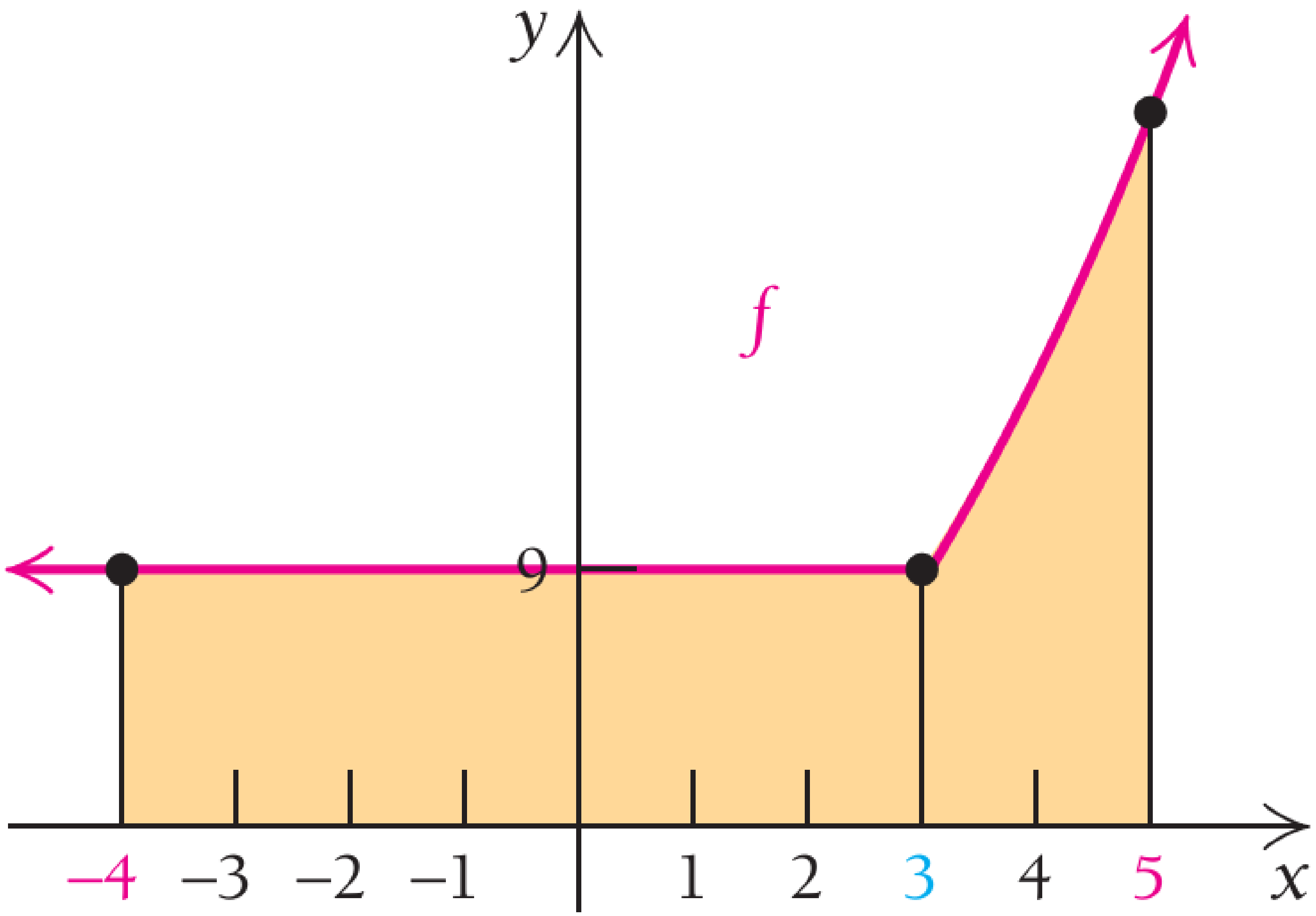
y
x
a
b
x=a
x=b
y=f(x)
Area = $\int_a^b f(x)\,dx$

Problem

Find the area under the graph of $y = f(x)$ from -4 to 5, where

$$f(x) = \begin{cases} 9, & \text{for } x < 3, \\ x^2, & \text{for } x \geq 3. \end{cases}$$

Solution



$$\int_{-4}^{5} f(x)\,dx = \int_{-4}^{3} f(x)\,dx + \int_{3}^{5} f(x)\,dx$$

$$= \int_{-4}^{3} 9\,dx + \int_{3}^{5} x^2\,dx$$

$$= 9[x]_{-4}^{3} + \left[\frac{x^3}{3}\right]_3^5$$

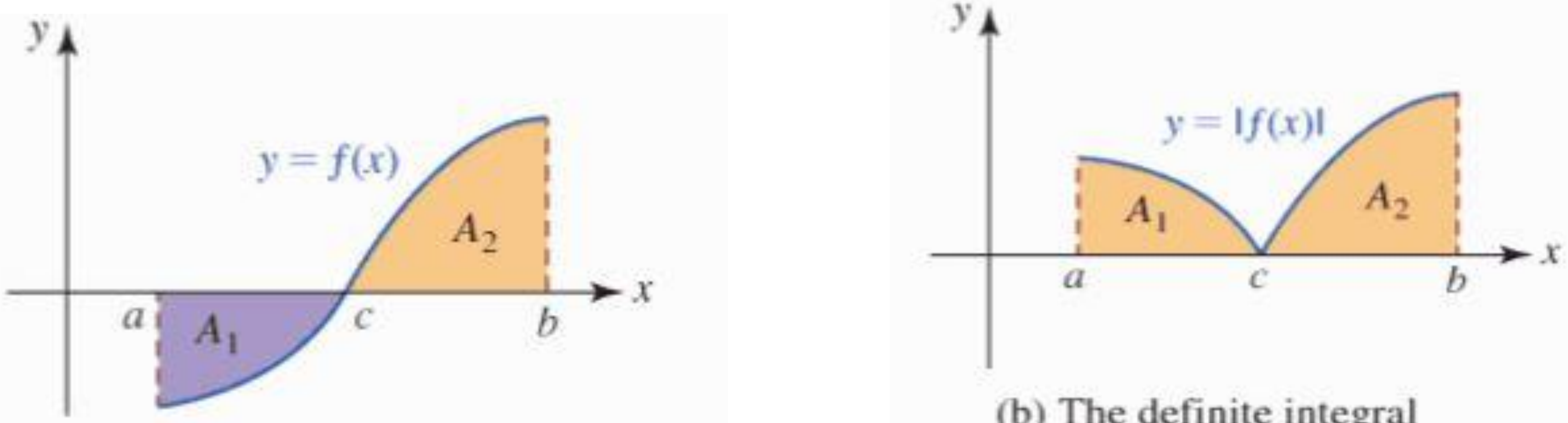
$$= 9(3 - (-4)) + \left(\frac{5^3}{3} - \frac{3^3}{3}\right)$$

$$= 95\tfrac{2}{3}$$

Definition 6.2.1 Total Area

If $y = f(x)$ is continuous on $[a, b]$, then the **total area** $\boldsymbol{A}$ bounded by its graph and the x-axis on the interval is given by

$$A = \int_a^b |f(x)|\, dx. \tag{2}$$



(a) The definite integral of f on $[a, b]$ is not area

(b) The definite integral of $|f|$ on $[a, b]$ is area

FIGURE 6.2.1 Total area is $A = A_1 + A_2$

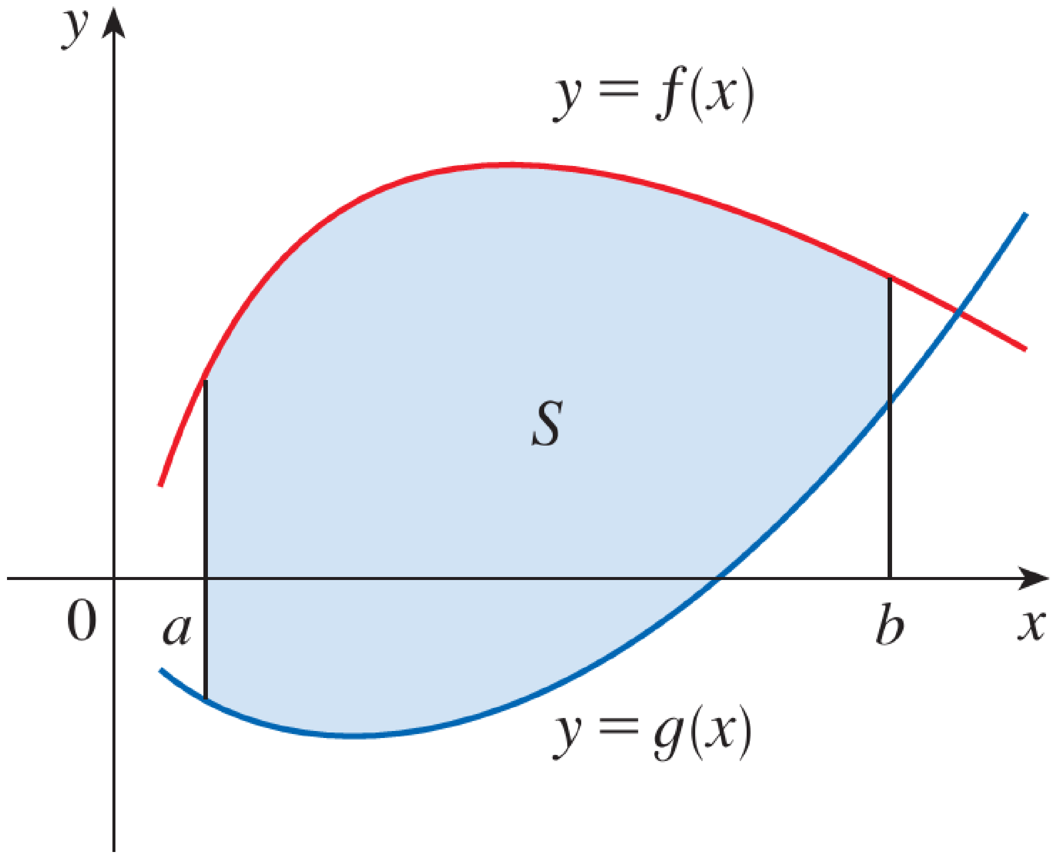
The Area of a Region Bounded by Two Graphs

$\boxed{2}$ The area A of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$, $x = b$, where f and g are continuous and $f(x) \geqslant g(x)$ for all x in $[a, b]$, is

$$A = \int_a^b [f(x) - g(x)]\,dx$$

OR

$$A = \int_a^b (\text{upper curve} - \text{lower curve})\,dx$$

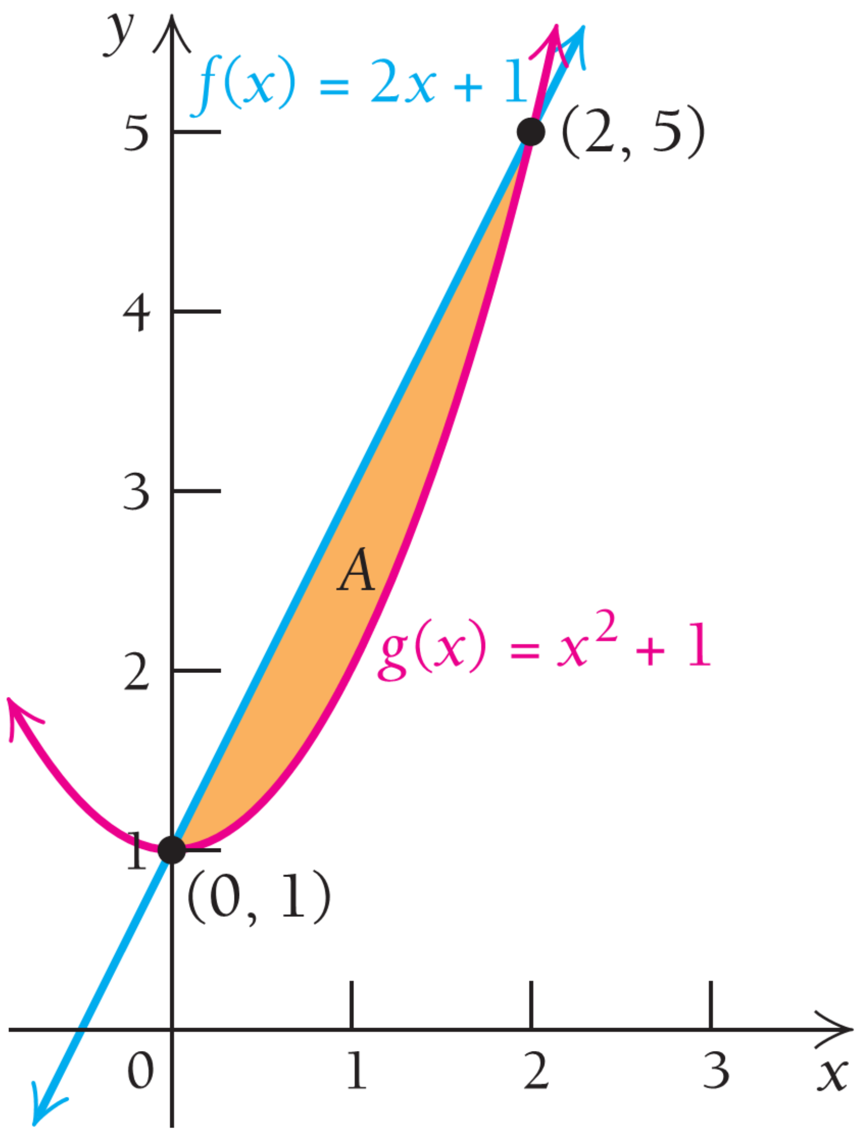


$$S = \{(x, y) \mid a \leqslant x \leqslant b,$$
$$g(x) \leqslant y \leqslant f(x)\}$$

Problem

Find the area of the region bounded by the graphs of $f(x) = 2x + 1$ and $g(x) = x^2 + 1$.

Solution First, we make a reasonably accurate sketch, as in the figure at the right, to determine which is the upper graph.



y
$f(x) = 2x + 1$
5
(2, 5)
4
3
A
$g(x) = x^2 + 1$
2
1
(0, 1)
0
1
2
3
x

To calculate the points of intersection, we set $f(x)$ equal to $g(x)$ and solve.

$$\begin{aligned} f(x) &= g(x) \\ 2x + 1 &= x^2 + 1 \\ 0 &= x^2 - 2x \\ 0 &= x(x - 2) \end{aligned}$$

$$x = 0 \quad \text{or} \quad x = 2$$

The graphs intersect at $x = 0$ and $x = 2$. We see that, over the interval $[0, 2]$, f is the upper graph.

We now compute the area as follows:

$$\int_0^2 [(2x + 1) - (x^2 + 1)]\, dx = \int_0^2 (2x - x^2)\, dx$$

$$= \left[x^2 - \frac{x^3}{3}\right]_0^2$$

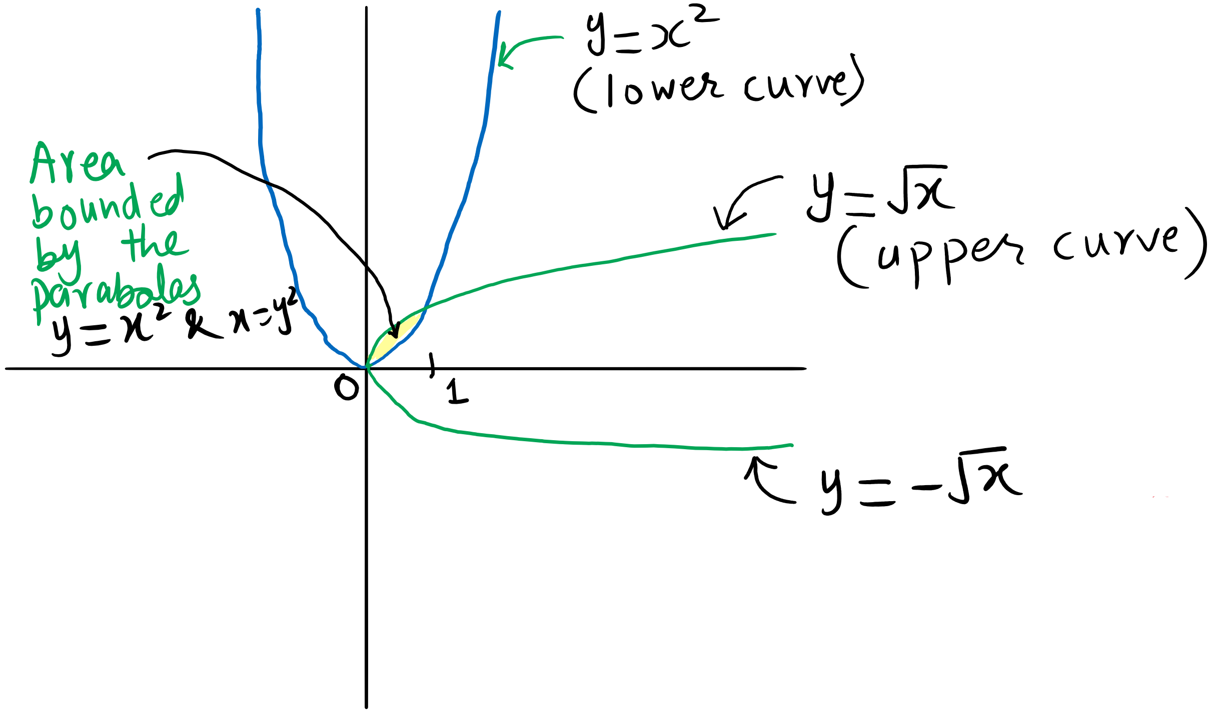
$$\int_0^2 [(2x + 1) - (x^2 + 1)]\,dx = \int_0^2 (2x - x^2)\,dx$$

$$= \left[x^2 - \frac{x^3}{3}\right]_0^2$$

Problem

Find area of the region bounded by the parabolas $y = x^2$ and $x = y^2$.

Solution



Note
$x = y^2$ is not a function of x, however it can represent two functions of x that is
$y = \sqrt{x}$ & $y = -\sqrt{x}$

$$A = \int_0^1 \left(\sqrt{x} - x^2\right) dx$$

$$= \left(\frac{2x^{3/2}}{3} - \frac{x^3}{3}\right)\Bigg|_0^1 = \left(\frac{2}{3} 1^{3/2} - \frac{1^3}{3}\right) - 0$$

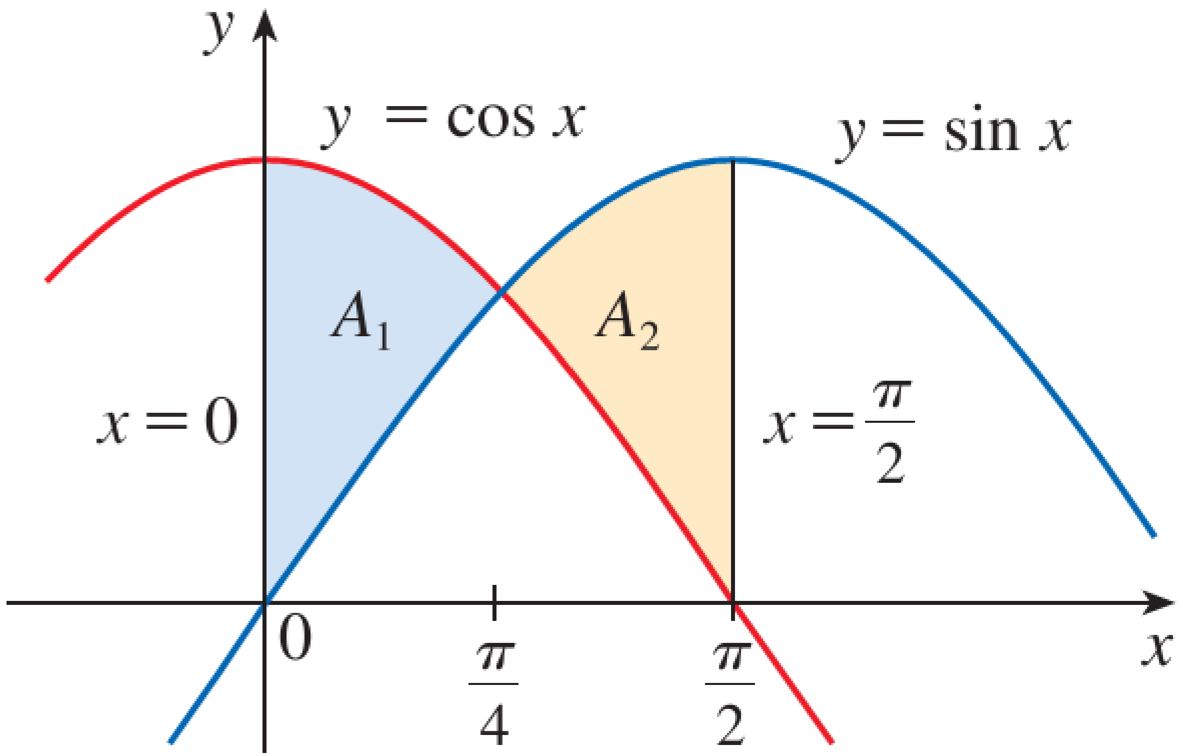
$$= \frac{2}{3} - \frac{1}{3} = \boxed{\frac{1}{3}}$$

Thus Area bounded by the parabolas $y = x^2$ and $x = y^2$ is $\frac{1}{3}$ units.

Problem

Find the area of the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \pi/2$.

Solution

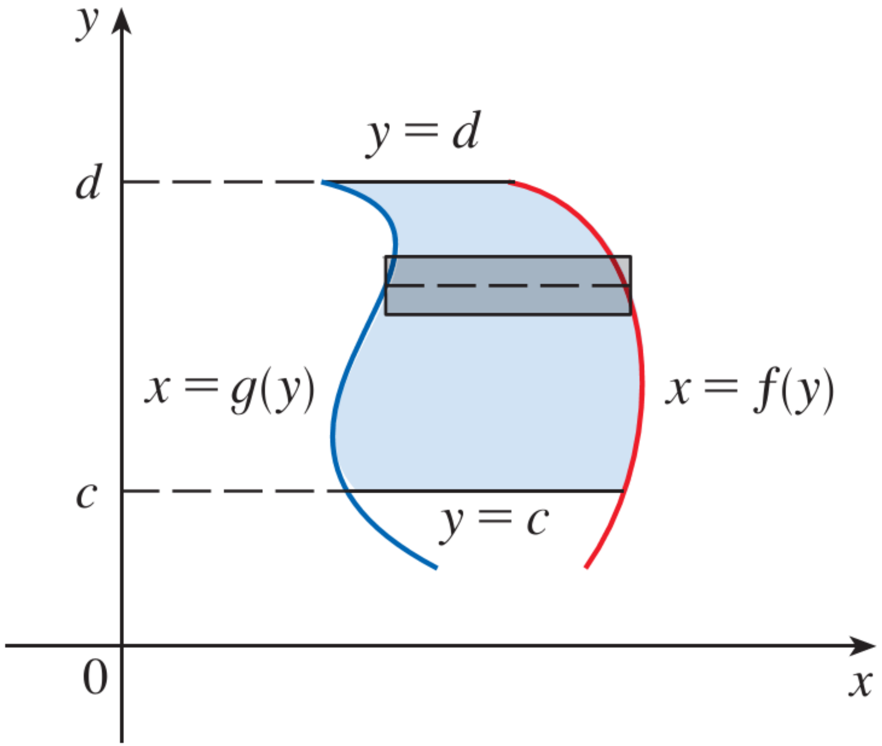


$$
\begin{aligned}
A &= \int_0^{\pi/4} (\cos x - \sin x)\,dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\,dx \\
&= \Big[\sin x + \cos x\Big]_0^{\pi/4} + \Big[-\cos x - \sin x\Big]_{\pi/4}^{\pi/2} \\
&= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 0 - 1\right) + \left(-0 - 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) \\
&= 2\sqrt{2} - 2
\end{aligned}
$$

Area Between Curves: Integrating With Respect to y

Some regions are best treated by regarding x as a function of y. If a region is bounded by curves with equations $x = f(y)$, $x = g(y)$, $y = c$, and $y = d$, where f and g are continuous and $f(y) \geqslant g(y)$ for $c \leqslant y \leqslant d$

FIGURE 11



$$A = \int_c^d [f(y) - g(y)]\, dy$$

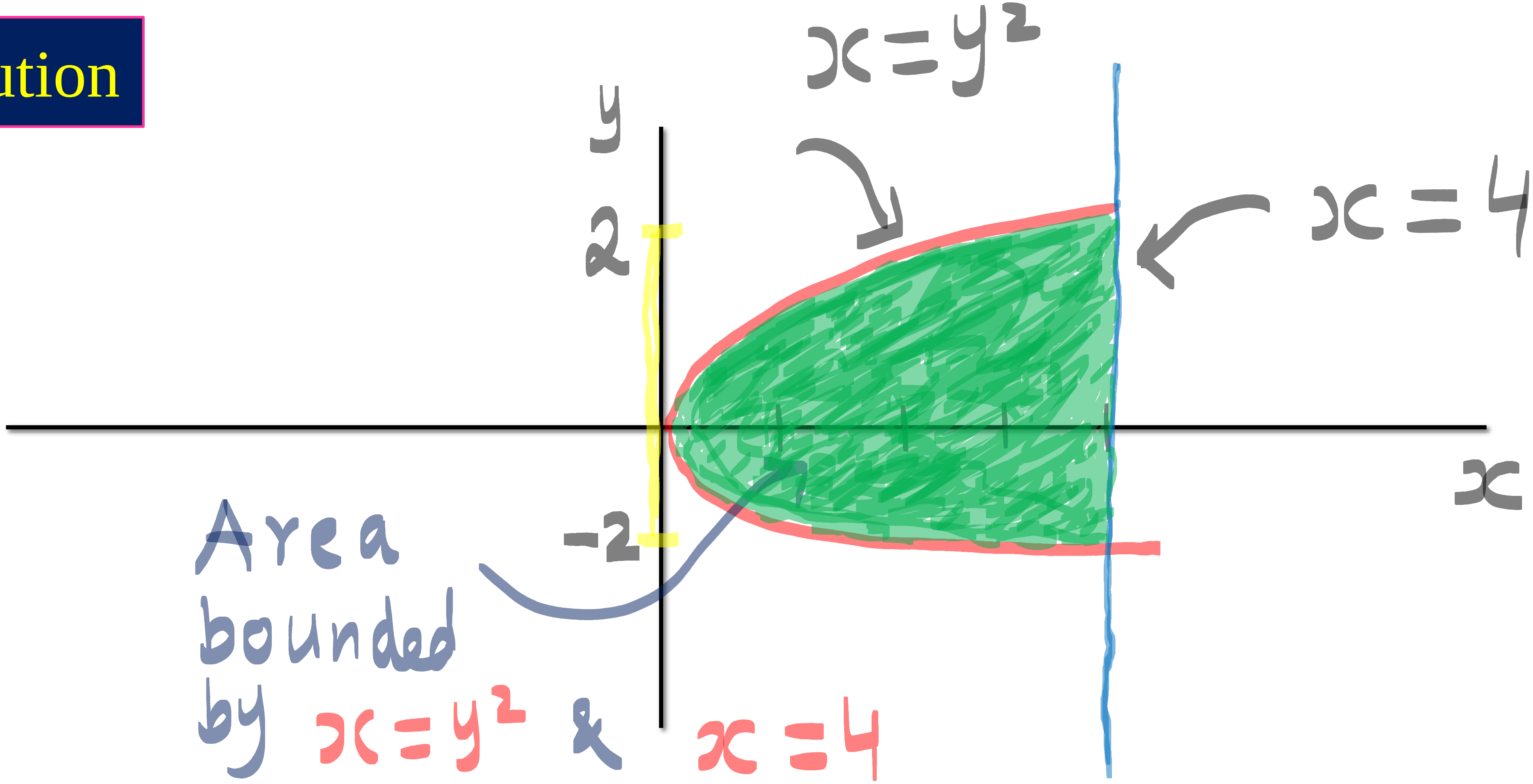
If we write x_R for the right boundary and x_L for the left boundary, then, as Figure 11 illustrates, we have

$$A = \int_c^d (x_R - x_L)\, dy$$

Problem

Find area of the region bounded by the parabolas $x = y^2$ and $x = 4$.

Solution



$$A = \int_{-2}^{2} (4 - y^2)\,dy$$

Since $4 - y^2$ is an even function

therefore $\int_{-2}^{2} (4 - y^2)\,dy = 2\int_{0}^{2} (4 - y^2)\,dy$

So

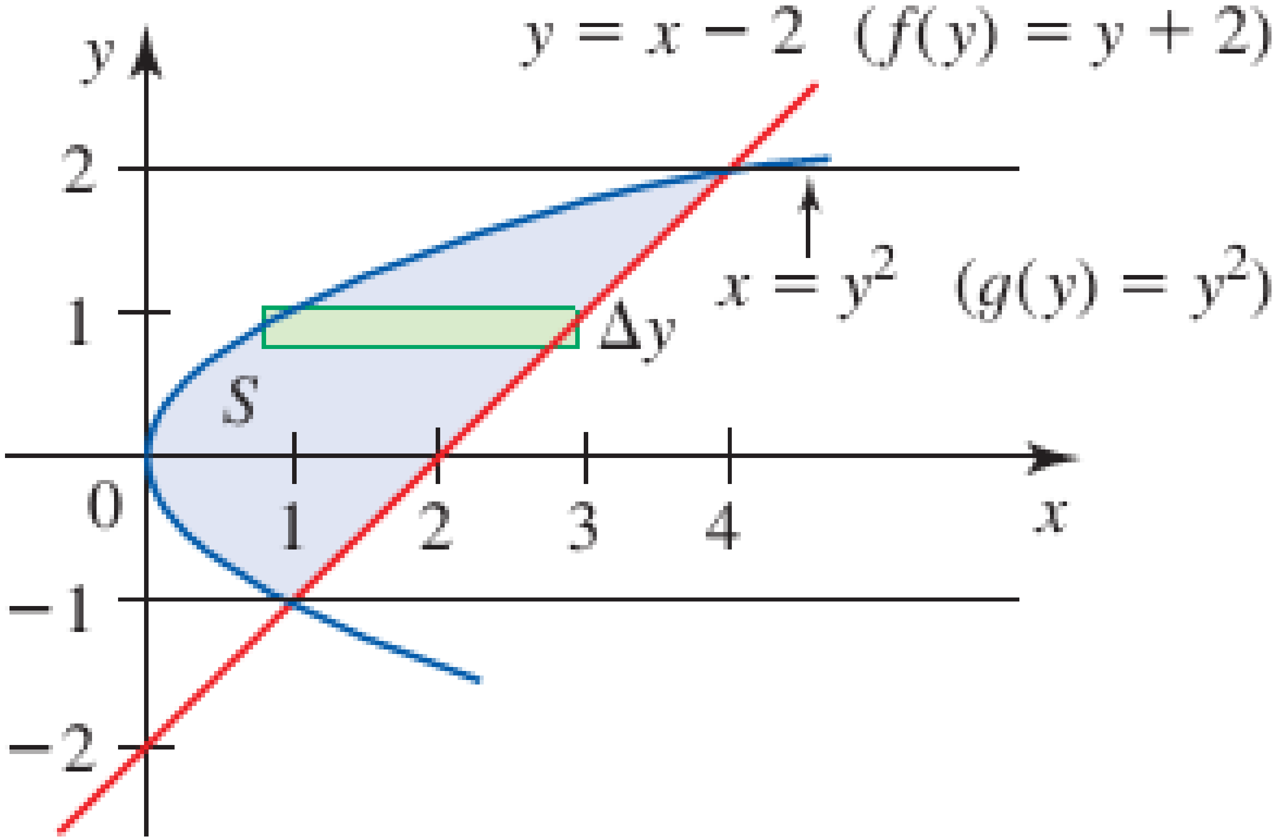
$$A = 2\int_{0}^{2} (4 - y^2)\,dy = 2\left(4y - \frac{y^3}{3}\right)\Big|_0^2$$

$$= 2\left[\left(4(2) - \frac{2^3}{3}\right) - 0\right]$$

$$= 2\left(8 - \frac{8}{3}\right) = 2\left(\frac{16}{3}\right) = \boxed{\frac{32}{3}}$$ Ans

Problem

Find the area of the region **bounded by the line $y = x - 2$ and the parabola $x = y^2$**



$$A = \int_{-1}^{2} (y + 2 - y^2)\, dy = \left[\frac{1}{2}y^2 + 2y - \frac{1}{3}y^3\right]_{-1}^{2}$$

$$= \left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right) = \frac{9}{2} \quad \text{or} \quad 4\frac{1}{2}$$

Net Change Theorem The integral of a rate of change is the net change:

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

Practice Problem

If $f(1)=12$, f' is continuous, and $\int_1^4 f'(x)\,dx = 17$. **Compute** the value of $f(4)$. [Hint: **Net Change Theorem**]

Average Value of a Function

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

For a positive function, we can think of this definition as saying

$$\frac{\text{area}}{\text{width}} = \text{average height}$$

Problem

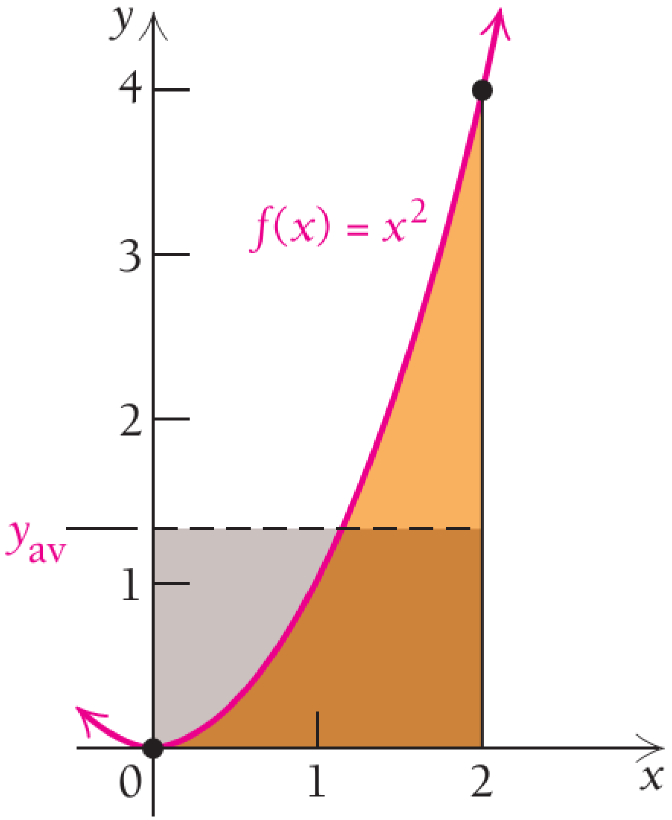
Find the average value of $f(x) = x^2$ over the interval $[0, 2]$.

Solution The average value is

$$\frac{1}{2-0}\int_0^2 x^2\,dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2$$

$$= \frac{1}{2}\left(\frac{2^3}{3} - \frac{0^3}{3}\right)$$

$$= \frac{1}{2}\cdot\frac{8}{3} = \frac{4}{3}$$



Problem

Find the average value of the function $f(x) = 1 + x^2$ on the interval $[-1, 2]$.

SOLUTION With $a = -1$ and $b = 2$ we have

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{2-(-1)}\int_{-1}^{2}(1+x^2)\,dx = \frac{1}{3}\left[x + \frac{x^3}{3}\right]_{-1}^{2} = 2$$

Practice Problems

Find the numbers b such that the average value of $f(x) = 2 + 6x - 3x^2$ on the interval $[0, b]$ is equal to 3.

In a certain city the temperature (in °F) t hours after 9 AM was modeled by the function

$$T(t) = 50 + 14 \sin \frac{\pi t}{12}$$

Find the average temperature during the period from 9 AM to 9 PM.

The height H (ft) of a palm tree after growing for t years is given by

$$H = \sqrt{t + 1} + 5t^{1/3} \quad \text{for} \quad 0 \leq t \leq 8.$$

a. Find the tree's height when $t = 0$, $t = 4$, and $t = 8$.

b. Find the tree's average height for $0 \leq t \leq 8$.

Water Level in Boston Harbor The water level (in feet) in Boston Harbor during a certain 24-hr period is approximated by the formula

$$H = 4.8 \sin\left[\frac{\pi}{6}(t - 10)\right] + 7.6 \qquad 0 \le t \le 24$$

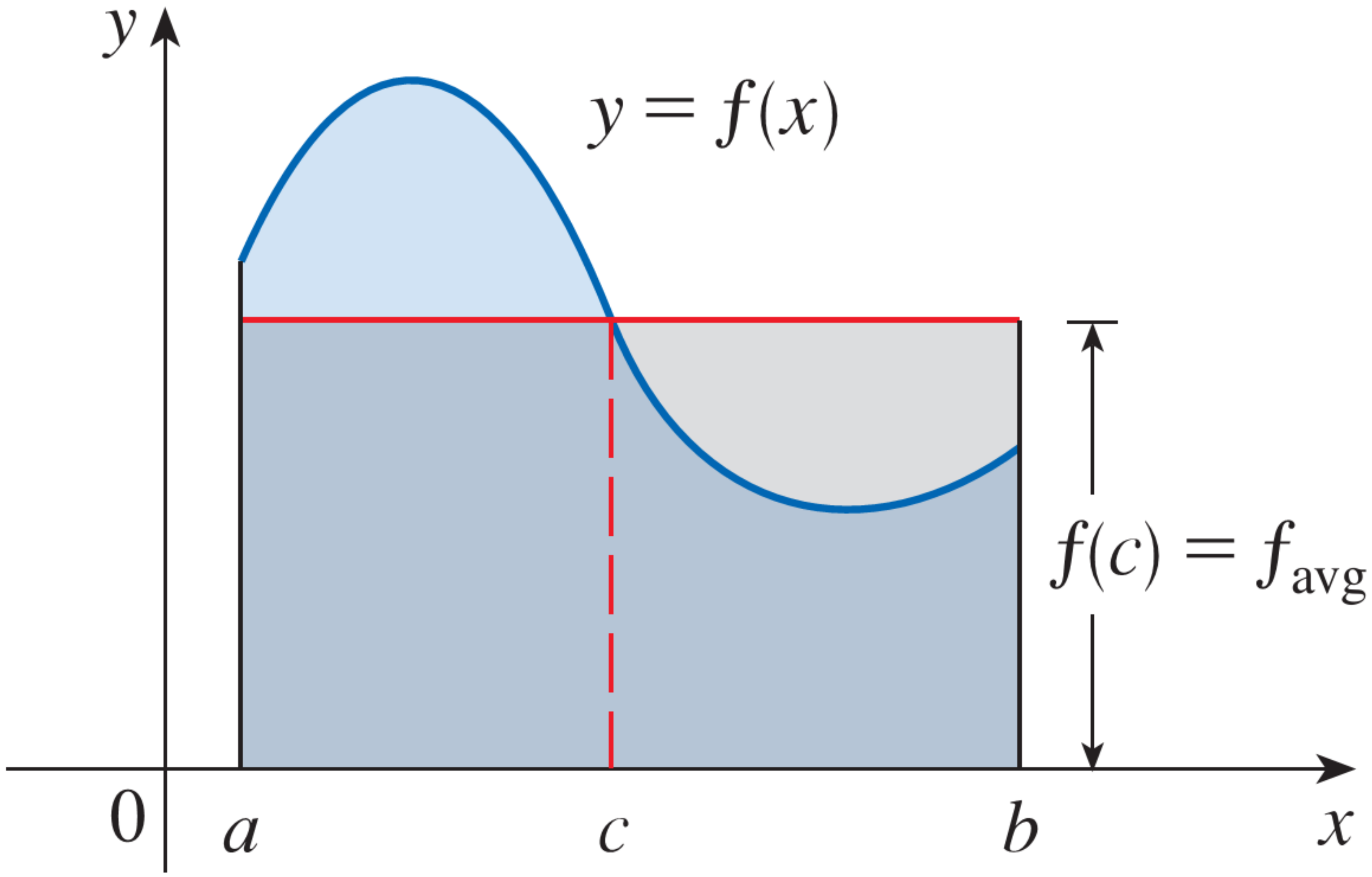
where $t = 0$ corresponds to 12 A.M. What is the average water level in Boston Harbor over the 24-hr period on that day?

The Mean Value Theorem for Integrals If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that

$$f(c) = f_{\text{avg}} = \frac{1}{b - a} \int_a^b f(x)\, dx$$

that is,
$$\int_a^b f(x)\, dx = f(c)(b - a)$$

The geometric interpretation of the Mean Value Theorem for Integrals is that, for *positive* functions f, there is a number c such that the rectangle with base $[a, b]$ and height $f(c)$ has the same area as the region under the graph of f from a to b.



y
y = f(x)
f(c) = f_avg
0
a
c
b
x

Problem

Find the value of c guaranteed by the Mean Value Theorem for Integrals for $f(x) = 4 - 2x$ on the interval $[0, 2]$.

Solution The function $f(x) = 4 - 2x$ is continuous on the interval $[0, 2]$. Therefore, the Mean Value Theorem for Integrals states that there is a number c in $[0, 2]$ such that

$$\frac{1}{b - a}\int_a^b f(x)\, dx = f(c)$$

where $a = 0$ and $b = 2$. Thus,

$$\frac{1}{2-0}\int_0^2 (4-2x)\,dx = 4-2c$$

but $$\int_0^2 (4-2x)\,dx = 4$$

So we have $$\frac{1}{2}(4) = 4-2c \quad \text{or} \quad \boxed{c = 1.}$$

Integration by Parts – Definite Integral

$$\int_a^b u\,dv = (uv)\Big|_a^b - \int_a^b v\,du$$

Let's Evaluate $\int_0^1 \tan^{-1} x \, dx$

Take $u = \tan^{-1} x$ & $dv = dx$

$\Rightarrow du = \dfrac{dx}{1+x^2}$ & $v = x$

So $\int_0^1 \tan^{-1}x\,dx = x\tan^{-1}x\Big|_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx$

$$= \left[1\cdot\tan^{-1}(1) - 0\right] - \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}\,dx$$

$$= \frac{\pi}{4} - \frac{1}{2}\Big[\ln|1+x^2|\Big]_0^1$$

$$= \frac{\pi}{4} - \frac{1}{2}\Big[\ln|1+1^2| - \ln|1+0|\Big]$$

$$\Rightarrow \int_0^1 \tan^{-1}x\,dx = \frac{\pi}{4} - \frac{1}{2}\left[\ln|2| - \ln|1|\right]$$

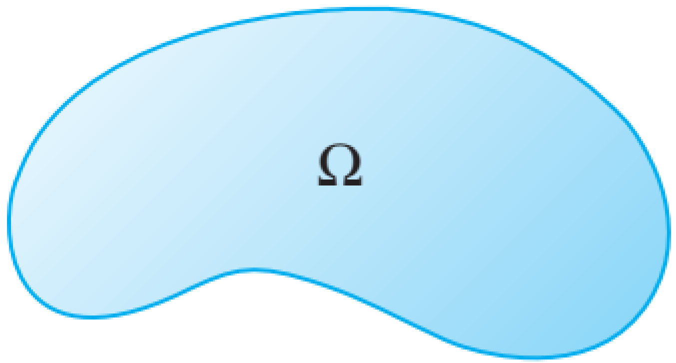
$$= \frac{\pi}{4} - \frac{1}{2}\left[\ln|2| - 0\right] = \frac{\pi}{4} - \frac{1}{2}\ln|2|$$

Thus $\boxed{\int_0^1 \tan^{-1}x\,dx = \frac{\pi}{4} - \frac{1}{2}\ln|2|}$ Ans

Volumes

In trying to find the volume of a solid we face the same type of problem as in finding areas. We have an intuitive idea of what volume means, but we must make this idea precise by using calculus to give an exact definition of volume.

Figure 1 shows a plane region Ω and a solid formed by translating Ω along a line perpendicular to the plane of Ω. Such a solid is called a *right cylinder with cross section* Ω.



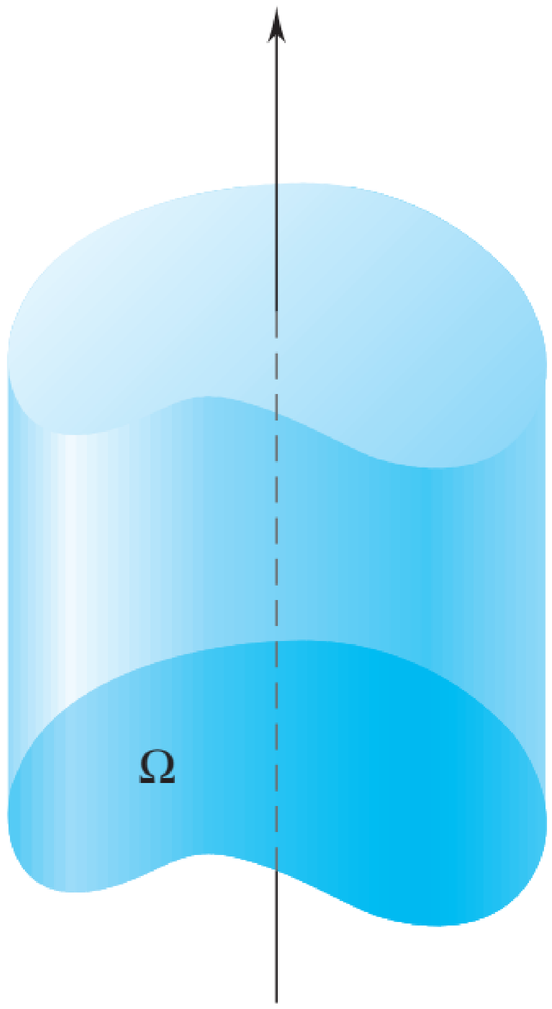
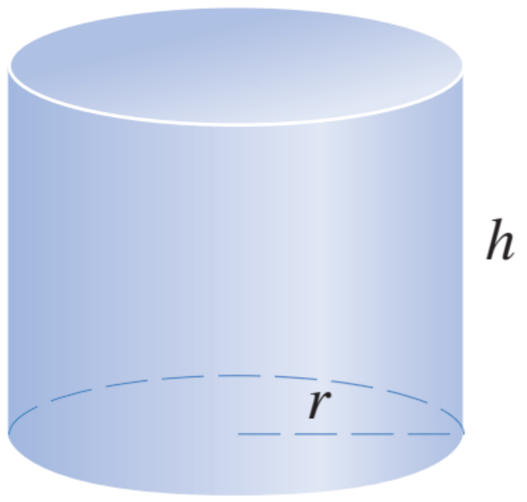


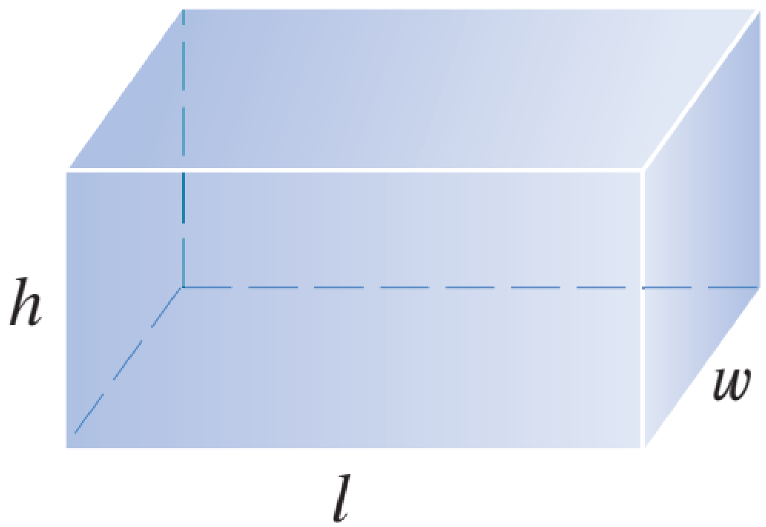
Figure 1

If Ω has area A and the solid has height h, then the volume of the solid is a simple product:

$$V = A \cdot h \qquad \text{(cross-sectional area} \cdot \text{height)}$$



Circular cylinder $V = \pi r^2 h$

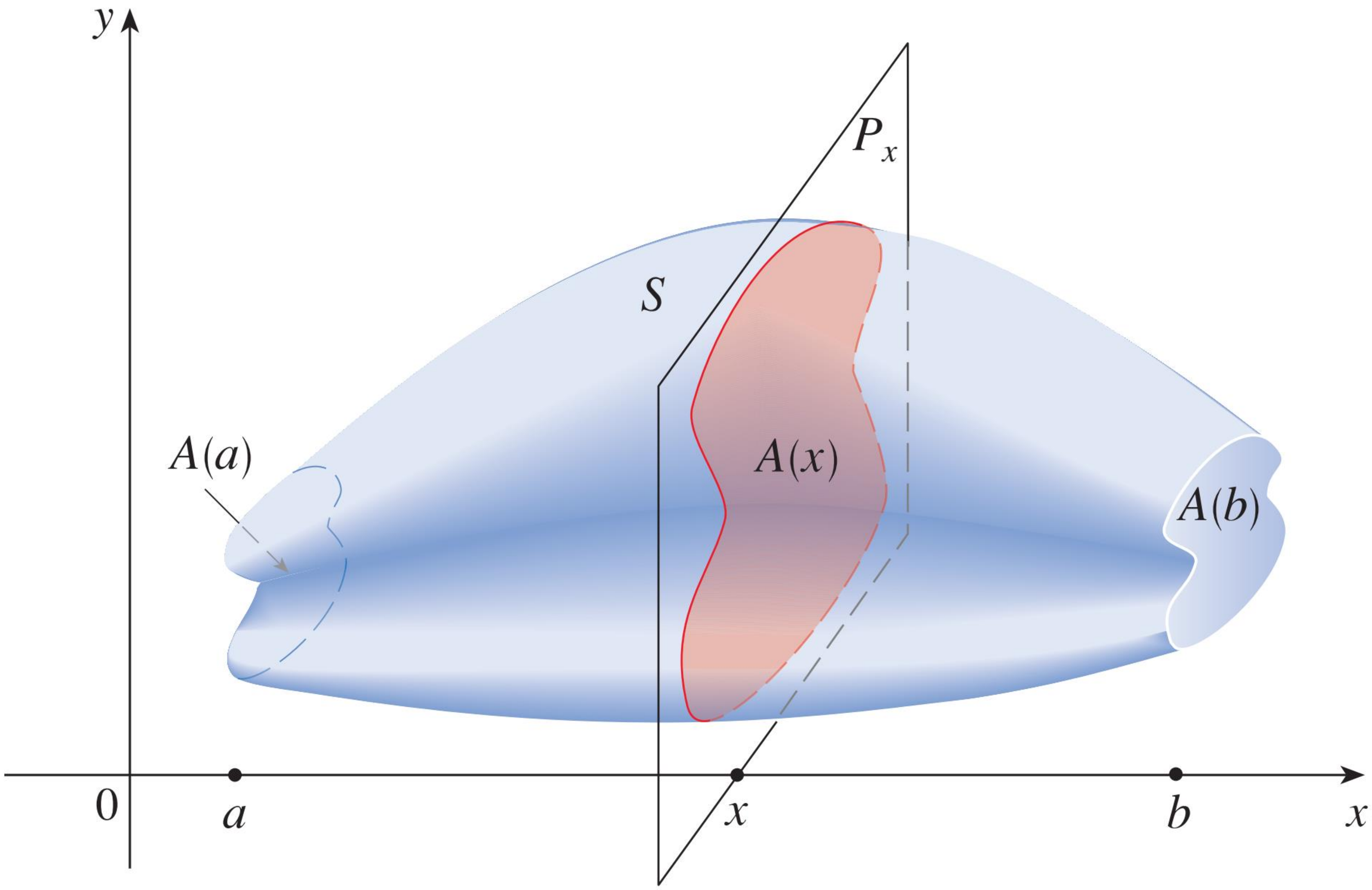


Rectangular box $V = lwh$

For a solid S that isn't a cylinder we first "cut" S into pieces and approximate each piece by a cylinder. We estimate the volume of S by adding the volumes of the cylinders. We arrive at the exact volume of S through a limiting process in which the number of pieces becomes large.

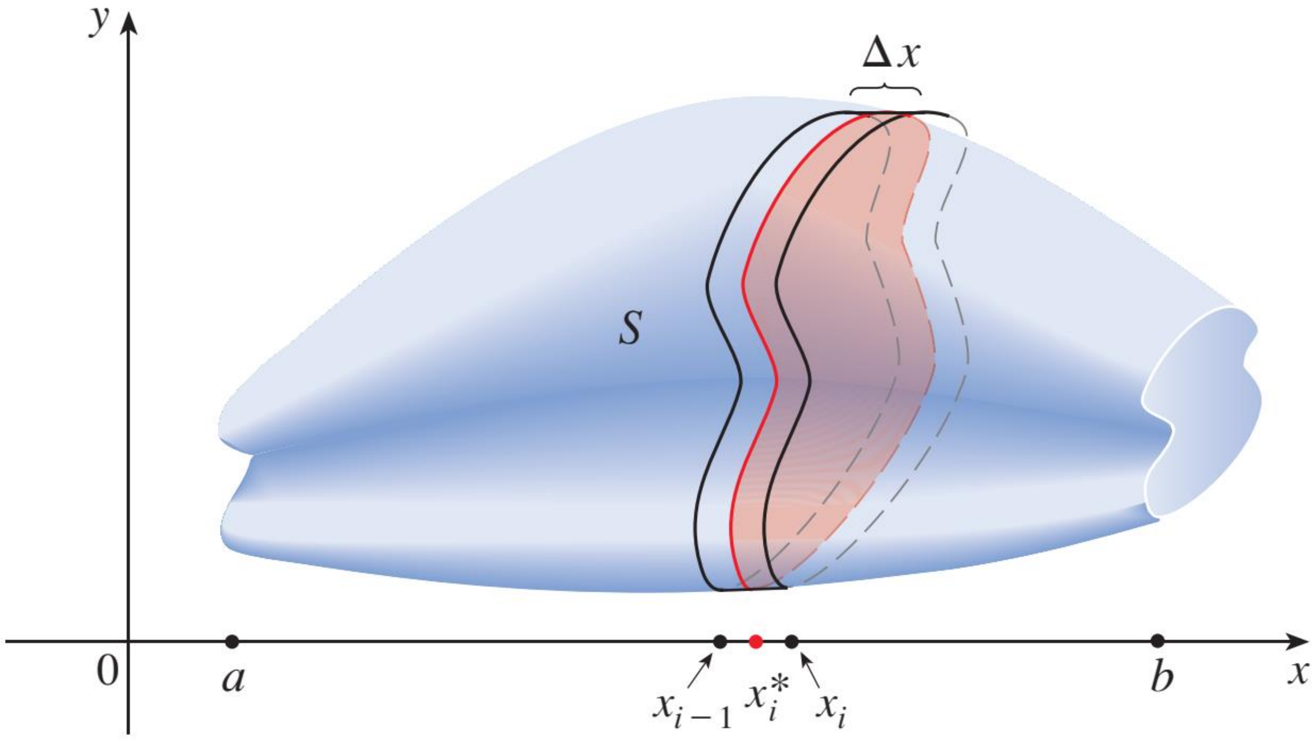
We start by intersecting S with a plane and obtaining a plane region that is called a **cross-section** of S. Let $A(x)$ be the area of the cross-section of S in a plane P_x perpendicular to the x-axis and passing through the point x, where $a \leqslant x \leqslant b$. (See Figure 2. Think of slicing S with a knife through x and computing the area of this slice.) The cross-sectional area $A(x)$ will vary as x increases from a to b.

FIGURE 2



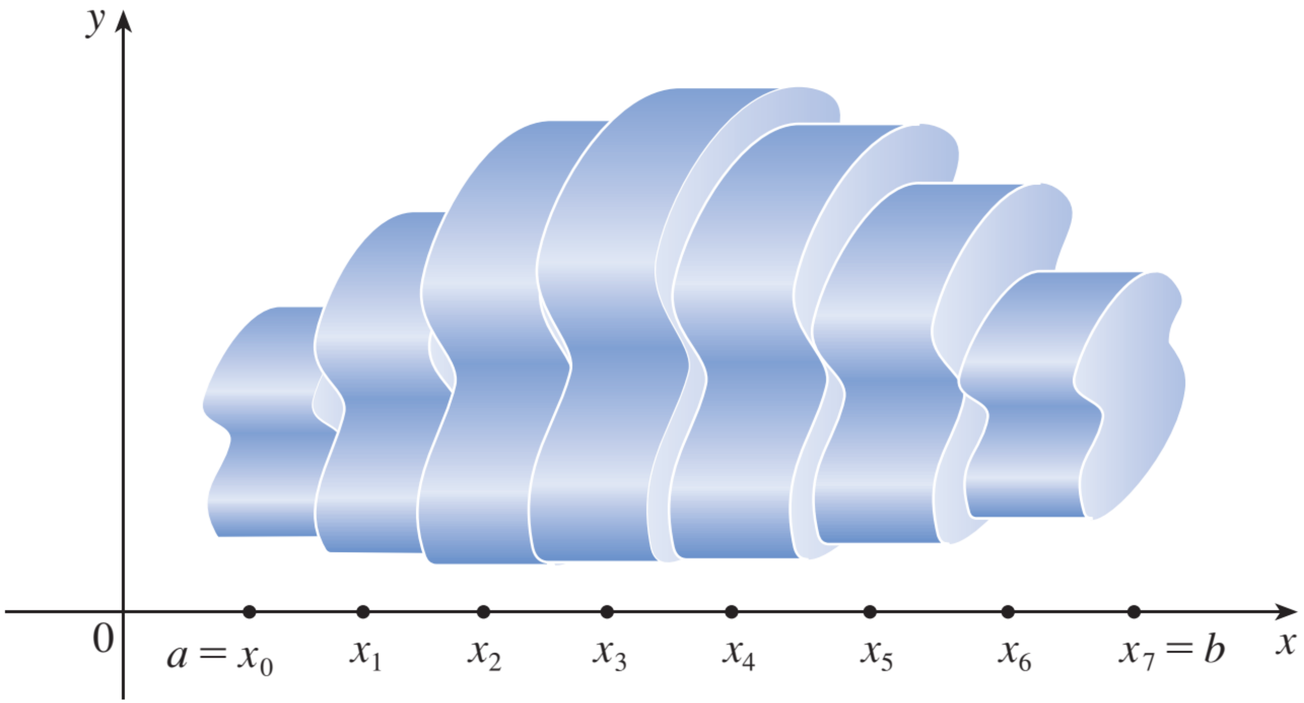
Let's divide S into n "slabs" of equal width Δx by using the planes P_{x_1}, P_{x_2}, . . . to slice the solid. (Think of slicing a loaf of bread.) If we choose sample points x_i^* in $[x_{i-1}, x_i]$, we can approximate the ith slab S_i (the part of S that lies between the planes $P_{x_{i-1}}$ and P_{x_i}) by a cylinder with base area $A(x_i^*)$ and "height" Δx. (See Figure 3.)

FIGURE 3



The volume of this cylinder is $A(x_i^*)\,\Delta x$, so an approximation to our intuitive conception of the volume of the ith slab S_i is

$$V(S_i) \approx A(x_i^*)\,\Delta x$$



Adding the volumes of these slabs, we get an approximation to the total volume (that is, what we think of intuitively as the volume):

$$V \approx \sum_{i=1}^{n} A(x_i^*)\,\Delta x$$

This approximation appears to become better and better as $n \to \infty$. (Think of the slices as becoming thinner and thinner.) Therefore we *define* the volume as the limit of these sums as $n \to \infty$. But we recognize the limit of Riemann sums as a definite integral and so we have the following definition.

Definition of Volume Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x, through x and perpendicular to the x-axis, is $A(x)$, where A is a continuous function, then the **volume** of S is

$$V = \lim_{n \to \infty} \sum_{i=1}^{n} A(x_i^*) \, \Delta x = \int_a^b A(x) \, dx$$

When we use the volume formula $V = \int_a^b A(x) \, dx$, it is important to remember that $A(x)$ is the area of a moving cross-section obtained by slicing through x perpendicular to the x-axis.

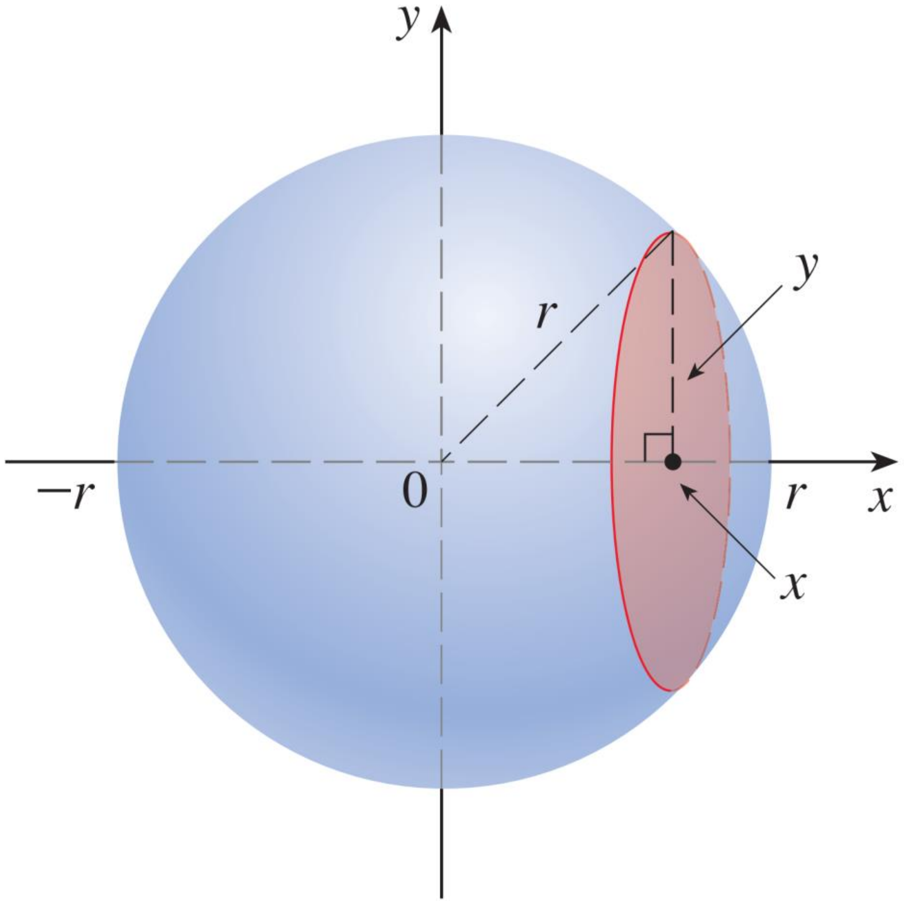
Notice that, for a cylinder, the cross-sectional area is constant: $A(x) = A$ for all x. So our definition of volume gives $V = \int_a^b A \, dx = A(b - a)$; this agrees with the formula $V = Ah$.

Problem Show that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.

Solution

If we place the sphere so that its center is at the origin, then the plane P_x intersects the sphere in a circle whose radius (from the Pythagorean Theorem) is

$$y = \sqrt{r^2 - x^2}$$



Using the definition of volume with $a = -r$ and $b = r$, we have

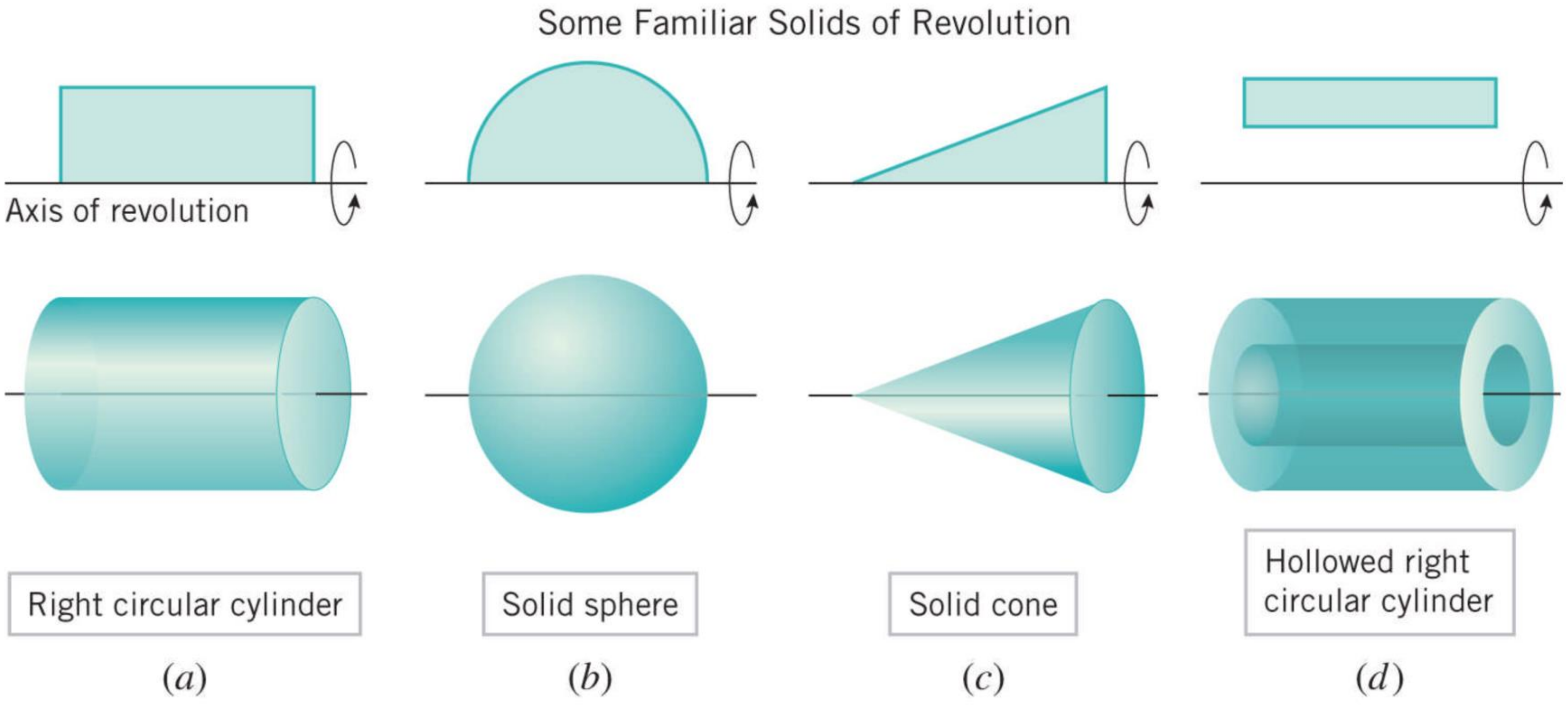
$$V = \int_{-r}^{r} A(x)\,dx = \int_{-r}^{r} \pi(r^2 - x^2)\,dx$$

$$= 2\pi \int_{0}^{r} (r^2 - x^2)\,dx \qquad \text{(The integrand is even.)}$$

$$= 2\pi\left[r^2x - \frac{x^3}{3}\right]_0^r = 2\pi\left(r^3 - \frac{r^3}{3}\right) = \tfrac{4}{3}\pi r^3$$

Volumes of Solids of Revolution

A ***solid of revolution*** is a solid that is generated by revolving a plane region about a line that lies in the same plane as the region; the line is called the ***axis of revolution***.



Problem

Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.

Solution

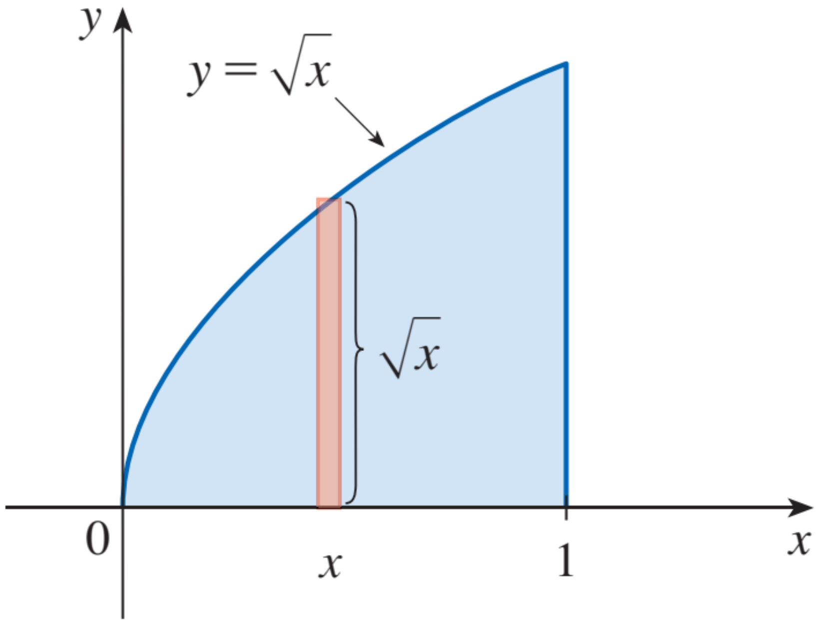
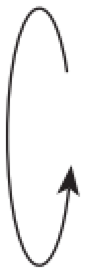


Figure 1



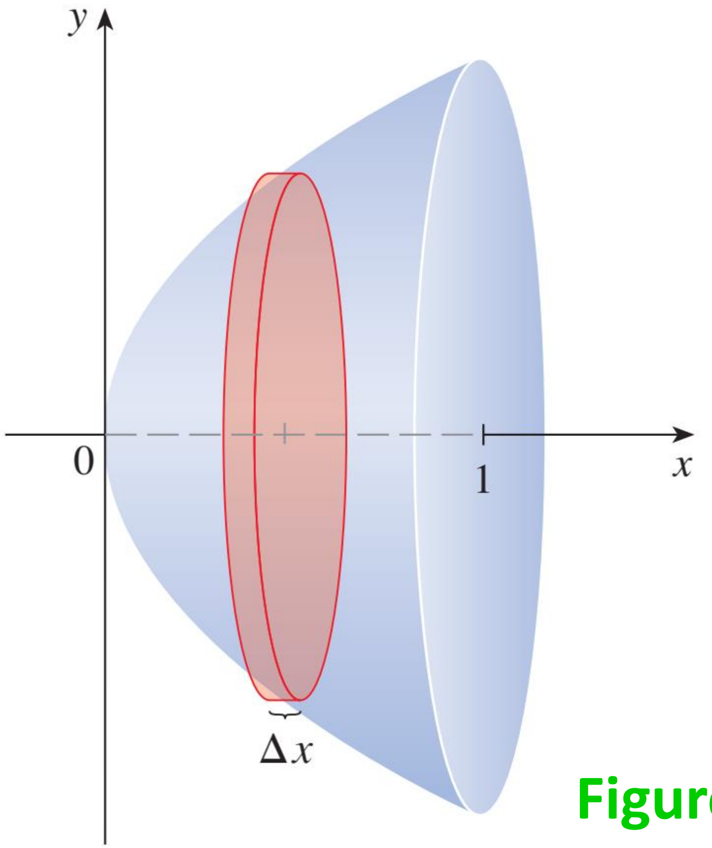


Figure 2

The region is shown in Figure 1. If we rotate about the x-axis, we get the solid shown in Figure 2. When we slice through the point x, we get a disk with radius $\sqrt{x}$. The area of this cross-section is

$$A(x) = \pi \left(\underbrace{\sqrt{x}}_{\text{radius}} \right)^2 = \pi x$$

The solid lies between $x = 0$ and $x = 1$, so its volume is

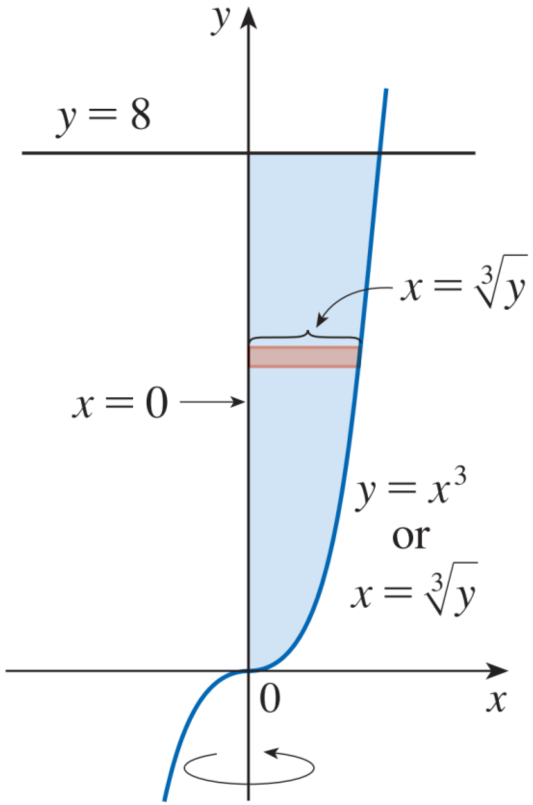
$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi x\,dx = \pi \frac{x^2}{2} \bigg]_0^1 = \frac{\pi}{2}$$

Problem

Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$, and $x = 0$ about the y-axis.

Solution

Figure 3



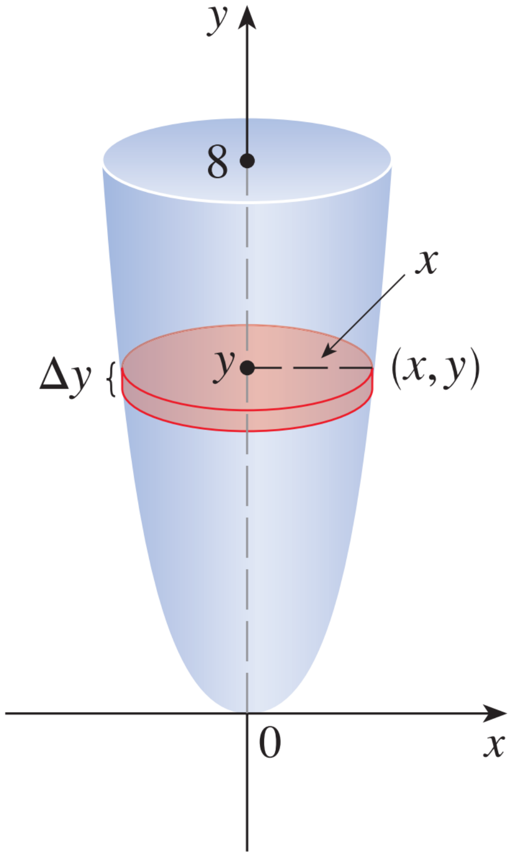


Figure 4

The region is shown in Figure 3 and the resulting solid is shown in Figure 4. Because the region is rotated about the y-axis, it makes sense to slice the solid perpendicular to the y-axis (obtaining circular cross-sections) and therefore to integrate with respect to y. If we slice at height y, we get a circular disk with radius x, where $x = \sqrt[3]{y}$. So the area of a cross-section through y is

$$A(y) = \pi \underbrace{(x)}_{\text{radius}}^2 = \pi \underbrace{\left(\sqrt[3]{y}\,\right)}_{\text{radius}}^2 = \pi y^{2/3}$$

Since the solid lies between $y = 0$ and $y = 8$, its volume is

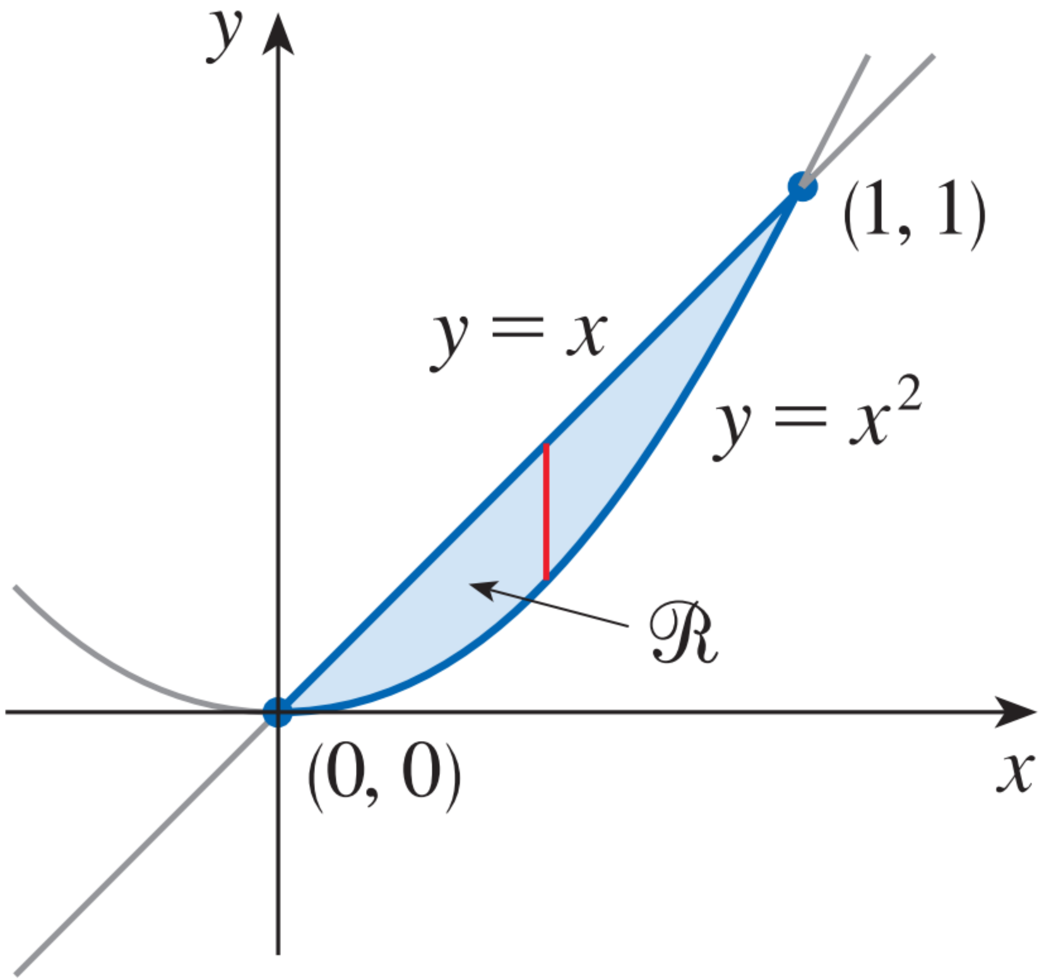
$$V = \int_0^8 A(y)\,dy = \int_0^8 \pi y^{2/3}\,dy = \pi\left[\tfrac{3}{5}y^{5/3}\right]_0^8 = \frac{96\pi}{5}$$

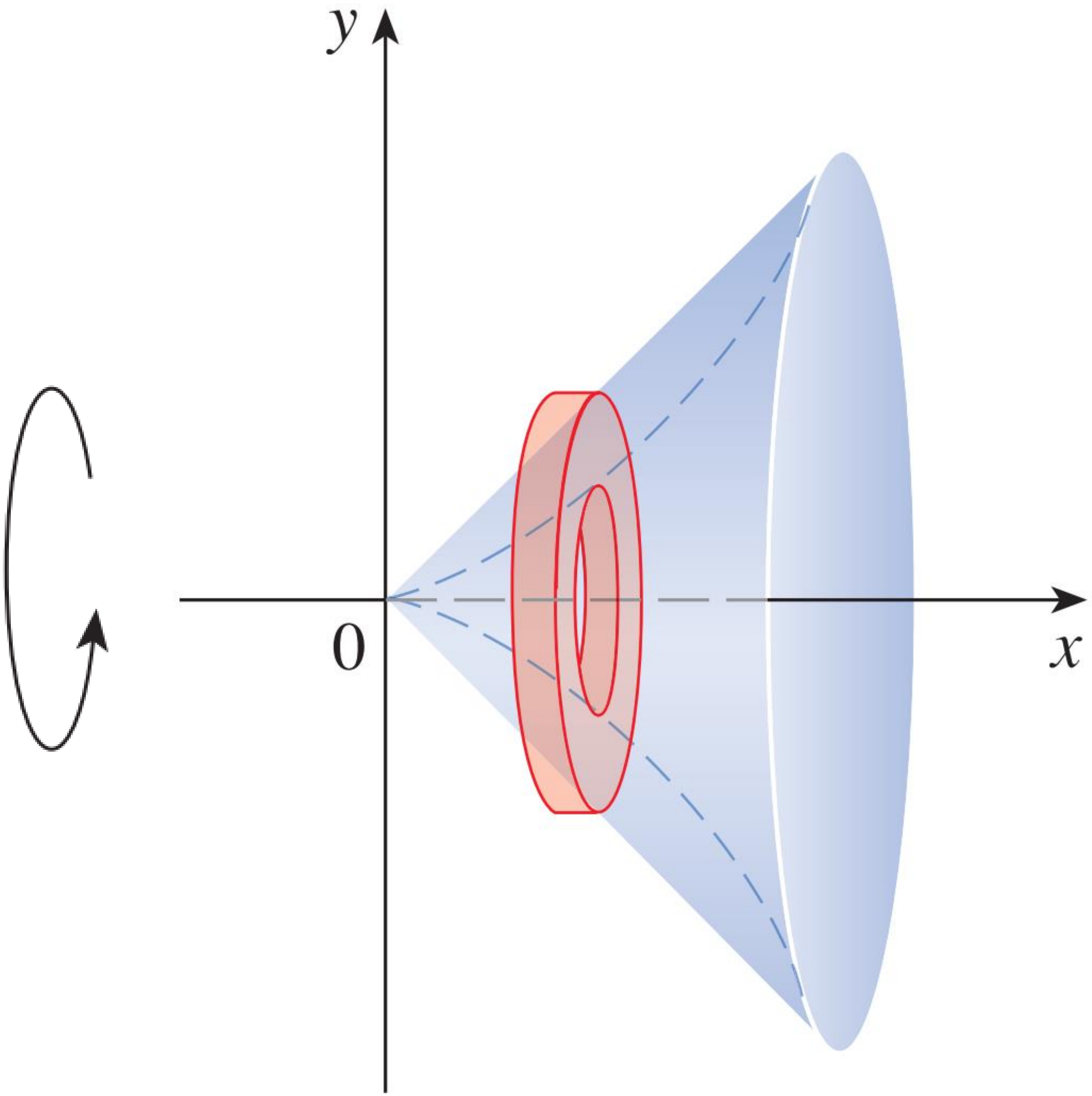
Problem

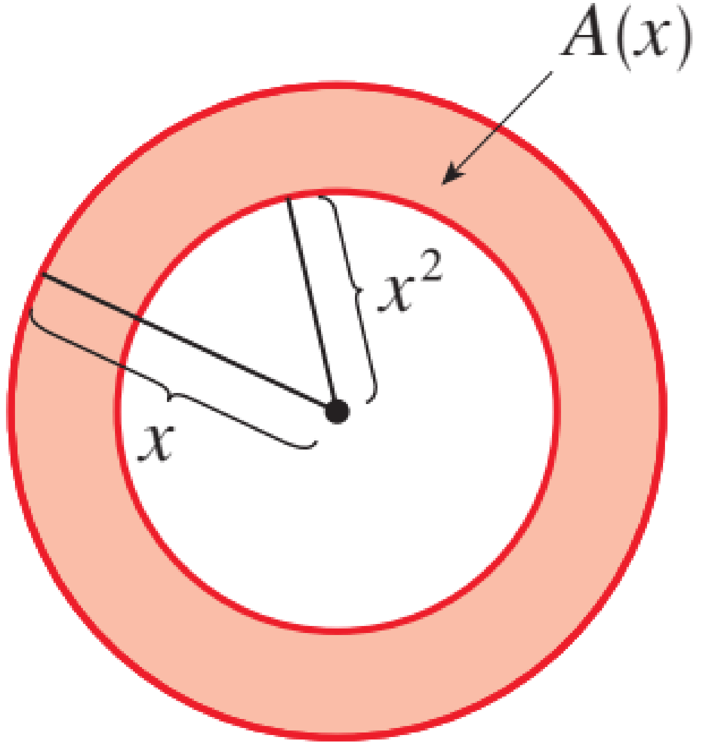
The region $\mathscr{R}$ enclosed by the curves $y = x$ and $y = x^2$ is rotated about the x-axis. Find the volume of the resulting solid.

Solution The curves $y = x$ and $y = x^2$ intersect at the points (0, 0) and (1,1) . The region between them, the solid of rotation, and a cross-section perpendicular to the x-axis are shown in the figure on the next slide. A cross-section in the plane P_x has the shape of a *washer* (an annular ring) with inner radius x^2 and outer radius x, so we find the cross-sectional area by subtracting the area of the inner circle from the area of the outer circle:

$$A(x) = \pi \underbrace{(x)^2}_{\substack{\text{outer}\\\text{radius}}} - \pi \underbrace{(x^2)^2}_{\substack{\text{inner}\\\text{radius}}} = \pi(x^2 - x^4)$$


y
x
(1, 1)
y = x
y = x^2
ℛ
(0, 0)


y
x
0


$A(x)$
x^2
x

Therefore we have

$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi(x^2 - x^4)\,dx$$

$$= \pi\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \frac{2\pi}{15}$$

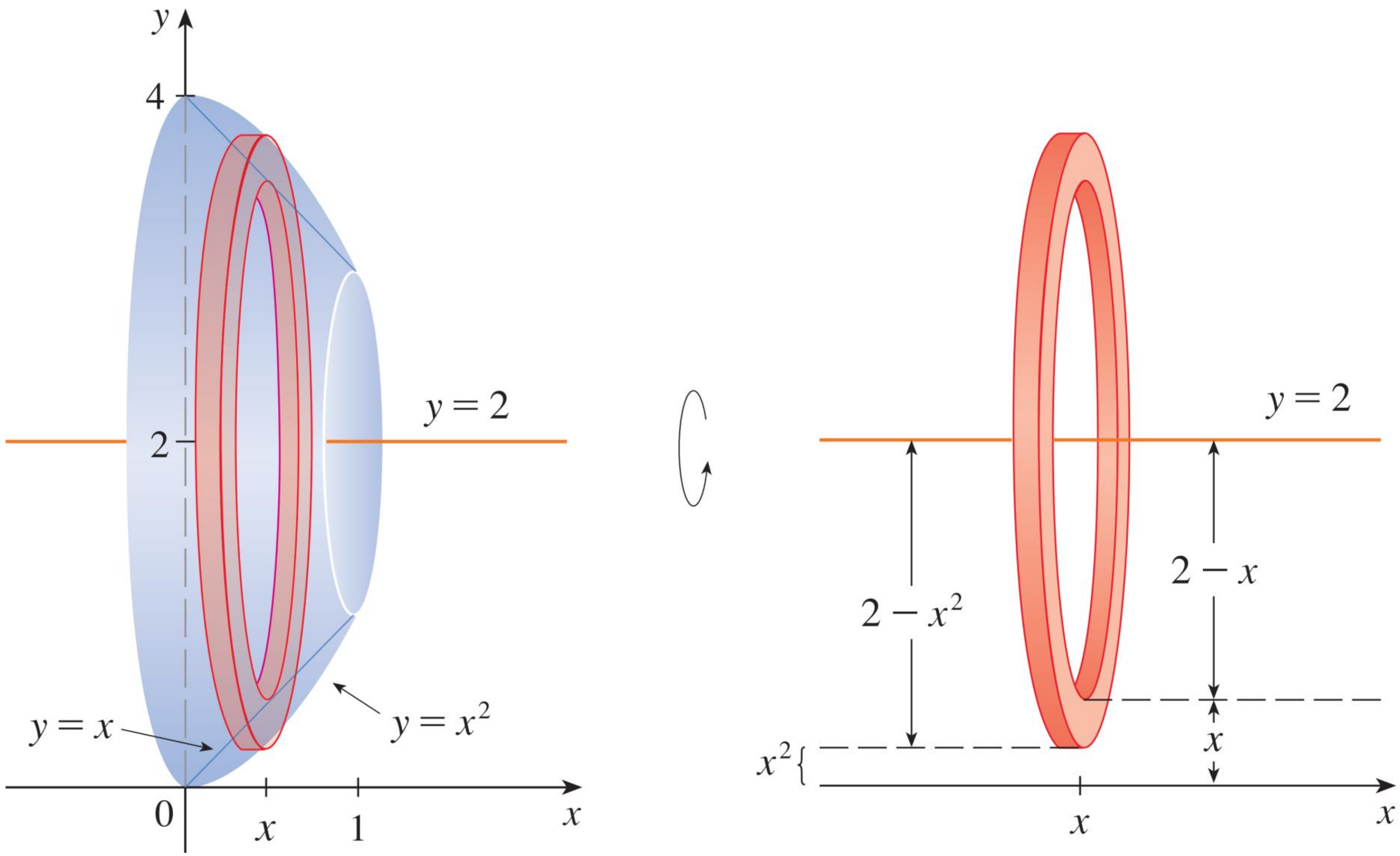
Problem

The region $\mathcal{R}$ enclosed by the curves $y = x$ and $y = x^2$ is rotated about the line $y = 2$. Find the volume of the resulting solid.

Solution

The solid and a cross-section are shown in the figure on the next slide. Again the cross section is a washer, but this time the inner radius is $2 - x$ and the outer radius is $2 - x^2$. The cross-sectional area is

$$A(x) = \pi\underbrace{(2 - x^2)}_{\substack{\text{outer}\\\text{radius}}}{}^2 - \pi\underbrace{(2 - x)}_{\substack{\text{inner}\\\text{radius}}}{}^2$$


y
4
2
y = 2
y = x
y = x^2
0
x
1
x
y = 2
2 − x
2 − x^2
x^2
x
x
x

and so the volume of S is

$$V = \int_0^1 A(x)\,dx = \pi \int_0^1 [(2 - x^2)^2 - (2 - x)^2]$$

$$= \pi \int_0^1 (x^4 - 5x^2 + 4x)\,dx$$

$$= \pi \left[\frac{x^5}{5} - 5\,\frac{x^3}{3} + 4\,\frac{x^2}{2} \right]_0^1 = \frac{8\pi}{15}$$

Summary

In general, we calculate the volume of a solid of revolution by using the basic defining formula

$$V = \int_a^b A(x)\,dx \qquad \text{or} \qquad V = \int_c^d A(y)\,dy$$

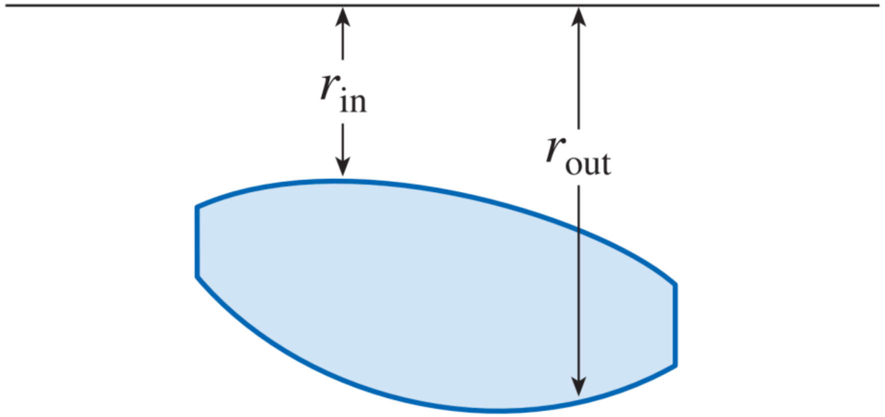
and we find the cross-sectional area $A(x)$ or $A(y)$ in one of the following ways:

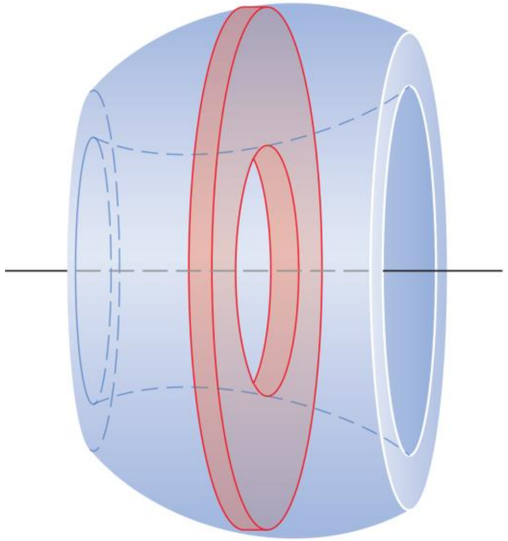
(1) If the cross-section is a disk , we find the radius of the disk (in terms of x or y) and use

$$A = \pi(\text{radius})^2$$

(ii) If the cross-section is a washer , we find the inner radius r_{in} and outer radius r_{out} from a sketch and compute the area of the washer by subtracting the area of the inner disk from the area of the outer disk:

$$A = \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2$$





Problem

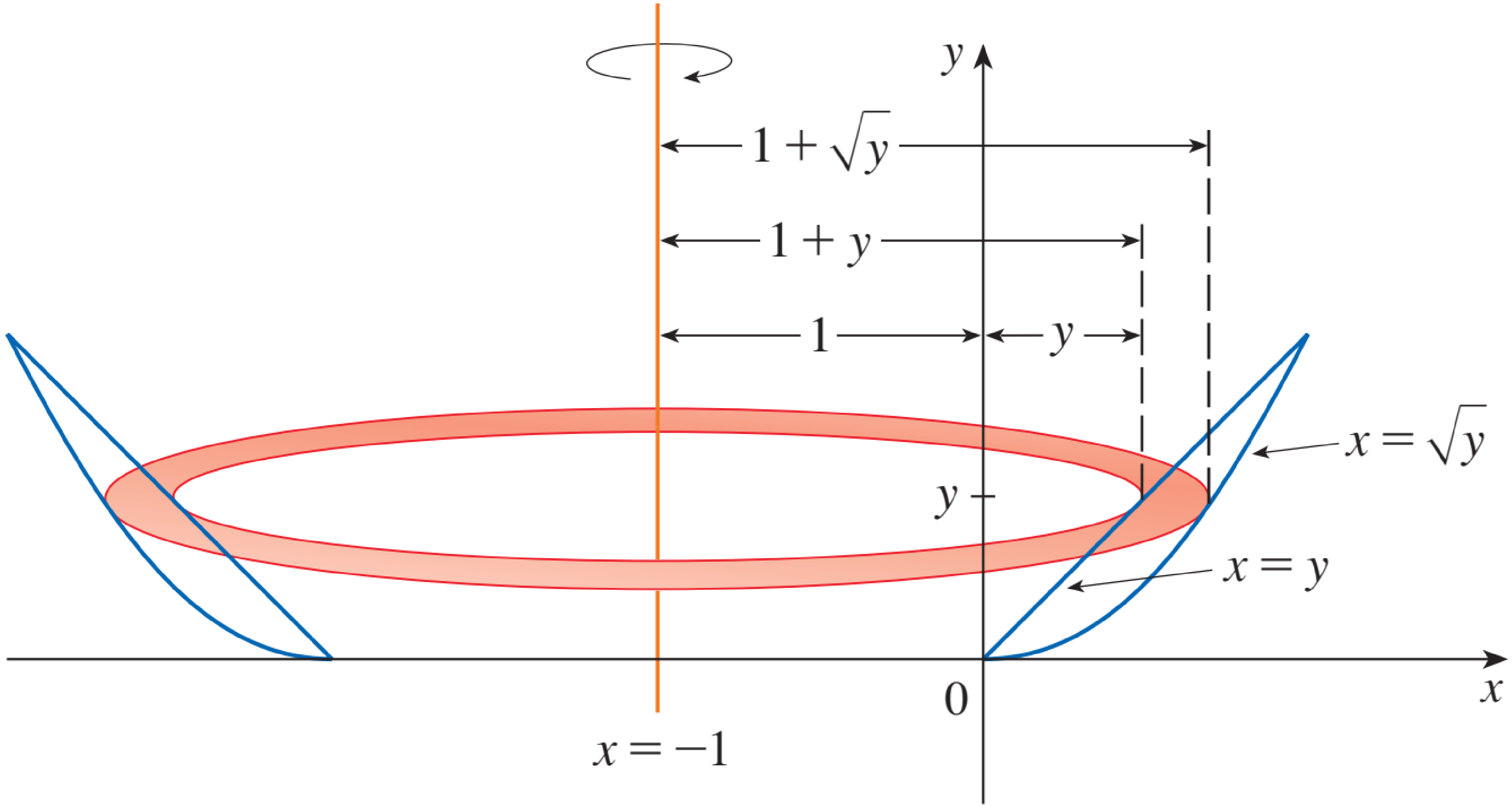
The region $\mathscr{R}$ enclosed by the curves $y = x$ and $y = x^2$ is rotated about the line $x = -1$. Find the volume of the resulting solid.

Solution

Figure on the next slide shows a horizontal cross-section. It is a washer with inner radius $1 + y$ and outer radius $1 + \sqrt{y}$, so the cross-sectional area is

$$A(y) = \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2$$

$$= \pi\left(1 + \sqrt{y}\right)^2 - \pi(1 + y)^2$$



y
$1+\sqrt{y}$
$1+y$
1
y
$x=\sqrt{y}$
y
$x=y$
0
x
$x=-1$

The volume is

$$V = \int_0^1 A(y)\,dy = \pi \int_0^1 \left[\left(1 + \sqrt{y}\right)^2 - (1 + y)^2\right] dy$$

$$= \pi \int_0^1 \left(2\sqrt{y} - y - y^2\right) dy = \pi \left[\frac{4y^{3/2}}{3} - \frac{y^2}{2} - \frac{y^3}{3}\right]_0^1 = \frac{\pi}{2}$$

We now find the volumes of solids that are not solids of revolution but whose cross-sections have areas that are readily computable. Figure 12 shows a solid with a circular base of radius 1. Parallel cross-sections perpendicular to the base are equilateral triangles and we want to compute the volume of the solid

.

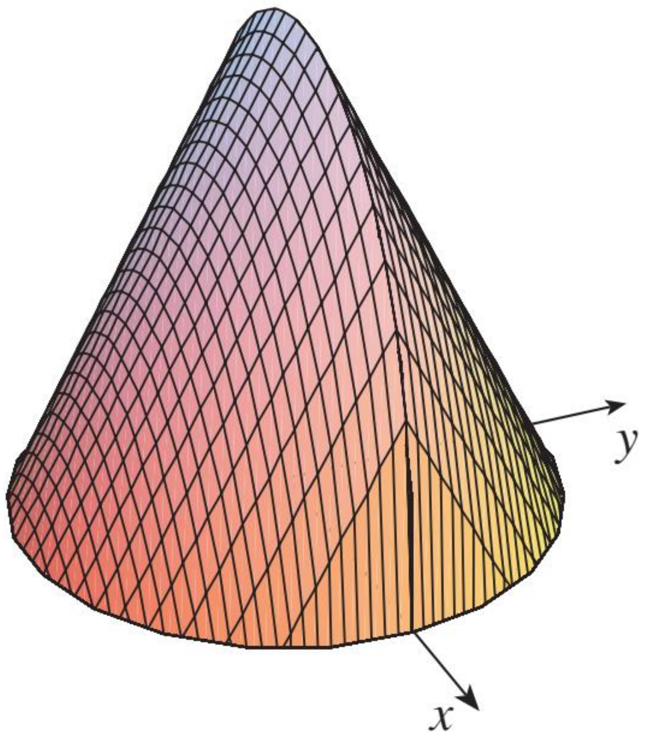


FIGURE 12

SOLUTION Let's take the circle to be $x^2 + y^2 = 1$. The solid, its base, and a typical cross-section at a distance x from the origin are shown in Figure 13.

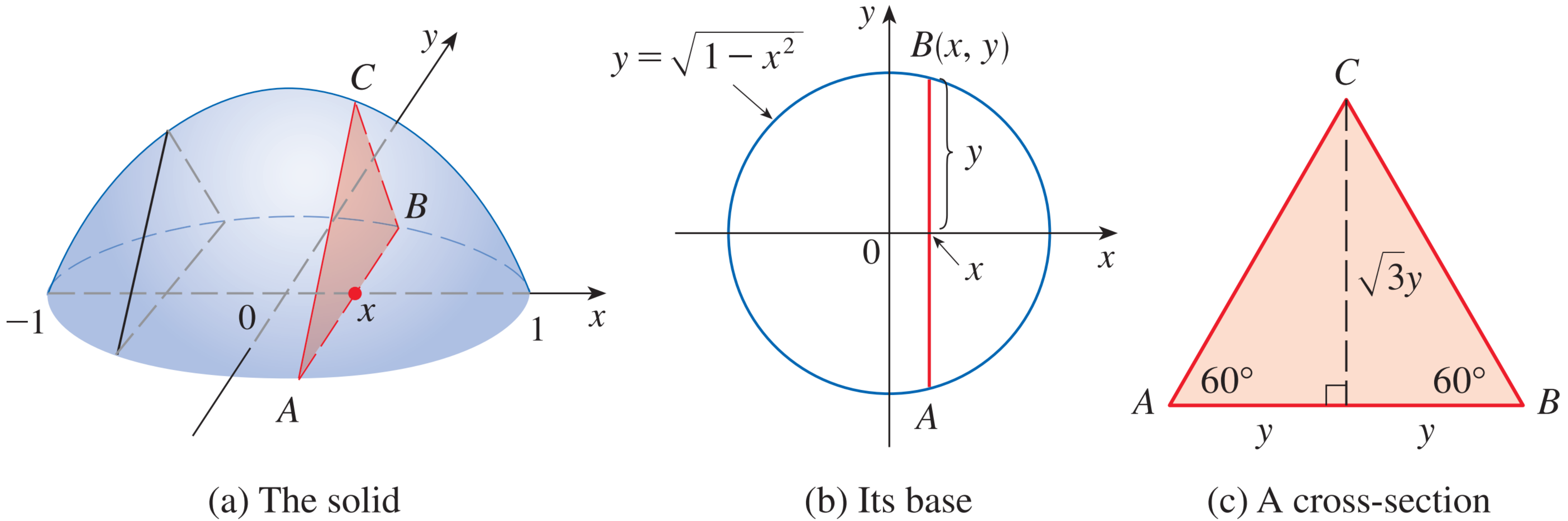


FIGURE 13

Since B lies on the circle, we have $y = \sqrt{1 - x^2}$ and so the base of the triangle ABC is $|AB| = 2y = 2\sqrt{1 - x^2}$. Since the triangle is equilateral, we see from Figure 13(c) that its height is $\sqrt{3}\,y = \sqrt{3}\sqrt{1 - x^2}$. The cross-sectional area is therefore

$$A(x) = \tfrac{1}{2} \cdot 2\sqrt{1 - x^2} \cdot \sqrt{3}\sqrt{1 - x^2} = \sqrt{3}\,(1 - x^2)$$

and the volume of the solid is

$$V = \int_{-1}^{1} A(x)\,dx = \int_{-1}^{1} \sqrt{3}\,(1 - x^2)\,dx$$

$$= 2\int_{0}^{1} \sqrt{3}\,(1 - x^2)\,dx = 2\sqrt{3}\left[x - \frac{x^3}{3}\right]_0^1 = \frac{4\sqrt{3}}{3}$$

Problem

Find the volume of a pyramid whose base is a square with side L and whose height is h.

SOLUTION We place the origin O at the vertex of the pyramid and the x-axis along its central axis as in Figure 14.

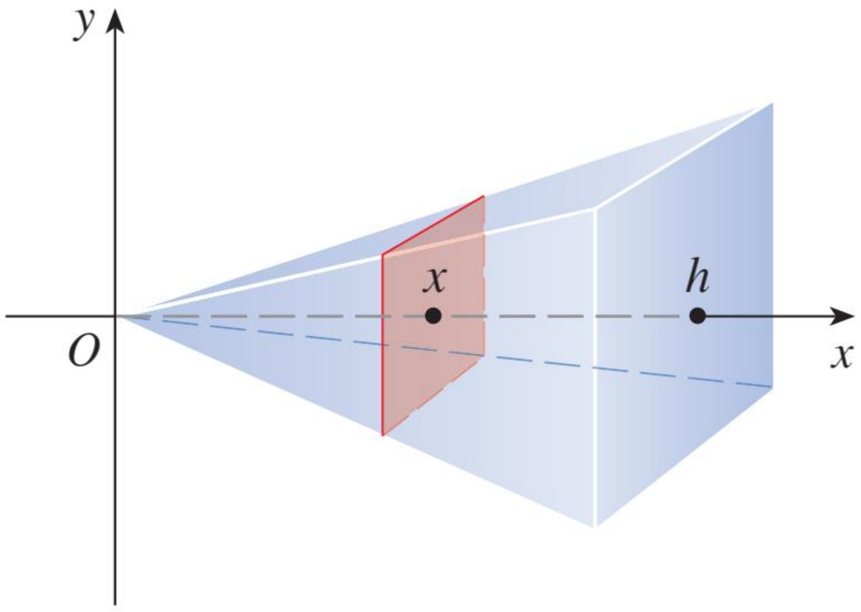


FIGURE 14

Any plane P_x that passes through x and is perpendicular to the x-axis intersects the pyramid in a square with side of length s, say. We can express s in terms of x by observing from the similar triangles in Figure 15 that

$$\frac{x}{h} = \frac{s/2}{L/2} = \frac{s}{L}$$

and so $s = Lx/h$.

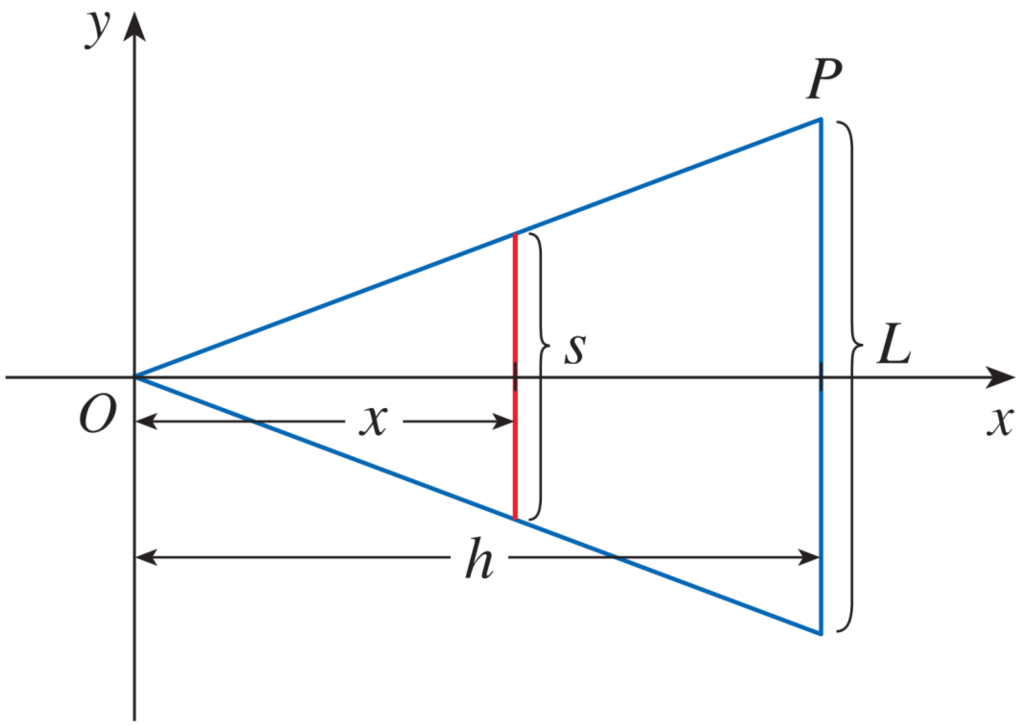


FIGURE 15

Therefore the cross-sectional area is

$$A(x) = s^2 = \frac{L^2}{h^2} x^2$$

The pyramid lies between $x = 0$ and $x = h$, so its volume is

$$V = \int_0^h A(x)\,dx = \int_0^h \frac{L^2}{h^2} x^2\,dx = \frac{L^2}{h^2} \frac{x^3}{3} \Bigg]_0^h = \frac{L^2 h}{3}$$

Improper Integrals

In defining a definite integral $\int_a^b f(x)\,dx$ we dealt with a function f defined on a finite interval $[a, b]$ and we assumed that f does not have an infinite discontinuity

In this section we extend the concept of a definite integral to the case where the interval is infinite and also to the case where f has an infinite discontinuity in $[a, b]$. In either case the integral is called an *improper integral*.

Type 1: Infinite Intervals

Consider the unbounded region S that lies under the curve $y = 1/x^2$, above the x-axis, and to the right of the line $x = 1$. You might think that, since S is infinite in extent, its area must be infinite, but let's take a closer look. The area of the part of S that lies to the left of the line $x = t$ (shaded in Figure 1) is

$$A(t) = \int_1^t \frac{1}{x^2}\,dx = -\frac{1}{x}\bigg]_1^t = 1 - \frac{1}{t}$$

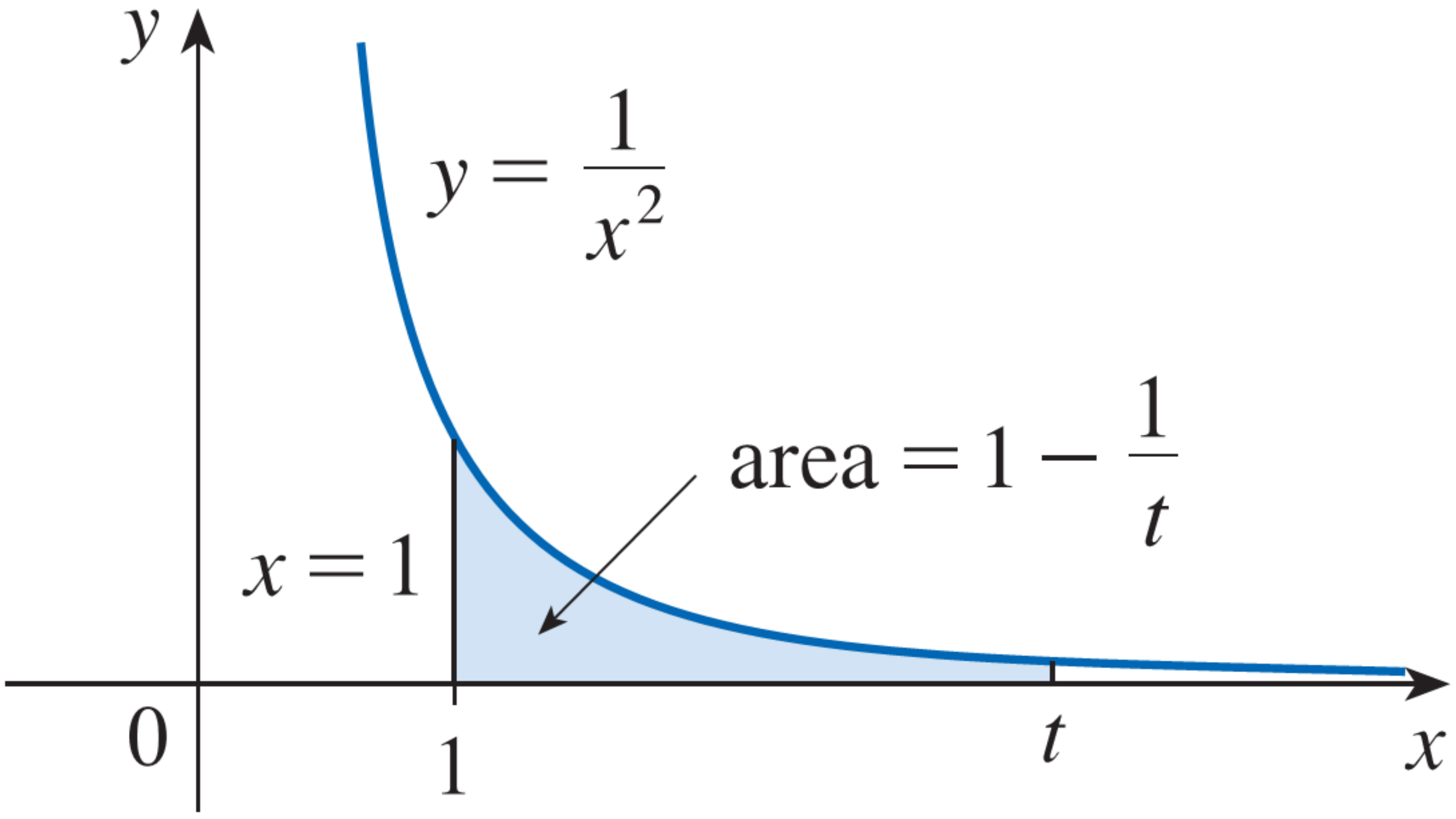


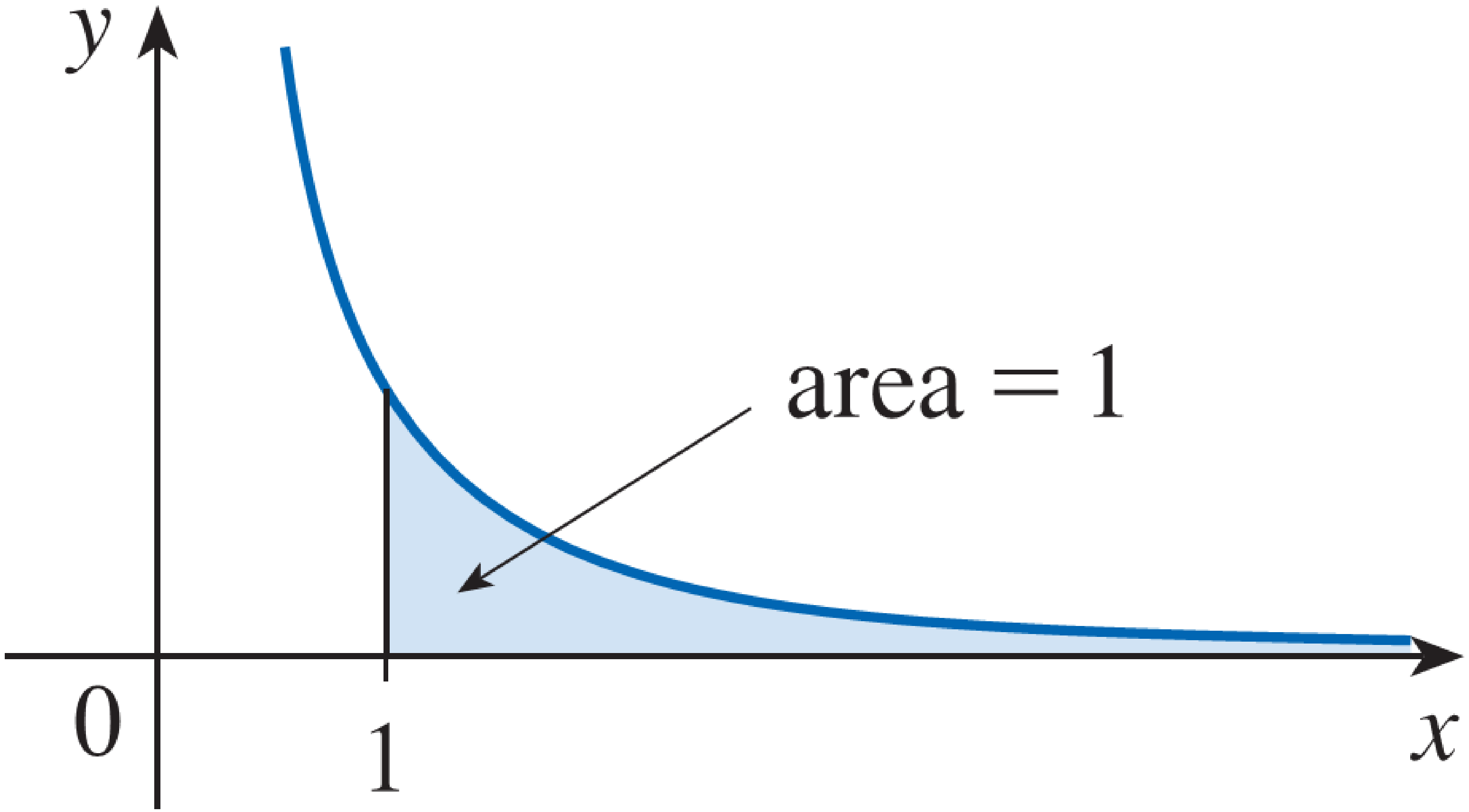
FIGURE 1

We also observe that

$$\lim_{t\to\infty} A(t) = \lim_{t\to\infty} \left(1 - \frac{1}{t}\right) = 1$$

The area of the shaded region approaches 1 as $t \to \infty$ (see Figure 2), so we say that the area of the infinite region S is equal to 1 and we write

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx = 1$$


y
area = 1
0
1
x

$\boxed{1}$ Definition of an Improper Integral of Type 1

(a) If $\int_a^t f(x)\,dx$ exists for every number $t \geqslant a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t \to \infty} \int_a^t f(x)\,dx$$

provided this limit exists (as a finite number).

(b) If $\int_t^b f(x)\,dx$ exists for every number $t \leqslant b$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t \to -\infty} \int_t^b f(x)\,dx$$

provided this limit exists (as a finite number).

The improper integrals $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent, then we define

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx$$

Let's Evalute $\displaystyle\int_{-\infty}^{0} x e^{x}\,dx = \lim_{t \to -\infty} \int_{t}^{0} x e^{x}\,dx$ —— $(\star)$

First we will evaluate

$$\int_{t}^{0} x e^{x}\,dx = \left(x e^{x}\right)\Big|_{t}^{0} - \int_{t}^{0} e^{x}\,dx = \left(x e^{x}\right)\Big|_{t}^{0} - e^{x}\Big|_{t}^{0}$$

$$= \left(0 - t e^{t}\right) - \left(1 - e^{t}\right)$$

$$\Rightarrow \boxed{\int_{t}^{0} x e^{x}\,dx = -t e^{t} - 1 + e^{t}}$$

So (*) becomes

$$\int_{-\infty}^{0} x e^{x}\,dx = \lim_{t \to -\infty}\left(-te^{t} - 1 + e^{t}\right)$$

$$= -\lim_{t \to -\infty}\left(te^{t}\right) - 1 + \lim_{t \to -\infty}\left(e^{t}\right) \quad \text{--- (i)}$$

Now

$$\lim_{t \to -\infty}\left(te^{t}\right) = \lim_{t \to -\infty}\left(\frac{t}{e^{-t}}\right) \quad \left(\frac{-\infty}{\infty}\right) \text{ form}$$

$$= \lim_{t \to -\infty}\left(\frac{1}{-e^{-t}}\right) = 0 \quad \text{[L-hospital Rule Used]}$$

Also $\lim\limits_{t\to-\infty}\left(e^{t}\right) = \lim\limits_{t\to-\infty}\left(\frac{1}{e^{-t}}\right) = 0$

Substituting values in eq (i) we get

$$\int_{-\infty}^{0} x e^{x}\, dx = 0 - 1 + 0 = -1$$

$$\Rightarrow \boxed{\int_{-\infty}^{0} x e^{x}\, dx = -1}$$ Ans

Problem

Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1 + x^2}\, dx.$

Solution

$$\int_{-\infty}^{\infty} \frac{1}{1 + x^2}\, dx = \int_{-\infty}^{0} \frac{1}{1 + x^2}\, dx + \int_{0}^{\infty} \frac{1}{1 + x^2}\, dx$$

We must now evaluate the integrals on the right side separately:

$$\int_0^\infty \frac{1}{1 + x^2}\,dx = \lim_{t\to\infty} \int_0^t \frac{dx}{1 + x^2} = \lim_{t\to\infty} \tan^{-1}x\Big]_0^t$$

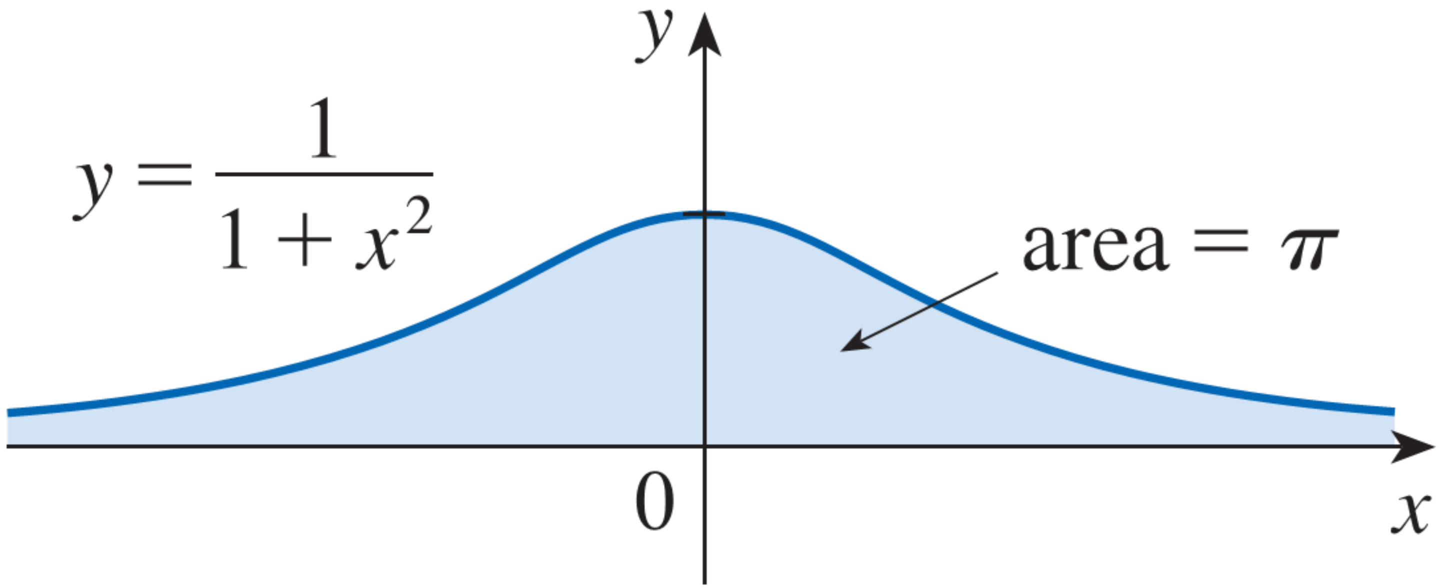
$$= \lim_{t\to\infty} (\tan^{-1}t - \tan^{-1}0) = \lim_{t\to\infty} \tan^{-1}t = \frac{\pi}{2}$$

$$\int_{-\infty}^0 \frac{1}{1 + x^2}\,dx = \lim_{t\to-\infty} \int_t^0 \frac{dx}{1 + x^2} = \lim_{t\to-\infty} \tan^{-1}x\Big]_t^0$$

$$= \lim_{t\to-\infty} (\tan^{-1}0 - \tan^{-1}t) = 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$

Since both of these integrals are convergent, the given integral is convergent and

$$\int_{-\infty}^{\infty} \frac{1}{1 + x^2}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$



Type 2: Discontinuous Integrands

Suppose that f is a positive continuous function defined on a finite interval $[a, b)$ but has a vertical asymptote at b. Let S be the unbounded region under the graph of f and above the x-axis between a and b. (For Type 1 integrals, the regions extended indefinitely in a horizontal direction. Here the region is infinite in a vertical direction.) The area of the part of S between a and t (the shaded region in Figure 7) is

$$A(t) = \int_a^t f(x)\,dx$$

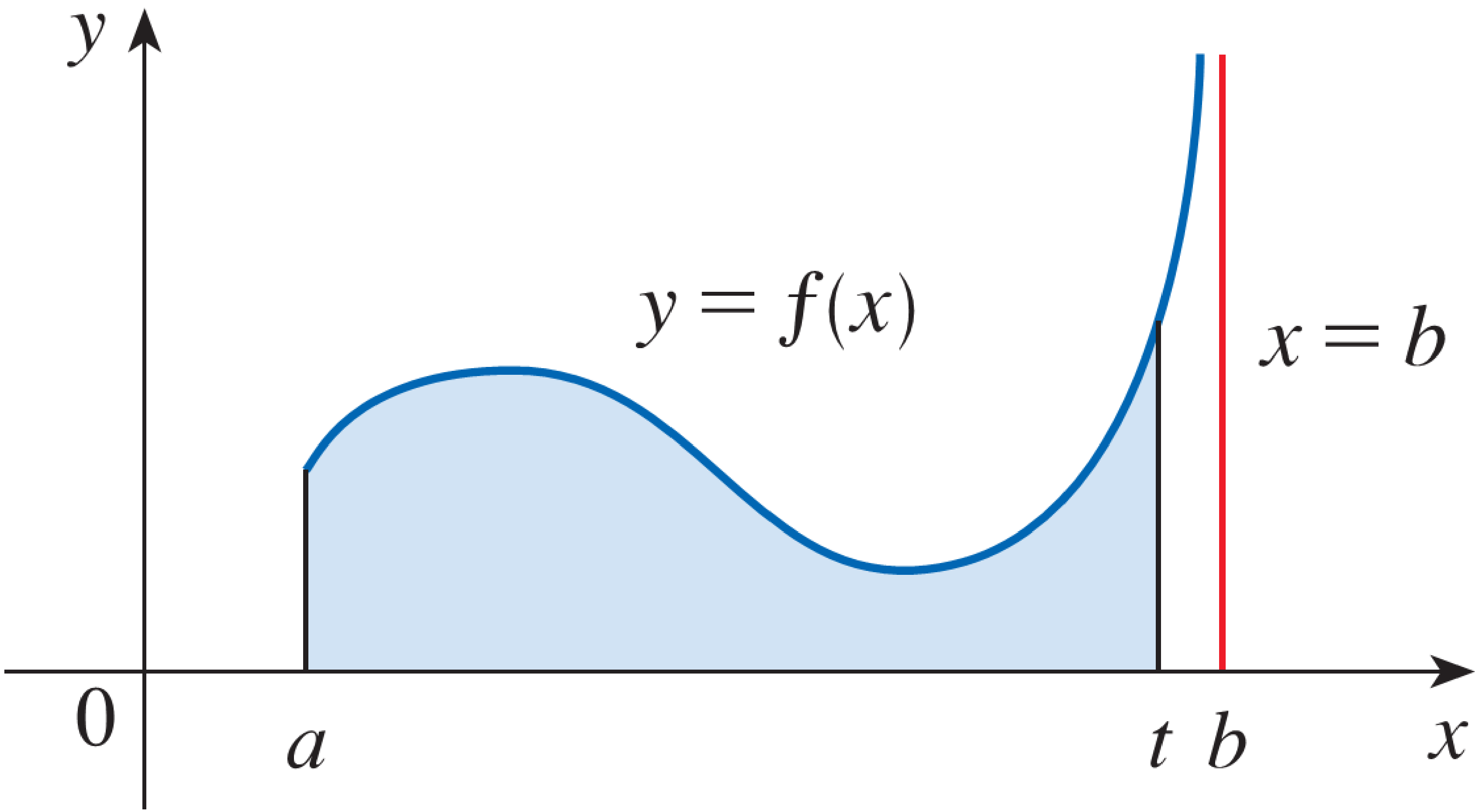


FIGURE 7

If it happens that $A(t)$ approaches a definite number A as $t \longrightarrow b^-$, then we say that the area of the region S is A and we write

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

$\boxed{3}$ Definition of an Improper Integral of Type 2

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

if this limit exists (as a finite number).

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$

if this limit exists (as a finite number).

The improper integral $\int_a^b f(x)\,dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Problem

$$\text{Find } \int_2^5 \frac{1}{\sqrt{x-2}}\,dx.$$

Solution

We note first that the given integral is improper because $f(x) = 1/\sqrt{x-2}$ has the vertical asymptote $x = 2$. Since the infinite discontinuity occurs at the left endpoint of $[2, 5]$, we use part (b) of Definition 3:

$$\int_2^5 \frac{dx}{\sqrt{x-2}} = \lim_{t \to 2^+} \int_t^5 \frac{dx}{\sqrt{x-2}} = \lim_{t \to 2^+} 2\sqrt{x-2}\,\Big]_t^5$$

$$\Rightarrow \quad \int_2^5 \frac{dx}{\sqrt{x-2}} = \lim_{t \to 2^+} 2\left(\sqrt{3} - \sqrt{t-2}\right) = 2\sqrt{3}$$

Problem

Evaluate $\int_0^3 \frac{dx}{x-1}$ if possible.

Solution

Observe that the line $x = 1$ is a vertical asymptote of the integrand. Since it occurs in the middle of the interval $[0, 3]$, we must use part (c) of Definition 3 with $c = 1$:

$$\int_0^3 \frac{dx}{x-1} = \int_0^1 \frac{dx}{x-1} + \int_1^3 \frac{dx}{x-1}$$

where $$\int_0^1 \frac{dx}{x-1} = \lim_{t \to 1^-} \int_0^t \frac{dx}{x-1} = \lim_{t \to 1^-} \ln|x-1|\Big]_0^t$$

$$= \lim_{t \to 1^-} \left(\ln|t-1| - \ln|-1|\right) = \lim_{t \to 1^-} \ln(1-t) = -\infty$$

because $1 - t \to 0^+$ as $t \to 1^-$. Thus $\int_0^1 dx/(x-1)$ is divergent. This implies that $\int_0^3 dx/(x-1)$ is divergent. [We do not need to evaluate $\int_1^3 dx/(x-1)$.]

Practice Problems

Determine whether the integral is convergent or divergent.

Evaluate integrals that are convergent.

$$\int_3^\infty \frac{1}{(x-2)^{3/2}}\,dx$$

$$\int_{-\infty}^0 \frac{x}{(x^2+1)^3}\,dx$$

$$\int_1^\infty \frac{e^{-1/x}}{x^2}\,dx$$

$$\int_0^\infty \sin\theta\, e^{\cos\theta}\,d\theta$$

$$\int_2^\infty \frac{dv}{v^2+2v-3}$$

$$\int_{-1}^{0} \frac{e^{1/x}}{x^3}\,dx$$

$$\int_{0}^{4} \frac{dx}{x^2 - x - 2}$$

$$\int_{-1}^{2} \frac{x}{(x+1)^2}\,dx$$

$$\int_{1}^{\infty} \frac{\ln x}{x^2}\,dx$$

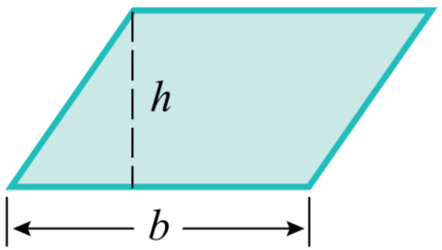
Formulas

GEOMETRY

A = area, S = lateral surface area, V = volume, h = height

B = area of base, r = radius, l = slant height, C = circumference, s = arc length

Parallelogram

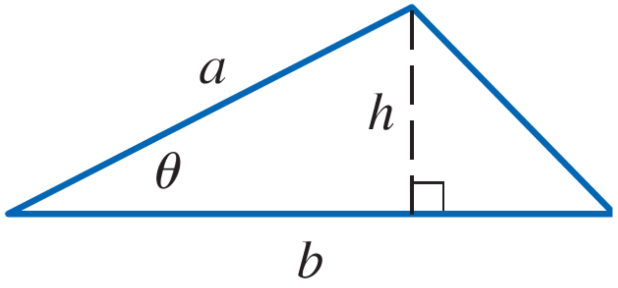


$A = bh$

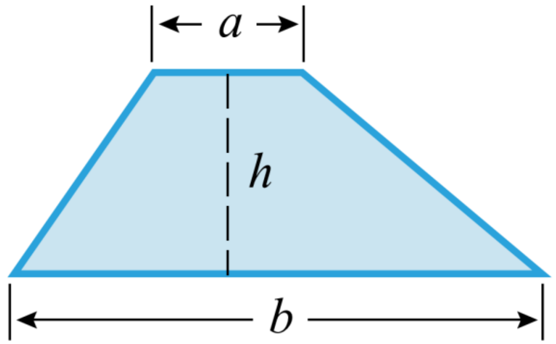
Triangle

$A = \frac{1}{2} bh$

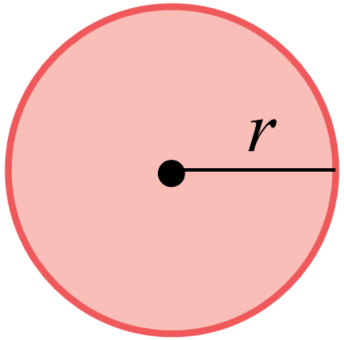
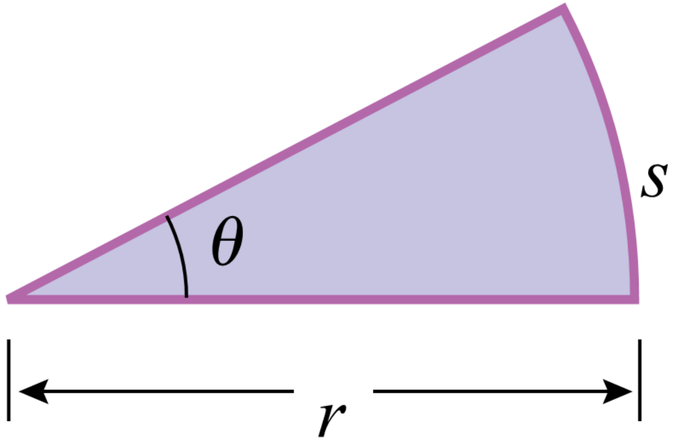
$= \frac{1}{2} ab \sin \theta$



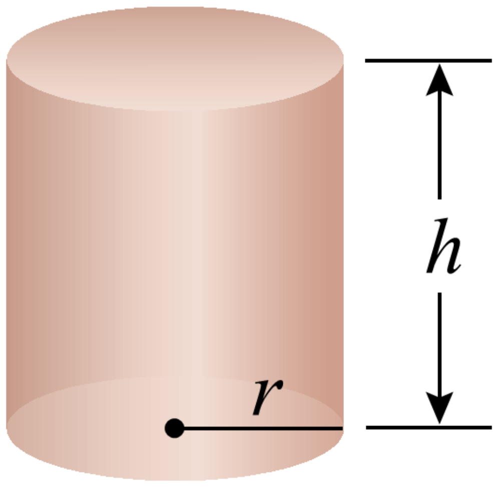
Trapezoid



$A = \frac{1}{2}(a + b)h$

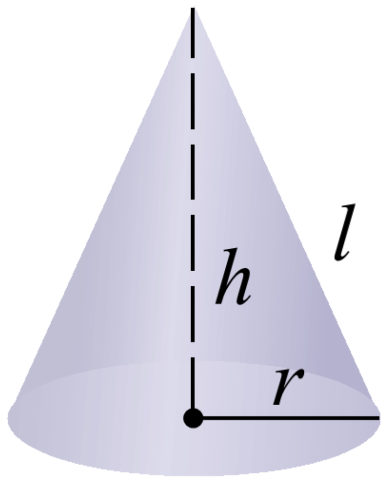
Circle	Sector
	
$A = \pi r^2, \; C = 2\pi r$	$A = \frac{1}{2} r^2\theta, \; s = r\theta$ (θ in radians)

Right Circular Cylinder



$$V = \pi r^2 h,\ S = 2\pi r h$$

Right Circular Cone



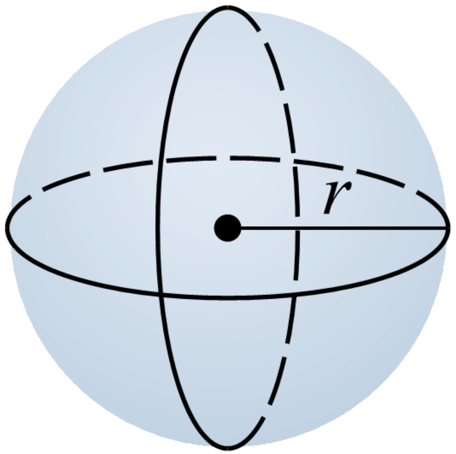
$$V = \frac{1}{3}\pi r^2 h,\ S = \pi r l$$

$$A = \pi r \sqrt{r^2 + h^2}$$

Any Cylinder or Prism with Parallel Bases

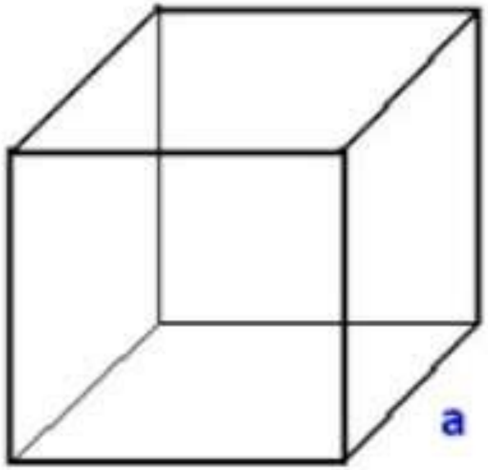
$$V = Bh$$

Sphere



$$V = \frac{4}{3}\pi r^3, \; S = 4\pi r^2$$

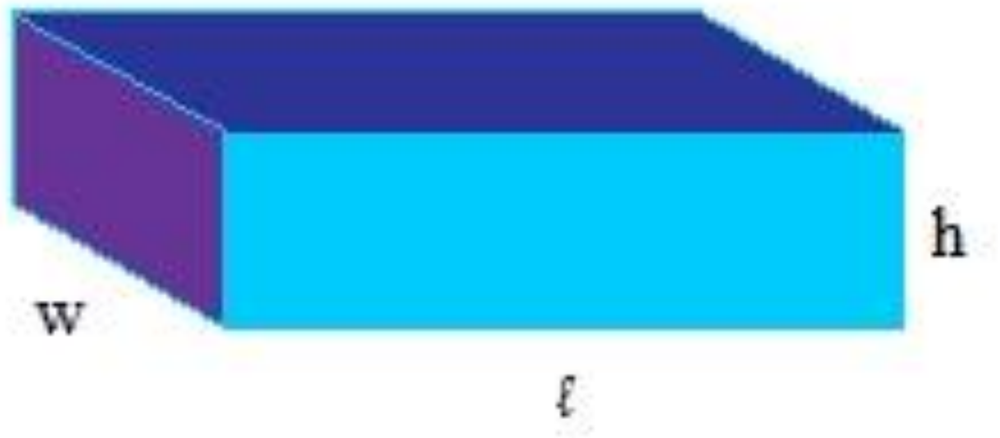
$$A = 4\pi r^2$$



Cube

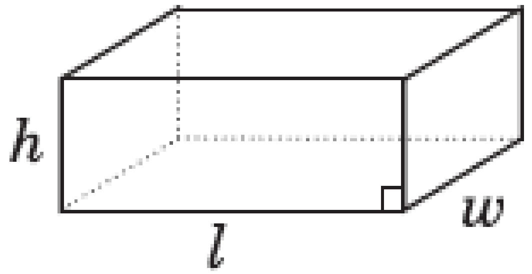
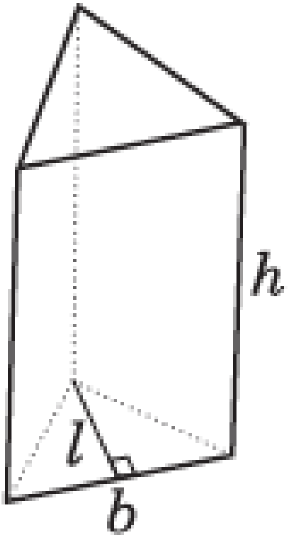
Volume of a Cube $= a^3$

Volume of a box



$$V_{box} = \text{length} \times \text{width} \times \text{height}$$

$$V_{box} = l \times w \times h$$

Rectangular prism 	$V = (\text{area of base})(\text{height})$ $V = lwh$
Triangular prism 	$V = (\text{area of base})(\text{height})$ $V = \frac{1}{2}\, blh$ or $V = \frac{blh}{2}$

Distance and Midpoint Formulas

Distance between $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Midpoint of $\overline{P_1P_2}$: $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$

Lines

Slope of line through $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Point-slope equation of line through $P_1(x_1, y_1)$ with slope m:

$$y - y_1 = m(x - x_1)$$

Slope-intercept equation of line with slope m and y-intercept b:

$$y = mx + b$$

Factoring Special Polynomials

$$x^2 - y^2 = (x + y)(x - y)$$

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$$

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$

Binomial Theorem

$$(x + y)^2 = x^2 + 2xy + y^2 \qquad\qquad (x - y)^2 = x^2 - 2xy + y^2$$

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$$

$$(x + y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2}x^{n-2}y^2 + \cdots + \binom{n}{k}x^{n-k}y^k + \cdots + nxy^{n-1} + y^n$$

where $\displaystyle\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{1 \cdot 2 \cdot 3 \cdot \cdots \cdot k}$

Inequalities and Absolute Value

If $a < b$ and $b < c$, then $a < c$.

If $a < b$, then $a + c < b + c$.

If $a < b$ and $c > 0$, then $ca < cb$.

If $a < b$ and $c < 0$, then $ca > cb$.

If $a > 0$, then

$$|x| = a \quad \text{means} \quad x = a \quad \text{or} \quad x = -a$$

$$|x| < a \quad \text{means} \quad -a < x < a$$

$$|x| > a \quad \text{means} \quad x > a \quad \text{or} \quad x < -a$$

Fundamental Identities

$$\csc\theta = \frac{1}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\cot\theta = \frac{1}{\tan\theta}$$

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

$$\sin(-\theta) = -\sin\theta$$

$$\cos(-\theta) = \cos\theta$$

$$\tan(-\theta) = -\tan\theta$$

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$$

Addition and Subtraction Formulas

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

$$\cos(x - y) = \cos x \cos y + \sin x \sin y$$

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

Double-Angle Formulas

$$\sin 2x = 2 \sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

Half-Angle Formulas

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

DIFFERENTIATION RULES

General Formulas

$$\frac{d}{dx}(c) = 0$$

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$$

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x) \quad \text{(Product Rule)}$$

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x) \quad \text{(Chain Rule)}$$

$$\frac{d}{dx}[cf(x)] = cf'(x)$$

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2} \quad \text{(Quotient Rule)}$$

$$\frac{d}{dx}(x^n) = nx^{n-1} \quad \text{(Power Rule)}$$

Exponential and Logarithmic Functions

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(b^x) = b^x \ln b$$

$$\frac{d}{dx}\ln|x| = \frac{1}{x}$$

$$\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$$

Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\csc^{-1}x) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$$

Hyperbolic Functions

$$\frac{d}{dx}(\sinh x) = \cosh x$$

$$\frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

$$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$$

$$\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

Inverse Hyperbolic Functions

$$\frac{d}{dx}(\sinh^{-1}x) = \frac{1}{\sqrt{1 + x^2}}$$

$$\frac{d}{dx}(\cosh^{-1}x) = \frac{1}{\sqrt{x^2 - 1}}$$

$$\frac{d}{dx}(\operatorname{csch}^{-1}x) = -\frac{1}{|x|\sqrt{x^2 + 1}}$$

$$\frac{d}{dx}(\operatorname{sech}^{-1}x) = -\frac{1}{x\sqrt{1 - x^2}}$$

$$\frac{d}{dx}(\tanh^{-1}x) = \frac{1}{1 - x^2}$$

$$\frac{d}{dx}(\coth^{-1}x) = \frac{1}{1 - x^2}$$

TABLE OF INTEGRALS

Basic Forms

$$\int u\,dv = uv - \int v\,du$$

$$\int e^u\,du = e^u + C$$

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$$

$$\int b^u\,du = \frac{b^u}{\ln b} + C$$

$$\int \frac{du}{u} = \ln|u| + C$$

$$\int \sin u\,du = -\cos u + C$$

$$\int \cos u \, du = \sin u + C$$

$$\int \sec^2 u \, du = \tan u + C$$

$$\int \csc^2 u \, du = -\cot u + C$$

$$\int \sec u \, \tan u \, du = \sec u + C$$

$$\int \csc u \, \cot u \, du = -\csc u + C$$

$$\int \tan u \, du = \ln|\sec u| + C$$

$$\int \cot u \, du = \ln|\sin u| + C$$

$$\int \sec u \, du = \ln|\sec u + \tan u| + C$$

$$\int \csc u \, du = \ln|\csc u - \cot u| + C$$

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a} + C, \quad a > 0$$

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$$

$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a} + C$$

$$\int \frac{du}{a^2 - u^2} = \frac{1}{2a} \ln \left| \frac{u + a}{u - a} \right| + C$$

$$\int \frac{du}{u^2 - a^2} = \frac{1}{2a} \ln \left| \frac{u - a}{u + a} \right| + C$$

$$\int \frac{du}{\sqrt{a^2 + u^2}} = \ln\left(u + \sqrt{a^2 + u^2}\right) + C$$

$$\int \frac{du}{\sqrt{u^2 - a^2}} = \ln \left| u + \sqrt{u^2 - a^2} \right| + C$$

Reduction Formulas

$$\int \sin^n u \, du = -\frac{1}{n} \sin^{n-1} u \cos u + \frac{n-1}{n} \int \sin^{n-2} u \, du$$

$$\int \cos^n u \, du = \frac{1}{n} \cos^{n-1} u \sin u + \frac{n-1}{n} \int \cos^{n-2} u \, du$$

$$\int \tan^n u \, du = \frac{1}{n-1} \tan^{n-1} u - \int \tan^{n-2} u \, du$$

$$\int \cot^n u \, du = \frac{-1}{n-1} \cot^{n-1} u - \int \cot^{n-2} u \, du$$

$$\int \sec^n u \, du = \frac{1}{n-1} \tan u \sec^{n-2} u + \frac{n-2}{n-1} \int \sec^{n-2} u \, du$$

$$\int \csc^n u \, du = \frac{-1}{n-1} \cot u \csc^{n-2} u + \frac{n-2}{n-1} \int \csc^{n-2} u \, du$$

$$\int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + C, \quad n \neq -1$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

Thanks (The End)