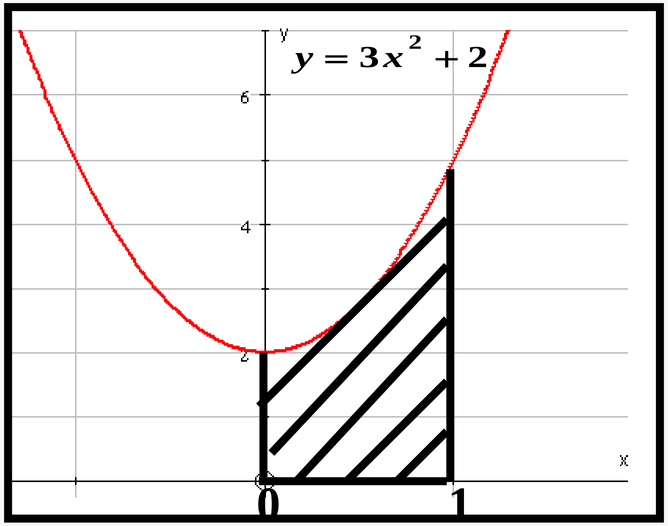
Numerical integration

Crash course...
Find the integration of a function

- Finding the ***integral*** of a function is useful (yawn, yawn!)
- Let's look at an example...
- If we plot *y* against *x* we can determine the area under the curve between the start and the end x- points.
- So, if we had a relationship (function) $y = 3x^2 + 2$ what is the area between the curve and x-values from *x*=0 to *x*=1?
- We can answer this by finding out the shaded area?



Crash course...
Find the integration of a function

- We can use 1mm x 1mm squared paper and count up all the little squares (remind you of maths class at school? boring?easy?)
- Or we can do some clever Maths called ***Integration*** and use some fancy symbols to describe what we want to do.

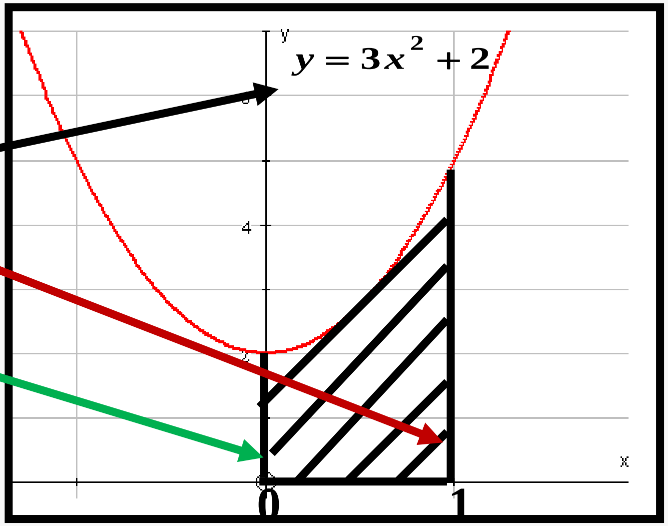
Upper limit

Integration sign

Lower limit

$$\int_{x=0}^{x=1} 3x^2 + 2\,dx$$

function



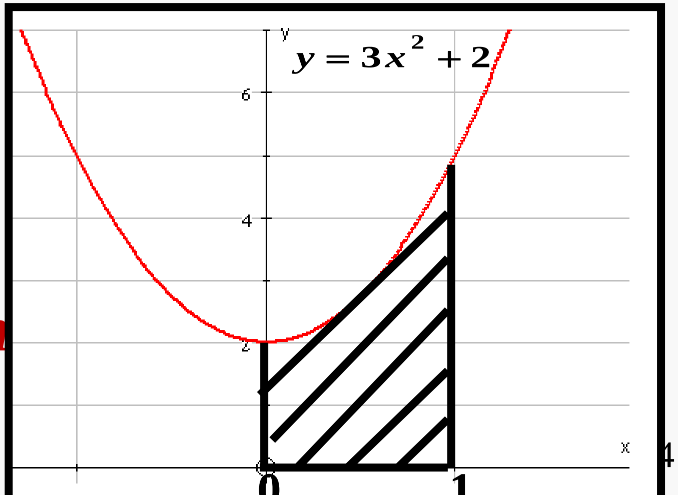
Crash course...
Find the integration of a function

- Now to solve it

$$\int_{x=0}^{x=1}(3x^2+2)\,dx = \left[\frac{3x^{3}}{3}+\frac{2x^{1}}{1}\right] = \left[\frac{3x^3}{3}+\frac{2x}{1}\right]_0^1$$

$$= \left[\frac{3(1)^3}{3}+\frac{2(1)}{1}\right]-\left[\frac{3(0)^3}{3}+\frac{2(0)}{1}\right]$$

$$=[1+2]=3$$



- So area=3units
- This is the ***analytic (actual) solution***

Crash course...
What is the basic rule?

- If we want to ***integrate*** a ***polynomial*** (something that has powers of x) then the rule is simple...
- RULE: **Add 1 to the power of *x* and then divide by the new power**
- Examples:

$$\int_{x=2}^{x=4} x^4 + 2x^3 \, dx = \left[\frac{x^5}{5} + \frac{2x^4}{4}\right]_2^4 = \left[\frac{4^5}{5} + \frac{2.4^4}{4}\right] - \left[\frac{2^5}{5} + \frac{2.2^4}{4}\right]$$

$$= 332.8 - 14.4 = 318.4$$

$$\int_{x=-1}^{x=3} 4x^7 + 6 \, dx = \left[\frac{4x^8}{8} + \frac{6x}{1}\right]_{-1}^3 = \left[\frac{4(3^8)}{8} + \frac{6(3)}{1}\right] - \left[\frac{4(-1)^8}{8} + \frac{6(-1)}{1}\right]$$

$$= 3298.5 - (-5.5) = 3304$$

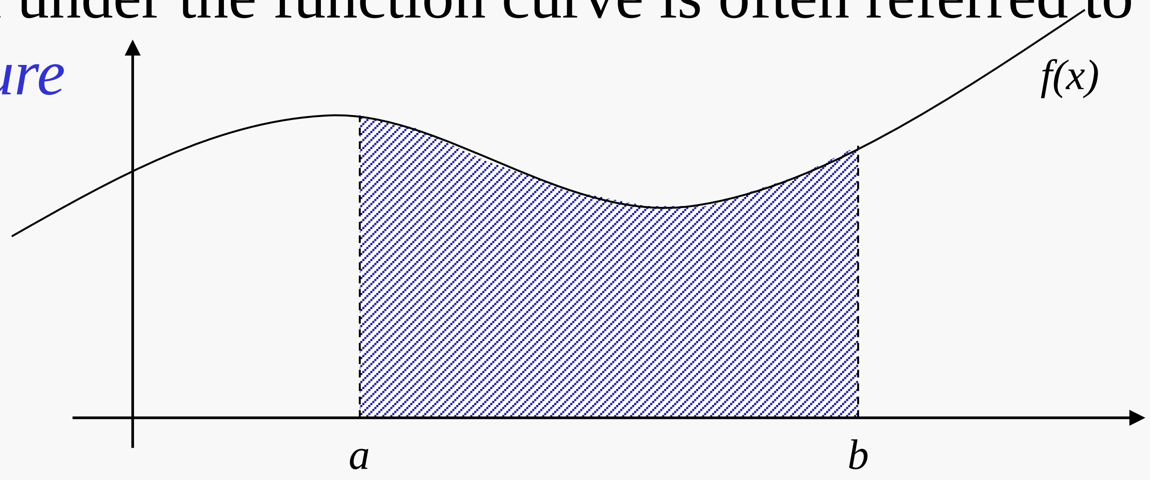
Numerical computation of the integration of a function

- Not always possible to get analytical solution, so the challenge is to compute an approximation

$$\int_a^b f(x)dx$$

by using values of the function at a finite number of discrete points

- The area under the function curve is often referred to as the *quadrature*



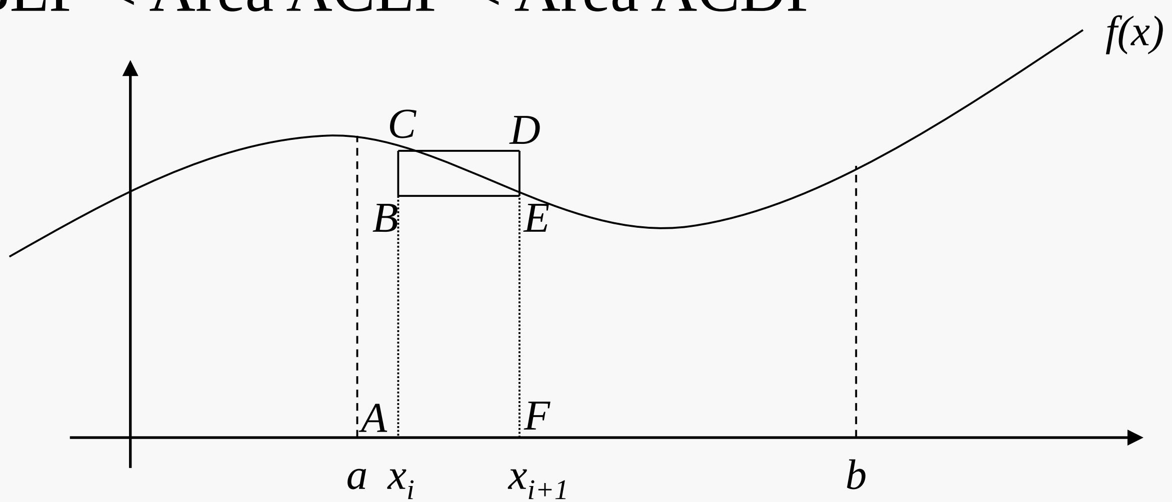
Numerical computation of the integration of a function

- Given the function $f(x)$, setting

$$x_0 = a \quad and \quad x_i = x_0 + i\Delta x \quad i = 1, 2, \ldots, n \quad where \quad \Delta x = \frac{b-a}{n}$$

- A typical vertical strip between x_i and x_{i+1} satisfies the inequality

  Area ABEF < Area ACEF < Area ACDF



- As $\Delta x \rightarrow 0$ ABEF and ACDF are equal for a well behaved function
- As n increases the accuracy of the solution improves

How good is the answer?

- "the error reduces when the accuracy improves..." this is not enough
- Need to be more precise about such phrases - try answering "by how much?"
- Need to have some idea about the *actual solution*, so given **true value** and our **approximate value** we can quantify some accuracy by describing the **error**
- There are two simple measurements we can compute
  - *absolute true error*
  - *absolute relative true error*

true and relative true error

$$absolute\ true\ error \quad = \quad \left| true\ value - approximate\ value \right|$$

$$absolute\ relative\ true\ error \quad = \quad \left| \frac{true\ value - approximate\ value}{true\ value} \right|$$

- Example: approximate value=12.20, true value=12.12. What are the *absolute true* and *absolute relative true* errors?

$$absolute\ true\ error \quad = \quad \left|12.12 - 12.20\right| = \left|-0.08\right| = 0.08$$

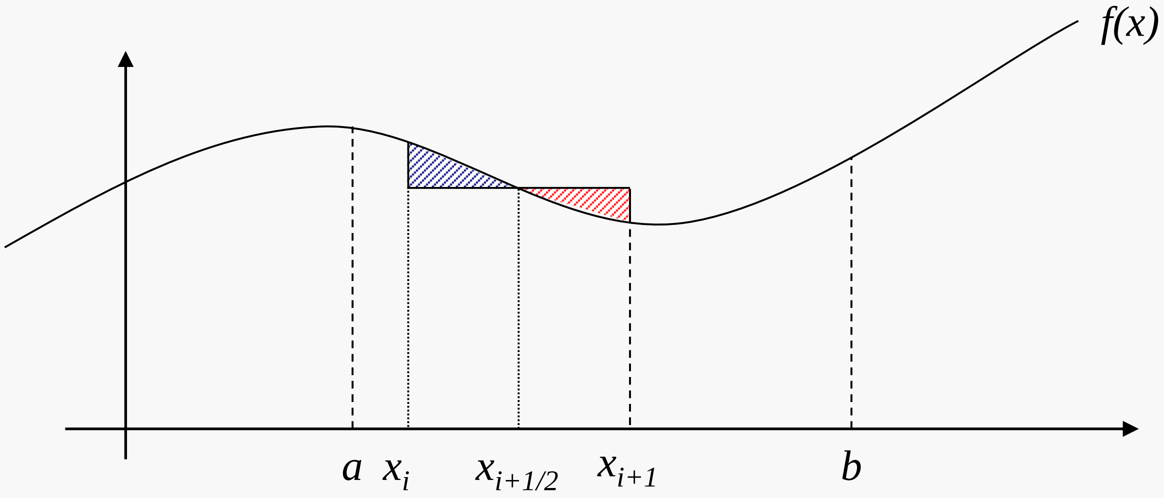
An absolute true error of 0.08

$$absolute\ relative\ true\ error \quad = \quad \left|\frac{12.12 - 12.20}{12.12}\right| = \left|\frac{-0.08}{12.12}\right| = 0.0066$$

As a percentage 0.0066x100=0.66% absolute relative true error

Rectangular rule

- The *rectangular rule* uses the values of f defined at the midpoints of the defined intervals and is based on a zero degree polynomial



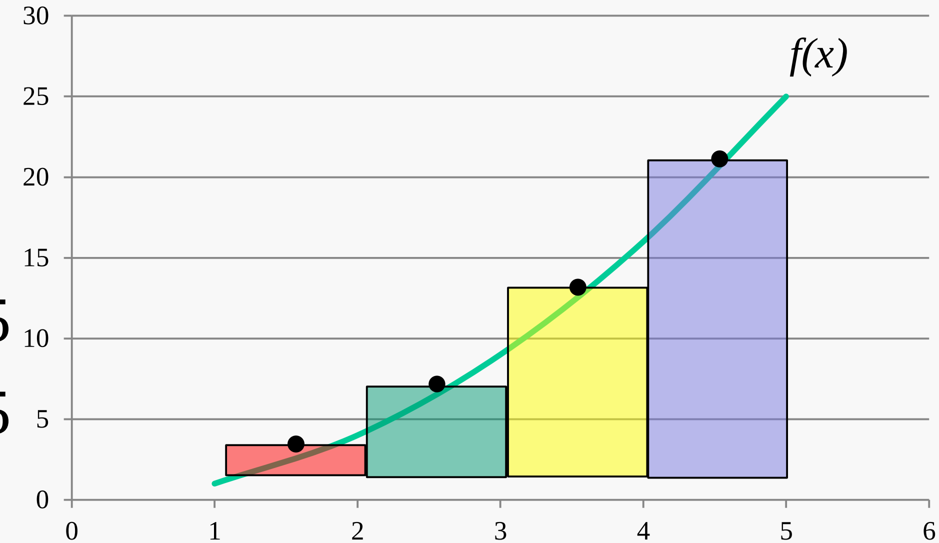
- For sufficiently small Δx, the two triangles in the diagram above will cancel each other out

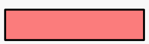
rectangular rule approximation is $$\int_a^b f(x)dx = \Delta x \sum_{i=0}^{n-1} f(x_{i+1/2})$$

Rectangular rule

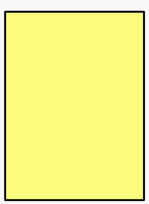
- Simple example for $\int_1^5 x^2 dx$

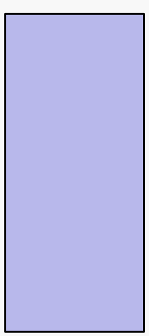
x	1	2	3	4	5
$f(x)$	1	4	9	16	25



Area of  is $1 \times 1.5^2 = 2.25$

Area of  is $1 \times 2.5^2 = 6.25$

Area of  is $1 \times 3.5^2 = 12.25$

Area of  is $1 \times 4.5^2 = 20.25$

so total area=

$2.25+6.25+12.25+20.25=41$

overestimate or underestimate ?

Rectangular rule

- Now to solve it analytically

$$\int_{x=1}^{x=5} x^2 \, dx = \left[\frac{x^3}{3}\right]_1^5 = \left[\frac{5^3}{3}\right] - \left[\frac{1^3}{3}\right]$$

$$= [41.66666 - 0.33333] = 41.33333$$

- So exact answer is 41.33333 and rectangular rule approximation with 4 strips is 41 (underestimate)

$$\textit{absolute relative true error} = \left|\frac{41.33333 - 41}{41.33333}\right| = 0.008$$

As a percentage 0.8% absolute relative true error

Rectangular rule

Problem

- Evaluate $\int_1^5 x^2 dx$ using 8 strips of the *rectangular rule*.

(The exact solution is 41.3333)

Solution

n=8, x_0=1 and x_8=5 so $\Delta x=(5-1)/8=0.5$

i	x_i	$X_{i+1/2}$	$f(x_{i+1/2})$
0	1.0	(1.0+1.5)/2=1.25	1.25^2=1.5625
1	1.0+0.5=1.5	(1.5+2.0)/2=1.75	1.75^2=3.0625
2	1.5+0.5=2.0	(2.0+2.5)/2=2.25	2.25^2=5.0625
3	2.0+0.5=2.5	(2.5+3.0)/2=2.75	2.75^2=7.5625
4	3.0	3.25	10.5625
5	3.5	3.75	14.0625
6	4.0	4.25	18.0625
7	4.5	4.75	22.5625
8	5.0		

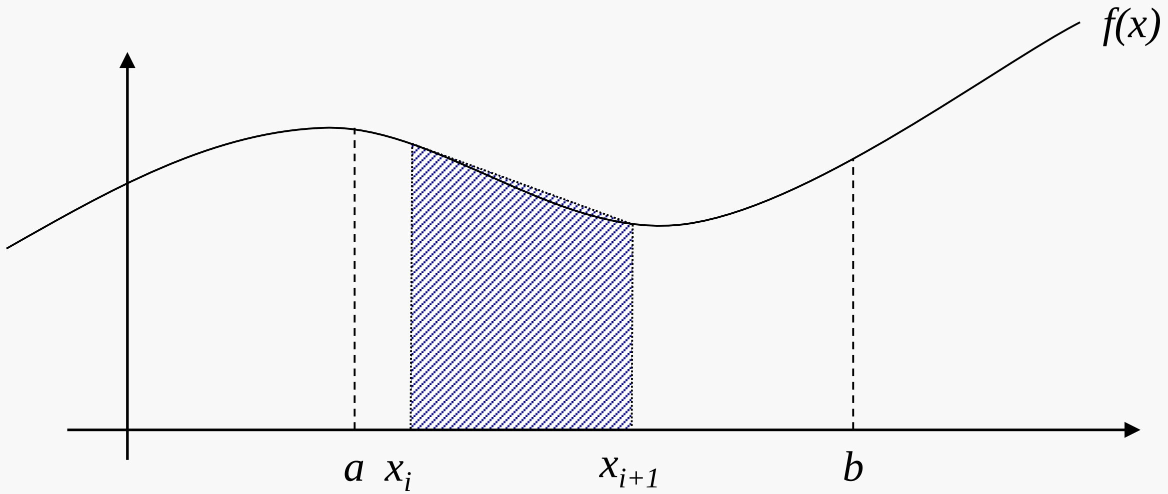
$$\int_a^b f(x)dx = \Delta x \sum_{i=0}^{n-1} f(x_{i+\frac{1}{2}}) = 0.5\text{x}82.5$$

$$= 41.25$$

absolute relative true error = 0.2%

Trapezoidal rule

- The *trapezoidal rule* uses a piecewise linear approximation and approximates to the area of a trapezium



- For sufficiently small Δx, the piecewise linear approximation provides improved accuracy

  trapezoidal rule approx is

$$\int_a^b f(x)dx = \Delta x\left(\frac{1}{2}f(x_0) + \sum_{i=1}^{n-1} f(x_i) + \frac{1}{2}f(x_n)\right)$$

Trapezoidal rule

- The approximation is sometimes (more conveniently) written as

$$\int_a^b f(x)dx = \frac{\Delta x}{2}\left(f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n) \right)$$

and can be remembered as

$$\int_a^b f(x)dx = \frac{\Delta x}{2}(first + 2*all\ intermediate + last)$$

Trapezoidal rule

Problem

- Evaluate $\int_1^5 x^2 dx$ using 8 strips of the *trapezoidal rule*.

(The exact solution is 41.3333)

Solution

n=8, x_0=1 and x_8=5 so $\Delta x=(5-1)/8=0.5$

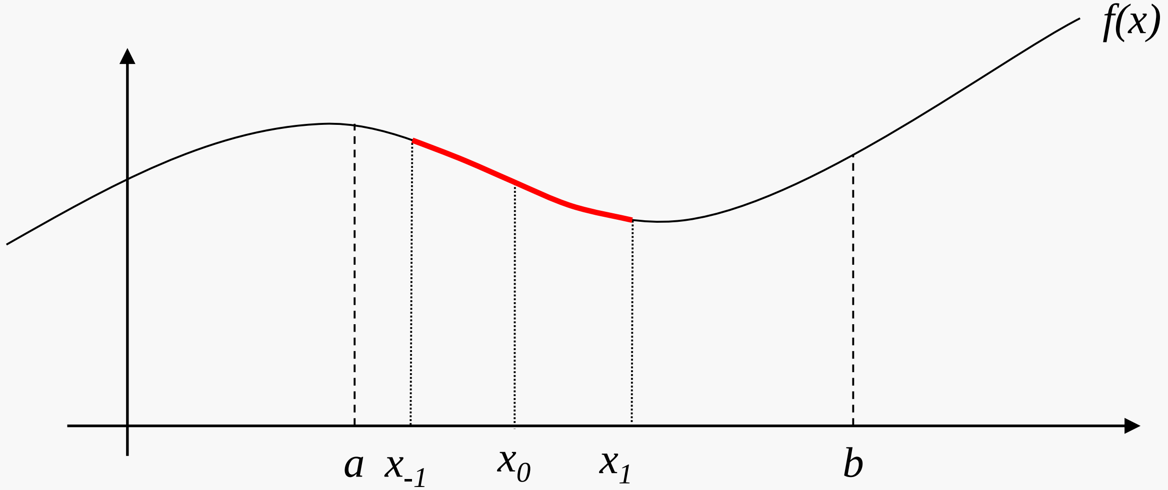
i	x_i	$f(x_i)$	$2*f(x_i)$
0	1.0	1.0	
1	1.0+0.5=1.5	2.25	4.50
2	1.5+0.5=2.0	4.00	8.00
3	2.0+0.5=2.5	6.25	12.50
4	3.0	9.00	18.00
5	3.5	12.25	24.50
6	4.0	16.00	32.00
7	4.5	20.25	40.50
8	5.0	25.00	

$$\int_a^b f(x)dx = \frac{\Delta x}{2}\left(f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n) \right) = 41.5$$

absolute relative true error = 0.4%

Simpson's rule

- *Simpson's rule* uses a piecewise quadratic approximation through the points $(x_{-1}, f(x_{-1}))$, $(x_0, f(x_0))$ and $(x_1, f(x_1))$



- The interval $[a,b]$ is divided into n strips, each of length

$$\Delta x = \frac{(b-a)}{n}$$

- For sufficiently small Δx, the piecewise linear approximation provides improved accuracy
- n must be even

Simpson's rule

- Simpson's (1/3) rule approx is

$$\int_a^b f(x)dx = \frac{\Delta x}{3}\left(f(x_0) + 4\sum_{i=1}^{n/2} f(x_{2i-1}) + 2\sum_{i=1}^{n/2-1} f(x_{2i}) + f(x_n) \right)$$

•This can be remembered as

$$\int_a^b f(x)dx = \frac{\Delta x}{3}\left(first + 4 * odd + 2 * even + last \right)$$

- Note again: need even number of strips

Simpson's rule

Problem

- Evaluate $\int_1^5 x^2 dx$ using 8 strips of *Simpson's rule*.

(The exact solution is 41.3333)

Solution

n=8, x_0=1 and x_8=5 so $\Delta x=(5-1)/8=0.5$

i	x_i	$f(x_i)$	$4*f(x_{odd})$	$2*f(x_{even})$
0	1.0	1.0		
1	1.5	2.25	9.0	
2	2.0	4.00		8.00
3	2.5	6.25	25.0	
4	3.0	9.00		18.00
5	3.5	12.25	49.0	
6	4.0	16.00		32.00
7	4.5	20.25	81.0	
8	5.0	25.00		

$$\int_a^b f(x)dx = \frac{\Delta x}{3}\left(f(x_0) + 4\sum_{i=1}^{n/2} f(x_{2i-1}) + 2\sum_{i=1}^{n/2-1} f(x_{2i}) + f(x_n) \right) = 41.3333$$

absolute relative true error = 0% - why?

Final remarks

- Numerical schemes can be used to provide an approximation to the integration of a function.
- The more strips that are used then the more accurate the solution becomes.
- Simpson's rule is generally more accurate than the trapezoidal rule and the rectangular rule.
- The implementation of these numerical techniques can be carried out by programming (using Java) or using a spreadsheet (such as Excel).