Wave packet dynamics in parabolic optical lattices: Bloch oscillations to quantum revivals

Usman Ali[1]
[1]Department of Physics, Quaid-i-Azam University, Islamabad 45320, Pakistan

Sara Medhet[2]
[2]Department of Electronics, Quaid-i-Azam University, Islamabad 45320, Pakistan

Farhan Saif[3]
[3]Department of Electronics, Quaid-i-Azam University, Islamabad 45320, Pakistan

E-mail: [1]usmanali@phys.qau.edu.pk

Abstract.
A primer on the dynamics of ultracold atoms in parabolic optical lattices is presented. The general description of the system is not yet available. Although quasimomentum representation allows easy estimates of the eigen states and energy spectrum, in the tight binding model. Exact behaviour of the eigen states segregates the system evolution into two prominent dynamical modes, that are discerned as Bloch and Dipole oscillations. The modes suffers, considerably, due to dephasing induced by unequal energy spacing of parabolic optical lattice. The corresponding atomic motions are heavily decohered. We analyse this phenomenon at longer evolutions by which wave packet revivals are reported. This we call as the CR oscillations owing to the periodic collapse and recurrence phenomenon. Landau-Zener tunnelling is also observed over Bloch mode beyond single band approximation, which is of a particular character like never considered before.

Uncorrected Proof

Contents

1 Introduction **2**

2 Single Band Tight Binding Dynamics **3**
2.1 *Tight-Binding Model: (basic features)* 4
2.2 Bloch Oscillations 9
2.3 Dipole Oscillations 11
2.4 CR Ocillations 12
2.5 Classicalization 13
2.6 Momentum Representation 15

3 Wave Packet Propagation **16**

4 Conclusion **18**

References **18**

1. Introduction

Atomic, Molecular and optical sciences is at a verge of transition to a giant sub-discipline of physics known as ultracold physics. For the past couple of decades, ultracold atoms took the central stage in ultracold research, and has been predominantly considered as the prime candidate for studying quantum many-body phenomena [1]. The use of Bose condensed atomic gases in this realm [2] translate the complete matter wave picture of de Broglie on a hundred thousand times bigger screen. This also bring about nearly ideal lattices which artificial and customizable. Thus ultracold atom is the leading quantum simulator [3] and the phenonmenons realized ranges from Band structure [4], Bloch oscillations [5], Anderson Localization [6] , Josephson Junction [7], Quantum Hall effect [8], Artificial gauge fields [9] , Bose-glass phase [10], to the phase transitions [11]. Bloch oscillations and some other phenonmenon in this domain has already contributed to high-precision determination of physical constants [12, 13] with atomic clocks providing stable and accurate time measurements [14].

Recently, there is an empirical resurgence to have a controlling influence on atoms in optical lattices by subjecting them to external modulation [15–20]. Most of the experiments performed uses a parabolic confining trap in addition. The parabolic potential is also utilized over optical lattice while loading and transference processes of trapped entities. This significantly revamps the overall potential profile giving rise to parabolic optical lattices [21]. In regard of the sublime use and high applicability of modulated parabolic optical lattices, we think that the modulation free system should be addressed more. Therefore, the present effort is devoted to fill the gaps and extend the theory of parabolic optical lattices.

The experiments performed work with both, Bosons as well as Fermions, which can be made interacting or non-interacting at will [22, 23]. We start our pursuit by considering the non-interacting case. Such a consideration is essential for an ideal single particle treatment. Furthermore, throughout this paper we stick to the actual parameteric values drawn from the Hamburg experiment [24] where the external modulation is presently suppressed. The experiment is preferred for its high customisability compared to others.

An ultracold atom constrained to 'hop' between contiguous sites of an optical lattice in the presence of a parabolic potential is an epitome of harmonic oscillator defined on a lattice [25]. The problem resembles to Bloch electrons in effect of a quadropole potential, and is often solved on the models borrowed from the solid-state-physics. An initial description [26] uses a quantum pendulum model to expound the dynamical features. Although the general behaviour of overall eigenstates which play crucial role in system evolution was not considered. In the tight binding approximation, the eigenstates of the system are found to be given by special Mathieu functions [27]. We proceed likewise, but from the not so well known momentum representation which effectively clarifies many

Uncorrected Proof

hidden aspects, unveiling modified dispersion relations and time evolution operators. It is also seen that some of the eigenstates of the system are localized far away from the center of the parabolic potential. The use of such spatially localized eigen states have already been proposed as prime candidate for precise quantum registers [28]. Our calculations suggest that these localized regions can be accessed by intelligently shifting the trap origin through a sudden displacement. Localization and an increasing quadratic force towards both spatial extremes allude to the Bloch oscillations over both wings of parabola. In the mean time, the phenomenon of dipole oscillations has already been ascribed to parabolic optical lattices [27, 29–33]. We inquire both these dynamics as the eigenstates interlace between extended to localized states.

In contrast to all the previous understandings of the system, which posit the dynamics to be continuously dephased, we report subsequent revivals of wave packet dynamics. Although, the revival phenomenon has already been predicted in reference [34], in the single band approximation, and the short time Talbot revivals have been gently discussed in conjuction with the tight binding approximation in reference [35], motivated by holes revival in [24] we believe that the steady state dynamical recurrences have not been discussed before. In addition to all these estimated results, we further turn to actual wave packet propagations, beyond all the approximation, which elicits Landau-Zener tunnelling phenomenon over Bloch dynamics. The tunnelling fraction perform Dipolar motion in higher bands. To the context of heavy research on the topic from past two decades, Dipole oscillations are merely studied for Landau-Zener tunnelling [36]. Therefore, the phenomenon is completely new which may open new doors towards sublime applications.

Uncorrected Proof

2. Single Band Tight Binding Dynamics

The tight binbing model allow an analytic treatment of parabolic optical lattices and provide valuable insight. In addition, the dynamics considerably simplifies on working inside a fixed single band. Such an approximation along with the usual tight binding axioms is known as single band tight binding model. In the following, we will briefly rederive the basic results, proceeding from the quasi-momentum representation, which offers various advantages compared to the traditional approach. The discussion is restrained to the simplest case where only nearest neighbouring coupling is considered. Though there are efforts to extend the tight binding model to higher level of coupling [37, 38], and it can also include coupling between the bands [39, 40], yet single band case is desirable here as Bloch oscillations are involved [41].

2.1. Tight-Binding Model: (basic features)

Let us start with a single atom in a one dimensional optical lattice in the presence of a symmetric parabolic trap potential. The Hamiltonian is:

$$H = H_o + H_T = \frac{p^2}{2m} + V_o \sin^2(\frac{\pi}{d}x) + \frac{1}{2}m\omega_T^2 x^2 \ , \tag{1}$$

here ω_T is the frequency of the parabolic trap, V_o is the optical lattice depth which is controllable through intensity of the laser beams, d is the lattice constant and m is the mass of atom.

The usual Wannier picture describe the most coveted positions of particles in such periodic potentials. The maximally locaized Wannier states $|n, j>$ delineate the whole weight of atoms around fixed lattice sites. Considering the atom to be an approximate harmonic oscillator at each potential minima, with finite tunneling between contiguous lattice sites,the system can be treated in tight binding approximation. Furthermore experiments with ultracold atoms are typically confined to the lowest band, $j = 0$, so the band index is also suppresed. With wavefunctions written in second quantization guis, $\hat{\psi} = \sum_n |n\rangle \hat{a}_n$, the hamiltonian 1 transforms like [42, 43],

$$H_{TB} = -\frac{J}{2}\sum_n (\hat{a}_n^\dagger \hat{a}_{n+1} + \hat{a}_{n+1}^\dagger \hat{a}_n) + \frac{1}{2}m\omega_\tau^2 d^2 \sum_n n^2 \hat{a}_n^\dagger \hat{a}_n \ , \tag{2}$$

where, $\hat{a}^\dagger$ creates a particle at $x/d = n$ and $\hat{a}$ obliterates, $J = -\langle n|\hat{H}_o|n+1\rangle$ is the nearest neighbouring hopping matrix element, i.e. $J = \frac{4}{\sqrt{\pi}}\left(\frac{V_o}{E_r}\right)^{\frac{3}{4}} e^{-2\sqrt{\frac{V_o}{E_r}}}$, for the ground band. The relation holds for deep lattices, $V_o >> E_R$ [44], with $E_R = h^2/8md^2$ being the recoil energy gained by the atoms.

Reverse engineering by taking $\hat{a}_n = \langle n|\hat{\psi}$, and replacing $m\omega_T^2$ by the trap strength κ , give rise to an alternate representation of 2,

$$H \ = -\frac{J}{2}\sum_n (|n\rangle\langle n+1| + |n+1\rangle\langle n|) + \frac{1}{2}\kappa d^2 \sum_n n^2 \ |n\rangle\langle n| \ . \tag{3}$$

This is the exact form of tight binding hamiltonian in Wannier picture, which substantially simplifies the analytical description. In addition, numerical efforts are also reduced bringing much easier and efficient computations.

Also, one can assign a description in terms of Bloch wave basis

$$|k\rangle = \sum_n |n\rangle\langle n|k\rangle = \sqrt{\frac{d}{2\pi}}\sum_n |n\rangle e^{inkd}, \tag{4}$$

which follows the identity $\langle n+1|k\rangle = e^{ikd}\langle n|k\rangle$, having quasi-momentum k limited to the first Brilloin zone $|k| \leq b$, with $b = 2\pi/d$. Using the relations

Uncorrected Proof

$$\sum_n \langle k'|n+1\rangle\langle n|k\rangle = e^{-ik'd}\frac{d}{2\pi}\sum_n e^{in(k-k')d} = \delta(k'-k)e^{-ikd}, \tag{5}$$

$$\sum_n n\langle k'|n+1\rangle\langle n|k\rangle = \frac{d}{2\pi}\sum_n ne^{in(k-k')d} = \delta(k'-k)\frac{x}{d}, \tag{6}$$

we find that the lattice Hamiltonian 3 reduces to $\langle k'|H|k\rangle = d\,\delta(k'-k)H(k)$ with modified dispersion relation $H(k)$ written as:

$$H(k) = -J\cos(kd) + F_1x^2, \tag{7}$$

Clearly the above quasi-momentum Hamiltonian is diagonal, where the operators $e^{\pm ikd}$ implicit in the kinetic-energy term produces discrete lattice translations of the hopping atom and leads to the dispersion relation for the case when there is no external trapping:

$$E(k) = -J\cos(kd) \tag{8}$$

Moreover, we have introduced the notation F_1 for the half of trap strength to highlight the notion of force on atom whenever it moves away from the trap origin. This force provided by the harmonic trap is inhomogeneous and increases quadratically from each side of the center of trap. To include the effects induced by position dependent forcing the trap origin is shifted by distance $s = \Delta x$ to new equilibrium position $l_o = \Delta x/d$. This will also set the particle into motion. Now, to include the initial displacement of trap we begin with

$$H(k) = -J\cos(kd) + F_1s^2, \tag{9}$$

here $s = x - x_o$. As we take the canonically conjugate variables $u = kd/2$, $s = i\frac{\partial}{\partial u}$ and solve the Schrodinger equation, we obtain a Mathieu equation:

$$\left[\frac{\partial^2}{\partial u^2} + \left(\frac{4E}{\Omega}\right) - 2\left(\frac{-2J}{\Omega}\right)\cos(2u)\right]\psi(u) = 0 \tag{10}$$

with corresponding Mathieu characteristic value $4E/\Omega = \alpha$ and kinetic energy parameter $-2J/\Omega = q$, mainly scaled with the parabolicity $\Omega = F_1d^2$. The eigenstates of the system are typically known periodic Mathieu Functions. Using Floquet solution [45–47] we write

$$\psi_l(u,\nu) = \sqrt{\frac{d}{2\pi}}\sum_m C_m^l(\nu)\, e^{[i(\nu+2m)u]}, \tag{11}$$

to the appropriate normalisation over first Brillouin zone. The coefficients $C_m^l(\nu)$ mainly depend upon ν, which is the characteristic Floquet exponent, and are given by difference equation,

$$[\alpha_l(q,\nu) - (\nu+2m)^2]C_m^l(\nu) = q[C_{m-1}^l(\nu) + C_{m+1}^l(\nu)]. \tag{12}$$

Here, $\alpha_l(q,\nu)$ represents the corresponding rescaled energy eigenvalues and the energy of a state with quantum number $'l'$ reads

$$E_l = \begin{cases} \frac{\Omega}{4}a_l(q) & ; \quad even \\ \frac{\Omega}{4}b_{l+1}(q) & ; \quad odd \end{cases}$$

Uncorrected Proof

Figure 1. Schematic illustration of the Mathieu characteristic energies as a function of parameter q. Solid dark blue (light red) curves are the eigen values related to a_l (b_l) whereas dotted-dashed lines correspond to the λ_l with $\nu = 0.5$.

(13)

where $a_l(q)$ and $b_{l+1}(q)$ describe the characteristic energies of α [46]. Note that the energy spectrum varies depending upon the choice of ν which commensurate with the order of Mathieu equation by relation $l = 2r \pm \nu$. Hence the quantum number l is rounded of to the nearest integer ν. For the case in above mentioned integral solution $l = \nu$. It is worth mentioning that there also exist a second solution to the Eq. 10 which is well known by the name of Mathieu functions for non-integer order. In that scenario the eigenstates are specified by $me_\nu(u, q)$ with corresponding characteristic eigen energies $\lambda_{\nu+2r}(q)$ uniquely describing both even and odd solutions [45–47].

The first few levels that appear from both these cases are shown in Fig. 1, plotted across the parameter q. Where the continuous solid lines represent energy spectrum for integer order corresponding to the even and odd Mathieu functions living in quasi-momentum space. Both of these solutions have definite parity with odd levels (dark blue) arising from both of these characteristics are 2π periodic whilst the even levels (Light red) are π periodic. Mathieu functions for non-integer order are also plotted to distinguish the energy spectra. They are shown as dotted-dashed lines in Fig. 1 in which each quantum number takes on an additional fractional constant number, i.e. $r = 0.5$. By this the spectrum modifies itself and the Eigen values mainly lie over a single curve which for our taken parameters fall exactly in between the first solution bands.

Some more insight can be drawn by looking at the Fig. 1 which also highlight a couple of other bounding features. One of which is the case of $q \to 0$ where the spectrum approach to the quardaticaly-spaced free particle energy levels. This behaviour is evident from the figure as well as Eq. 10 which reduces to the free particle Schrodinger equation whenever q vanishes. To further elaborate the trends in this region we have

Uncorrected Proof

the following asymptotic expansion for small q [47],

$$E_l = \frac{\Omega}{4}\left\{l^2 + \frac{1}{2(l^2-1)}q^2 + \frac{5l^2+7}{32(l^2-1)^3(l^2-4)}q^4 + \dots\right\}. \tag{14}$$

Another important point upon which we would like to emphasise on is the origin of localized Eigen states in this regime. Wavefunctions approaching free particle solution in momentum space will bring maximally localized Eigen states in the position space and the particle will sit on a single lattice site with maximum probability amplitude on it. Localization of the wavepacket is a prerequisite for the observation of some key dynamical phenomenons like Bloch oscillation which we construe in detail in next subsections.

Second bounding case appears in the $q \to \infty$ limit of this problem which is singular in the sense that the spectrum is entirely different from that of the translationally invariant lattice ($\Omega = 0$). In this regime for very large values of kinetic energy parameter q we have the asymptotic expansions [47]

$$E_l = \frac{\Omega}{4}\left\{-2q + 4\sqrt{q}\left(l+\frac{1}{2}\right) - \frac{(2l+1)^2+1}{8} + \frac{[(2l+1)^3+3(2l+1)]}{128\sqrt{q}} + \dots\right\}, \tag{15}$$

ignoring higher order terms and restricting ourselves to the quantum numbers linear in l above expression approximates to,

$$E_l = -J + \sqrt{2J\Omega}\left(l+\frac{1}{2}\right). \tag{16}$$

These are the well known equi-spaced harmonic oscillator eigen energies displaced by amount $-J$ from the bottom. Also the Hermite eigenfunctions corresponding to these energies are shifted to new equilibrium position at l_o. The particle will perform simple dipolar motion with an effective dipole frequency $\omega_D = \sqrt{2\Omega J}/\hbar$. These transitions are also evident by looking at the Fig. 1 where one can see the energy curves merging over a single line which becomes equally spaced as well at the right extreme. This is a unique behaviour because in the absence of the harmonic trap the eigen functions are Bloch states but here for a diminishing trap strength one encounters something different. This phenomenon is explained in [48] where the authors reveal another striking characteristic of the system in this regime, i.e. eigenstates becomes more and more localized by increasing l. Now lets examine the dynamical modes of the system as the eigenstates interlace between localized to extended states considering both the cases.

Before describing time propagation, let us analyse the eigenfunctions in real space [49],

$$\begin{aligned}\psi_l(s,\nu) = \langle s|\psi_l\rangle &= \int_{-\frac{b}{2}}^{\frac{b}{2}} dk \langle s|k\rangle\langle k|\psi_l\rangle = \frac{d}{2\pi}\sum_m C_m^l(\nu)\int_{-\frac{b}{2}}^{\frac{b}{2}} dk\, e^{iks}.e^{\frac{i}{2}(\nu+2m)kd} \\ &= \frac{d}{2\pi}\sum_m C_m^l(\nu)\int_{-\frac{b}{2}}^{\frac{b}{2}} dk\, e^{i[(x-l_o+\frac{\nu d}{2}+md)k]} = \sum_m C_m^l(\nu)\,\delta(x-l_o+\nu d/2+md).\end{aligned} \tag{17}$$

Uncorrected Proof

Figure 2. Eigen functions for n = 0, 5, 40, 75 are shown in quasimomentum representation starting from left, each with their corresponding spatial analog placed right together it, at fixed value of $q = -500$, as obtained by solving Eq. 12.

This represent a collection of delta functions defining the discrete Hilbert subspace $x = nd$ and the vanishing argument of the Dirac Delta function in above expression deduce that the characteristic Floquet exponent proportionate with the initial displacement l_o by relation $\nu = 2l_o/d$. Thus eigenstates become a function of the initial displacement of the trap and the coefficients $C_m^l(\nu)$ describe the real space amplitude which sweeps across the whole lattice as m varies. Therefore l being integer or non-integer decides the points where maximum amplitudes reside. This for integers lie exactly above or half way between each lattice site corresponding to the 2π or π periodic solutions, respectively when odd or even values of l are considered. For ν non-integer, the non-trivial solution discussed in above text is dropped with the understanding that the real space eigenstates vanishes uniformly over each lattice site. Hence only π and 2π periodic Mathieu functions are considered as valid solutions in k-space which give rise to even and odd amplitudes $C_m^l(\nu)$ in position space. Generally, we can say that the presence of external trapping takes the Bloch states to the k-space with the equivalent delta functions envelopes give rise to the maximally localized Wannier states in real space.

The overall behaviour of Eigenstates as obtained via exact diagonalization of Eq. 12 is shown in Fig. 2 in both the quasimomentum and spatial description. All these plots associate with constant value of kinetic energy parameter $q = -500$, which depict similar behaviour to discretized harmonic oscillator for lower lying states, but are still not exactly the same. However, it can be shown that for large values of q the eigen states took the exact shape of Hermite polynomials, as known analytically. The wiggles appearing in the spatial profiles related to first two quasimomentum eigenstates are because of the fact that there will be many such periodic functions of the same form in k-space and they sums up to a single localized packet when connected by the fourier transform. The localization effect of high energy states away from the center of parabolic

Unchecked Proof

potential is also visible by looking at the right extreme of Fig. 2. In [27] the authors quantify the critical eigen number, $l_c = 2||\sqrt{q/2}||$, being the threshold at which the eigen states change their behaviour. Remember that here we have chosen a moderate value of kinetic energy parameter q which can be controlled in experiment and for very small values of q we can see the localization effect emerging right from the ground state (not shown).

2.2. Bloch Oscillations

The time evolution operator $\hat{U}(t)$ in Bloch wave basis is,

$$\begin{aligned}\hat{U}_{k',k}(t) &= \langle k'|\hat{U}(t)|k\rangle = \sum_r \langle k'|\psi_r\rangle e^{-iE_r t/\hbar}\langle\psi_r|k\rangle \\ &= \frac{d}{2\pi}\sum_{r,m} C_m^r(\nu)C_m^{r*}(\nu)e^{-i(k-k')\nu}e^{-2im(k-k')}e^{-iE_r t/\hbar}\ ,\end{aligned} \tag{18}$$

this propagator changes based on the inherent eigenenergies, which were earlier segregated into two prominent regimes. Keeping in mind that for the solutions that sustains in our problem, $r = \nu$. Using Eq. **??** we write,

$$\hat{U}_{k',k}(t) = \frac{d}{2\pi}\sum_{r,m} C_m^\nu C_m^{\nu*} e^{-2im(k-k')}e^{-i(k-k')\nu}\{e^{-i\Omega\nu^2 t/4\hbar}g(\nu,t)\}, \tag{19}$$

where, $g(\nu,t)$ represents the dynamical phase determined by nonlinear terms in q. Substituting $\nu = 2l_o/d$, above equation reduces to,

$$\hat{U}_{k',k}(t) = \sum_m C_m^\nu C_m^{\nu*}\ g(\nu,t)e^{-2im(k-k')}\delta(k-k'+\Omega l_o t/\hbar d), \tag{20}$$

which defines a local static force $F = \Omega l_o/d$, and we see that the quasi-momentum k' follows the classical acceleration,

$$k' = k_t = k - Ft/\hbar\ . \tag{21}$$

The usual lattice band picture restricts the quasimomentum k to first Brillouin zone. Hence in low q regime the atom will perform Bloch oscillations with frequency $\omega_B = \Omega l_o/\hbar$, remaining constrained to one side of the parabolic lattice.

In the basis of Wannier states propagator takes the form

$$\begin{aligned}\hat{U}_{n,m}(t) &= \langle n|\hat{U}(t)|m\rangle = \sum_l \langle n|\psi_l\rangle e^{-iE_l t/\hbar}\langle\psi_l|m\rangle \\ &= \sum_l C_n^l(\nu)C_m^{l*}(\nu)e^{-iE_l t/\hbar},\end{aligned} \tag{22}$$

the Fourier counterpart of (16). The time evolution operator can be simplified by substituting asymptotic expressions 14 and 15 which divides the system evolution into two

Uncorrected Proof

Figure 3. Time evolution of the absolute value of the wavefunction $|\langle n|\psi(t)\rangle|$ as a function of t/T_B and n, exhibiting Bloch Breathing mode of a sharply localized wavepacket initially placed at $n = 24$ (left) and center of mass Bloch oscillations for a broad Gaussian wavepacket with variance $\sigma = 3.16$ and mean $a = 24$ (right) in the tight binding model at $|q|= 160$.

dynamical modes, namely Bloch mode and Dipole mode. Further one can see that evolution operator is periodic in both regimes with Bloch period $T_B = 2\pi\hbar/\Omega l_o$ and Dipole period $T_D = \sqrt{2}\pi\hbar/\sqrt{\Omega J}$. Lets examine wave packet evolvolution in both scenarios one after the other.

Consider the complete time evolution operator with discrete basis, acting upon an initial state ψ_o as,

$$\begin{aligned} |\psi(t)\rangle &= \left(\sum_{n,m} \hat{U}_{n,m}(t)|n\rangle\langle m|\right)|\psi(0)\rangle = \sum_{n,m,l} C_n^l(\nu) C_m^{l*}(\nu) e^{-iE_l t/\hbar}|n\rangle\langle m|\psi(0)\rangle \\ &= \sum_{n,m,l} C_n^l(\nu) C_m^{l*}(\nu) e^{-iE_l t/\hbar} C_m(0)|n\rangle = \sum_n C_n(t)|n\rangle, \end{aligned} \tag{23}$$

where $C_m(0)$ counts for the spatial distribution of initial wave packet over the finite lattice Hilbert subspace. The time evolution is governed by $C_n(t)$ connate with the ingrained quasi-energies.

Bloch oscillations of an initial state ψ_o is shown in Fig. 3 which depict a couple of illustrating instances in the low q regime, used to expound the dynamical sensitivity of the system towards the initially projected wave function. That is, if the overall amplitude of the initial state particularly reside in a single quantum well then the wave packet will perform coherent Bloch breathing, uniformly spreading towards both spatial extremes, after which a complete reconstruction is observed on the same outset at each Bloch period. Other than that is the case of an extended initial distribution populating more than one nearby lattice sites. Such a broad Gaussian distribution with variance $\sigma = 3.16$ and mean $a = 24$ is taken as the initial state in right of Fig. 3 having

Uncorrected Proof

non-zero probability amplitude over several contiguous wells. In this case, the initial distribution undergoes dispersionless center of mass Bloch oscillations. The parametric values used in these numerical simulations are those of Hamburg experiment [24] with a lattice depth of $10E_R$ and parabolic trap frequency $\omega_T = 2\pi \times 50Hz$. These values correspond to $J = 0.0227E_R$, $\Omega = 2.83 \times 10^{-4}E_R$ and $q \simeq -160$. One can easily note the wavepacket hampering on one side of the parabolic optical lattice although there is no physical boundary.

2.3. Dipole Oscillations

Dipole oscillations of atomic wave packets have been largely studied in parabolic optical lattices which gather more attention right after the realisation of BECs. A lot of work has been done in this regard both theoretically [27, 29–31] and experimentally [32, 33]. This harmonic oscillator character is pretty evident from the underneath energy spectrum in the high q regime. And the Hermite eigenfunction are simply displaced from the bottom, by amount J, with centre on the new equilibrium position l_o. With Eq. 16 the real space propagator takes the form

$$\hat{U}_{n,m}(t) = \sum_l C_n^l(\nu) C_m^{l*}(\nu) e^{-\left[\frac{\sqrt{2J\Omega}}{\hbar}\left(l+\frac{1}{2}\right)-J\right]t} . \tag{24}$$

Thus the wavepacket transverses both sides of parabolic optical lattice in the high q regime, performing center of mass dipolar motion with an effective frequency $\omega_D = \sqrt{2\Omega J}/\hbar$. The dynamics generated are shown in Fig 4. The time axis is scaled with the corresponding dipole period.

Figure 4. Time evolution of the absolute value of the wavefunction $|\langle n|\psi(t)\rangle|$ as a function of t/T_D and n, exhibiting dispersive dipole oscillations for a sharply localized wavepacket at $n = 24$ (left) and simple dipolar motion for a broad Gaussian wavepacket with variance $\sigma = 1.5$ and mean $a = 24$ (right) in the tight binding model with $|q|= 1604$.

Uncorrected Proof

Prominent changes in the dynamics are observed at $|q|= 1604$ which approaches to coherent dipole oscillations for a broad initial distribution as shown above. While a sharply localized distribution suffers significantly expanding through the contiguous wells, however, the overall transport is still very much the same. In our numerical simulation the high q regime is achieved by reducing the parabolicity, i.e. $\Omega = 2.83 \times 10^{-5} E_R$, by order of ten which is the only parameter we posit to be adjusted in real experiment. Such an amendment is feasible as Hamburg experiment is versatile in the range of trap frequencies. Although, the dynamics could also be changed either by enhancing the lattice tunnelling J through depth V_o or by carefully shifting the trap center to the regions of extended eigenstates $l_o < l_c$. We infer that instead of readjusting the lattice depth each time or displacing Parabolic trap in real experiments, tuning the external trap frequency is more viable. Slight changes in the trap frequency ω_T will be enough to change the q value, which will correspondingly levitate the critical eigen number and a change in dynamics could easily be observed.

2.4. CR Ocillations

One can also note the additional dephasing, induced by nonlinear spectrum of energies, just after few evolution periods. Unlike tilted optical lattices the energy levels inside each adjacent well are unequally spaced. Thus the wavepacket suffers significantly during its evolution, lossing the overall shape due to phase mixing. Both oscillatory modes, Bloch oscillations as well as Dipole oscillations, are found to be effected. Remaining inside the tight binding model the phenomenon is analysed at longer evolutions where high fidelity quantum revivals are observed.

The results obtained are shown in Fig. 5, which shows steady state evolution of the initially prepared wavepacket. Collapse and revival phenomenon can be observed. The periodic dephasing and reappearance of coherence can be thought of as an oscillation, with much longer period, running over both dynamical modes. This we call as the CR oscillations [35]. It can be seen that a much quicker revival is observed for Bloch oscillations than the Dipole oscillations. Also the CR oscillation period decreases at longer evolutions. Thus, it is perspicacious to examine the related effects in detail with appropriate analytical description. So for now we end this section and ascribe that theoretical treatment a topic of another separate paper [50].

Uncorrected Proof

Figure 5. Time evolution of the absolute value of the wavefunction $|\langle n|\psi(t)\rangle|$ as a function of $t/T_{B(D)}$ and n, exhibiting CR oscillations over Bloch (top) and Dipole (bottom) dynamics. Based Gaussian wavepacket is evolved with all the same parameters, where the dynamics are changed by controlling $q = 160$ and $q = 1604$.

2.5. Classicalization

In this section we present a semiclassical model, by considering the tight binding Hamiltonian 7, which leads to the Bloch acceleration

$$\hbar\dot{k} = -\frac{\partial E(k)}{\partial x} = -2F_1 x, \tag{25}$$

and the group velocity,

$$\dot{x} = \frac{\partial E(k)}{\hbar\partial k} = \frac{Jd}{\hbar}\sin kd\ . \tag{26}$$

The above coupled equations results in an oscillator equation

$$\hbar\ddot{k}(t) = -\frac{2dJF_1}{\hbar}\sin(kd)\ , \tag{27}$$

which can be solved considering the time dependent ansatz $K(t) = k\sin(\eta t)$, where η is an unknown frequency. Substituting the ansatz above equation reduces to

$$\begin{aligned} k\eta^2\sin(\eta t) &= \frac{2dJF_1}{\hbar^2} sin(kd\sin(\eta t)) \\ &= \frac{2dJF_1}{\hbar^2}\sum_{n=0}^{\infty} J_{2n+1}(kd)\sin[(2n+1)\eta t]\ . \end{aligned} \tag{28}$$

Uncorrected Proof

Figure 6. Classical phase space of the quasimomentum Hamiltonian 7. The green contours marks the separatix trajectory as the boundary at which the dynamics changes.

On comparing the coefficients of $\sin(nk)$ on both sides, we get

$$\eta^2 = \frac{2J\Omega}{\hbar^2} J_1(kd) \ , \tag{29}$$

Thus the frequency of the oscillator depends upon quasimomentum k. For k approaching to zero, we get the well known dipole oscillations frequency

$$\eta_o = \omega_D = \sqrt{\frac{2J\Omega}{\hbar^2}} \ . \tag{30}$$

Therefore, the solution of $k(t)$ and $x(t)$ is given as

$$k(t) = k \sin(\eta t) \ , \tag{31}$$

$$x(t) = -\frac{nk}{2F_1} \eta \cos(\eta t) \ . \tag{32}$$

Hence in the phase space, $(x, p = \hbar k)$, the particle with a constant initial quasimomentum $k = k_o$, effectively follows rotational dynamics with frequency η and maximum displacement $x = \pm \frac{\pi \hbar d}{2\Omega} \eta$. The phase space generated is shown in Fig. 6 which elicits liberation over circular dynamics. The region of closed contours below the separatix is the threshold of simple dipolar motion near the center of parabolic optical lattice. On the other hand, the liberation allude towards periodic switching of quasimomentum vector which quantum mechanically results in Bloch oscillations.

Uncorrected Proof

2.6. Momentum Representation

In this section we will discuss the transformations from the quasi-momentum representations to the true atomic momentum. As we know that one can typically use the wannier function index $n = x/d$ as the discretization of space. But for the momentum the choice is critical and a clarification is needed to distinguish between the quasi-momentum k and the momentum p. Therefore, we write the transformation:

$$\Psi_r(p) = \int_{-\frac{\pi}{d}}^{\frac{\pi}{d}} dk\ \langle p|k\rangle\langle k|\psi_r\rangle = \int_{-\frac{\pi}{d}}^{\frac{\pi}{d}} dk\ \psi_r(k)\phi_k(p), \tag{33}$$

with the momentum-space Bloch wavefunctions given by

$$\phi_k(p) = \sqrt{\frac{2\pi}{d}}\ \omega_o(p) \sum_n \delta(p - k - nb), \tag{34}$$

where, $\omega_o(p)$ is the Wannier state at origin in momentum representation, which is well localized in the interval $|p|< b/2$.

$$\Psi_r(p) = \sqrt{\frac{2\pi}{d}} \int_{-\frac{\pi}{d}}^{\frac{\pi}{d}} dk\ \psi_r(k)\ \omega_o(p) \sum_n \delta(p - k - nb), \tag{35}$$

Therefore, the only way 34 to be non-zero is if $n = 0$, since all other n will be outside the Brillioiun zone. Hence, $\Psi_r(p) = \sqrt{2\pi/d}\ \psi_r(p)\ \omega_o(p)$. Therefore, we can write the following simple relation for the momentum space eigenstates:

$$\Psi_r(p, \nu) = \ \omega_o(p) \sum_m C_m^r(\nu) e^{[i(\nu+2m)p]}, \tag{36}$$

that are the Mathieu functions in momentum space, where the overall distributions are determined by the envelope $|\omega_o(p)|$ and the phase factors are periodic in p with period b.

Figure 7. Wigner Distribution Function of the lowest Mathieu function.

Uncorrected Proof

3. Wave Packet Propagation

Important features of the dynamics of parabolic optical lattices can be understood proceeding towards the actual wave packet evolutions. This allows us to probe beyond tight binding or single-band approximation. Such numerical evaluations are common and we adduce to [41] for similar studies.

The dynamics of a single atom in effect of the Hamiltonian 1 are governed by the time dependent schrödinger equation

$$i\hbar_{eff}\frac{\partial\psi}{\partial t} = -\frac{\hbar_{eff}^2}{2}\frac{\partial^2\psi}{\partial y^2} + (\sin^2(y) + \nu y^2)\psi, \tag{37}$$

where y, $\hbar_{eff}$ and ν denote the rescaled position $y = kx$, the rescaled Planck's constant $\hbar_{eff} = \sqrt{2/s}$ and the scaled parabolicity $\nu = \Omega/\pi^2 s$, respectively with $s = V_o/E_R$.

Above equation can fairly be solved by means of an operator splitting method [51]. Keeping the same value of parabolicity as previously chosen to be in tight binding approach, parameters of the physical system reduces to only one variable, i.e. scaled depth of the optical potential s. For $s \geq 4$ the system strictly follows tight binding dynamics and at the same initial displacement, the results obtained here exactly coincide with the previously depicted ones.

Figure 8. Wave packet evolutions at $s = 10E_R$ (Hamburg Experiment) under the same values as used before (Fig. 3, 4).

Uncorrected Proof

Figure 9. Bloch mode in co-ordinate space for $s = 3$ and $\nu = 2.83 \times 10^{-5}$.

Further the depth of optical potential is reduced to see beyond the tight binding approximation. The value of scaled Planck constant was chosen as $\hbar_{eff} = 0.816$ which corresponds to the $s = 3$ and $\nu = 2.83 \times 10^{-5}$. As an initial state, we use the displaced ground state of the system and, hence, width of the wave packet is no more an independent parameter. The ground state of the Hamiltonian 1, with center of the harmonic potential shifted to Δx, is found by imaginary time propagation of some trial wavefunction. For the real time propagations then, equation 37 is solved. This has the same effect as suddenly displacing the ground state wave function. In the Catiliotti experiment [32], this has been achieved by means of a weakly interacting (dilute) Bose-Einstein condensate, where the atomic cloud is shifted by displacement of the magnetic trap. It could also be achieved by initial driving as engineered in [52]. Therefore, we posit that Hamburg experiment [24] may serve equally well, where the parametric values are more favourable for observing the discussed dynamical phenomenons.

Figure 9 shows the bloch mode in real space as a function of time (shown is the probability density $|\psi(x,t)|^2$ over initial four Bloch periods where the bright regions corresponds to higher density). At $t = 0$, the wavepacket splits instantly and the subsequent dynamics results in the superposition of Bloch and Dipole oscillations. The splitting is repeated with a decreasing amplitude at each Bloch period. This behaviour can be explained on the basis of band structure involved. Band structure reflects the periodicity of the lattice and it is well known that band gap increases at higher depths of the optical potentials. Therefore at higher band gaps the interband coupling could be neglected, which leads to the single band or, more restrictively, to the tight binding system described in section 2. However, as we probe beyond tight binding approximation there is a significant interband coupling which gives rise to Landau-Zener tunnelling at smaller band gaps. Thus numerically observed splitting is due to contributions from higher bands.

Till now the research literature ascribes the motion of Landau Zener tunnelling fraction to uniform acceleration, as the course of homogeneously tilted lattices. However here we

Uncorrected Proof

report the dipole oscillations of the accelerated part of the wavepacket due to symmetry of the parabolic optical lattice. It can easily be seen from Fig 9 that while performing Bloch oscillations, on one arm of the symmetric structure wave packet splits and a small fraction starts oscillating in the higher bands, traversing both sides of parabola. This is due to different bandwidths and fraction of the wavepacket excited undergoes Dipole oscillations. Hence in parabolic optical lattices there will be total oscillations and a properly trapped atom can never escape from the lattice during its propagation.

4. Conclusion

We study the single-particle spectrum and dynamics of an ultracold atomic gas trapped in a parabolic potential over an optical lattice in one dimension. The eigenenergies and eigenstates of the system are found to be given in terms of Mathieu special functions. We used asymptotic expansions of the eigenenergies and eigenstates of the system to fully characterise the single-atom tight binding dynamics. Our analytical treatment complements the previous numerical and semi-classical analysis of the same system. The advantage is that we could calculate the time evolution operators. The knowledge of these operators allowed us also to derive analytic expression for the acceleration of the quasi-momentum vector. This confirms the Bloch oscillations of atomic wavepacket induced by the position dependent forcing, when trapped ideal gas is suddenly displaced from the trap center. For weaker forcing this turn into dipole oscillations of single-atom system. This is also evident from the semi-classical model presented. Further additional dephasing and revival of these oscillatory modes are shown. By studying wave packet propagations we confirm the previously observed dynamics, remaining inside tight binding parametric domain. As we probe beyond tight binding approximation Landau-Zener tunnelling is observed over Bloch oscillations, where dipole oscillations of the tunnelling fraction are reported. Hence the wavepacket dynamics interlace between Dipole and Bloch oscillations.

Uncorrected proof

Acknowledgements

Usman Ali thanks Prof. Hans Jurgen Korsch for useful discussions. F. S. thanks Prof. Shinichi Watanabe and Dr. M Abdul for useful comments and suggestions.

References

[1] Bloch I, Dalibard J and Zwerger W 2008 *Rev. Mod. Phys.* **80**

[2] Morsch O and Oberthaler M 2006 *Rev. Mod. Phys.* **78**

[3] Gross C 2017 *Science Journal* **357** 995–1001

[4] Fläschner N, Tarnowski M, Rem B S, Vogel D, Sengstock K and Weitenberg C 2018 *Phys. Rev. A* **97**

[5] Reeves J B, Gadway B, Bergeman T, Danshita I and Schneble D 2014 *New J. Phys.* **16**

[6] Roati G, D'Errico C, Fallani L, Fattori M, Fort C, Zaccanti M, Modugno G, Modugno M and Inguscio M 2008 *Nature* **453** 895–898

[7] Gati R 2007 *J. Phys. B* **40**

[8] Palmer R N, Klein A and Jaksch D 2008 *Phys. Rev. A* **78**

[9] Goldman N, Juzeliūnas G, Öhberg P and Spielman I B 2014 *IOP Science* **77**

[10] Habibian H, Winter A, Paganelli S, Rieger H and Morigi G 2013 *Phys. Rev. Lett.* **110**

[11] Dubius N 2009 *Phys. B: Condense Matter* **404**

[12] Ramos A, Moore K and Raithel G 2017 *Phys. Rev. A* **96**

[13] Sacchetti A 2017 *Physics Letters A* **381**

[14] Bloom B J, Nicholson T L, Williams J R, Campbell S L, Bishof M, Zhang X, Zhang W, Bromley S L and Ye J 2014 *Nature* **506**

[15] Pedersen P L, Gajdacz M, Winter N, Hilliard A J, Sherson J F and Arlt J 2013 *Phys. Rev. A* **88**

[16] Heinze J, Götze S, Krauser J S, Hundt B, Fläschner N, Lühmann D S, Becker C and Sengstock K 2011 *Phys. Rev. Lett* **107**

[17] Zhai Y, Yue X, Wu Y, Chen X, Zhang P and Zhou X 2013 *Phys. Rev. A* **87**

[18] Jotzu G, Messer M, Desbuquois R, Lebrat M, Uehlinger T, Greif D and Esslinger T 2014 *Nature* **515**

[19] Li-Chung Ha L W C, Parker C V, Anderson B M and Chin C 2015 *Phys. Rev. Lett.* **114**

[20] Fujiwara K M, Singh K, Geiger Z A, Senaratne R, Rajagopal S, Lipatov M and Weld D M 2018 *arXiv:1806.07858*

[21] Yamakoshi T and Watanabe S 2015 *Phys. Rev. A* **91**

[22] Pollack S E, Dries D, Junker M, Chen Y P, Corcovilos T A and Hulet R G 2009 *Phys. Rev. Lett.* **102**

[23] Chin C, Grimm R, Julienne P and Tiesinga E 2010 *Rev. Mod. Phys.* **83**

[24] Heinze J, Krauser J S, Fläschne N, Hundt B, Götze S, Itin A P, Mathey L, Sengstock K and Becker C 2013 *Phys. Rev. Lett.* **110**

[25] Mattis D C 1986 *Rev. Mod. Phys.* **58**

[26] Brand J and Kolovsky A R 2007 *The European Physical Journal D* **41**

[27] Rey A M, Pupillo G, Clark C W and Williams C J 2005 *Phys. Rev. A* **72**

[28] Viverit L, Menotti C, Calarco T and Smerzi A 2004 *Phys. Rev. Lett.* **93**

[29] Pezzè L, Pitaevskii L, Smerzi A, Stringar S, Modugno G, de Mirandes E, Ferlaino F, Ott H, Roati G and Inguscio M 2004 *Phys. Rev. Lett.* **93**

[30] Ruuska V and Törmä P 2004 *New J. Phys.* **6**

Uncorrected Proof

[31] Polkovnikov A, Sachdev S and Girvin S M 2002 *Phys. Rev. A* **66**

[32] Cataliotti F S, Fallani L, Ferlaino F, Fort C, Maddaloni P and Inguscio M 2003 *New J. Phys.* **5**

[33] Fertig C D, O'Hara K M, Huckans J H, Rolston S L, Phillips W D and Porto J V 2005 *Phys. Rev. Lett.* **94**

[34] Ponomarev A V and Kolovsky A R 2005 *Laser Physics* **16**

[35] Mahmud K, Jiang L, Tiesinga E and Johnson P R 2014 *Phys. Rev. A* **89**

[36] Zheng M J, Chan Y S and Yu K W 2011 *Journal of the Optical Society of America B* **28**

[37] Wohlthat S, Reimers J R and Hush N S 2010 *Phys. Rev. B* **81**

[38] Liu G B, Shan W Y, Yao Y, Yao W and Xiao D 2013 *Phys. Rev. B* **88**

[39] Modugno M, Ibañez-Azpiroz J and Pettini G 2016 *Science China Physics* **56**

[40] Li X and Liu W V 2016 *Reports on Progress in Physics* **79**

[41] Hartmann T, Keck F, Korsch H J and Mossmann S 2004 *New J. Phys.* **6**

[42] Jaksch D, Bruder C, Cirac J I, Gardiner C W and Zoller P 1998 *Physical Review Letters* **81**

[43] Cataliotti F S, Burger S, Fort C, Maddaloni P, Minardi F, Trombettoni A, Smerzi A and Inguscio M 2001 *Science* **293** 843–846

[44] Zwerger W 2003 *J. Opt. B* **5**

[45] McLachlan N W 1964 *Theory and Application of Mathieu Functions* (New York: Dover)

[46] Abramowitz I and Stegun I A 1964 *Handbook of Mathematical Functions, National Bureau of Standards*

[47] Meixner J and Schäfke F W 1954 *Mathieusche Funktionen und Sphroidfunktionen* (Berlin: Springer-Verlag)

[48] Hooley C and Quintanilla J 2004 *Phys. Rev. Lett.* **93**

[49] Torres-Vegadag G, Morales-Guzmánddag J D and Zúñiga-Segundo§ A 1998 *Journal of Physics A: Mathematical and General* **31**

[50] Ali U and Saif F *To be published*

[51] MDFeit, JAFleck and Steiger J 1982 *Journal of Computational Physics* **47**

[52] Sherson J F, Park S J, Pedersen P L, Winter N, Gajdacz M, Mai S and Arlt J 2012 *New J. Phys.* **14**

Uncorrected Proof