Unit No. 1

Real Numbers

Basic Concepts

Real Numbers (R):

Real numbers are all the numbers that can be represented on a number line. This includes all rational and irrational numbers. They can be positive, negative, or zero, and can be whole numbers, fractions, or decimals (terminating or non-terminating, repeating or non-repeating).

Examples:

1. -10 (an integer)
2. 3.14 (a terminating decimal, which is rational)

Rational Numbers (Q):

Rational numbers are any numbers that can be expressed as a fraction $\frac{p}{q}$, where p and q are integers and

$q \neq 0$. Their decimal representations either terminate or repeat.

Examples:

1. $\frac{1}{2} = 0.5$ (a terminating decimal)
2. $\frac{3}{5} = 1.6666...=1.\overline{6}$ (a repeating decimal)

Irrational Numbers:

Irrational numbers are real numbers that cannot be expressed as a fraction $\frac{p}{q}$, where p and q are integers and

$q \neq 0$. Their decimal representations are non-terminating and non-repeating.

Examples:

1. $\sqrt{3} \approx 1.7320508...$
2. $e \approx 2.7182818284...$ (Euler's number)

Terminating Numbers:

Terminating numbers are numbers whose decimal representation ends after a finite number of digits. They can always be expressed as a fraction where the denominator is a power of 10.

Examples:

1. $0.25 = \frac{25}{100} = \frac{1}{4}$
2. $1.78 = \frac{178}{100} = \frac{89}{50}$

Non-Terminating Numbers:

Non-terminating numbers are numbers whose decimal representation goes on infinitely. These can be further divided into repeating and non-repeating decimals.

Examples:

1. $0.1111... = 0.\bar{1} = \frac{1}{9}$

(a non-terminating repeating decimal, which is rational)

2. $\pi \approx 3.14159265...$

(a non-terminating non-repeating decimal, which is irrational)

Recurring Decimal Fractions:

A recurring decimal fraction (also known as a repeating decimal) is a decimal fraction in which one or more digits repeat infinitely. This repeating sequence of digits is called the repetend or repeating block.

Examples:

1. $\frac{1}{3} = 0.3333... = 0.\bar{3}$ (The digit 3 repeats)
2. $\frac{2}{7} = 0.285714285714... = 0.\overline{285714}$ (The sequence 285714 repeats)
3. $\frac{11}{6} = 1.8333... = 1.8\bar{3}$ (The digit 3 repeats after a non-repeating part '8')

Recurring decimal fractions are always **rational numbers**, meaning they can be expressed as a fraction $\frac{p}{q}$ where p and q are integers and $q \neq 0$.

Non-Recurring Decimal Fractions:

A **non-recurring decimal fraction** is a decimal fraction in which the digits after the decimal point do not repeat in a pattern and continue infinitely.

Examples:

1. $\sqrt{2} = 1.41421356237...$

(The digits continue without any repeating pattern)

2. $\pi = 3.14159265358...$

(The digits continue without any repeating pattern)

3. 0.101001000100001...

(Although there's a pattern in the number of zeros, the sequence of the entire decimal does not repeat)

Non-recurring decimal fractions are always **irrational numbers**, meaning they cannot be expressed as a simple fraction $\frac{p}{q}$ where p and q are integers and $q \neq 0$. Their decimal expansions go on forever without repeating.

Properties of Real Numbers:

All calculations involving addition, subtraction, multiplication and division of real numbers are based on their properties.

Additive Properties:

Name of the property	**$\forall$ a, b, c $\in$ R**	**Examples**
Closure	**$a + b \in R$**	**$2+3 = 5 \in R$**
Commutative	**$a + b = b + a$**	**$2+5 = 5+2$** **$7 = 7$**
Associative	**$a+(b+c) = (a +b)+c$**	**$2+(3+5) = (2+3)+5$** **$2+8 = 5+5$** **$10 = 10$**
Identity	**$a+0 = 0+a = a$**	**$5+0 = 0+5 = 5$**
Inverse	**$a+(-a) = -a+a = 0$**	**$6+(-6) = (-6) + 6 = 0$**

Multiplicative Properties:

Name of the property	$\forall$ a, b, c $\in$ R	Examples
Closure	$ab \in R$	$2\times5 = 10 \in R$
Commutative	$ab = ba$	$2\times3 = 3\times2$ $6 = 6 \in R$
Associative	$a(bc) = (ab)c$	$2\times (3\times5) = (2\times3)\times5$ $2\times15 = 6\times5$ $30 = 30$
Identity	$a\times1 = 1\times a = a$	$5\times1 = 1\times5$ $= 5$
Inverse	$a\times \left(\frac{1}{a}\right) = \frac{1}{a}\times a = 1$	$7\times \left(\frac{1}{7}\right) = \frac{1}{7}\times7 = 1$

Distributive Properties:

For all real numbers a, b, c:

(i) $a(b+c) = ab+ac$ is called left distributive property of multiplication over addition.

(ii) $a(b-c) = ab-ac$ is called left distributive property of multiplication over subtraction.

(iii) $(a+b)c = ac+bc$ is called right distributive property of multiplication over addition.

(iv) $(a-b)c = ac-bc$ is called right distributive property of multiplication over subtraction.

Do you know?

0 and 1 are the additive and multiplicative identities of real numbers respectively.

Remember!

$0 \in R$ has no multiplicative inverse.

Properties of Equality of Real Numbers:

No.	Name of Property	Symbolic Form
I	Reflexive property	$\forall\, a \in R,\ a = a$
II	Symmetric property	$\forall\, a, b \in R,\ a = b \Rightarrow b = a$
III	Transitive property	$\forall\, a, b, c \in R,\ a = b \wedge b = c \Rightarrow a = c$
IV	Additive property	$\forall\, a, b, c \in R,\ a = b \Rightarrow a + c = b+c$
V	Multiplicative property	$\forall\, a, b, c \in R,\ a = b \Rightarrow ac = bc$
VI	Cancellation property w.r.t addition	$\forall\, a, b, c \in R,\ a + c = b+c \Rightarrow a = b$
VII	Cancellation property w.r.t multiplication	$\forall\, a, b, c \in R$ and $c \neq 0,\ ac = bc \Rightarrow a = b$

Order Properties:

No.	Name of Property	Symbolic Form
I	Trichotomy Property	$\forall\, a, b \in R$, either $a = b$ or $a > b$ or $a < b$
II	Transitive Property	$\forall\, a, b, c \in R$ ➢ $a > b \wedge b > c \Rightarrow a > c$ ➢ $a < b \wedge b < c \Rightarrow a < c$
III	Additive Property	$\forall\, a, b, c \in R$ ➢ $a > b \Rightarrow a+c > b+c$ ➢ $a < b \Rightarrow a+c < b+c$
IV	Multiplicative Property	$\forall\, a, b, c \in R$ ➢ $a > b \Rightarrow ac > bc$ if $c > 0$ ➢ $a < b \Rightarrow ac < bc$ if $c > 0$ ➢ $a > b \Rightarrow ac < bc$ if $c < 0$ ➢ $a < b \Rightarrow ac > bc$ if $c < 0$ ➢ $a > b \wedge c > d \Rightarrow ac > bd$ ➢ $a < b \wedge c \Rightarrow ac < bd$
V	Division Property	$\forall\, a, b, c \in R$ ➢ $a < b \Rightarrow \frac{a}{c} < \frac{b}{c}$ if $c > 0$ ➢ $a < b \Rightarrow \frac{a}{c} > \frac{b}{c}$ if $c < 0$ ➢ $a > b \Rightarrow \frac{a}{c} > \frac{b}{c}$ if $c > 0$ ➢ $a > b \Rightarrow \frac{a}{c} < \frac{b}{c}$ if $c < 0$
VI	Reciprocal Property	$\forall\, a, b \in R$ and have same sign $a < b \Rightarrow \frac{1}{a} > \frac{1}{b}$ $a > b \Rightarrow \frac{1}{a} < \frac{1}{b}$

Radical Expressions:

If n is a positive integer greater than 1 and a is a real number, then any real number x such that $x = \sqrt[n]{a}$ is called the nth root of a.

Here, $\sqrt{\ }$ is called the radical and n is the index of the radical. A real number under the radical sign is called a radicand.

$\sqrt[3]{5}, \sqrt[5]{7}$ are the examples of radical form.

Exponential form of $x = \sqrt[n]{a}$ is $x = (a)^{\frac{1}{n}}$.

Laws of Radicals and Indices:

#	Laws of Radical	Laws of Indices
i	$\sqrt[n]{ab} = \sqrt[n]{a}.\sqrt[n]{b}$	$a^m . a^n = a^{m+n}$
ii	$\sqrt[n]{(a/b)} = \sqrt[n]{a} / \sqrt[n]{b}$	$(a^m)^n = a^{mn}$
iii	$\sqrt[n]{a^m} = (\sqrt[n]{a})^m$	$(ab)^n = a^n. b^n$
iv	$(\sqrt[n]{a})^n = (a^{\frac{1}{n}})^n = a$	$(\frac{a}{b})^n = \frac{a^n}{b^n}$
v		$\frac{a^m}{a^n} = a^{m-n}$
vi		$a^0 = 1$

Surds and their Applications:

An irrational radical with a rational radicand is called a surd.

For example, if we take the n^{th} root of any rational number a, then $\sqrt[n]{a}$ is a surd. $\sqrt{5}$ is a surd because the square root of 5 does not give a whole number. But $\sqrt{9}$ is not a surd because it simplifies to a whole number 3, and our result is not an irrational number. Therefore, the radical $\sqrt[n]{a}$ is irrational, $\sqrt{7}$, $\sqrt{2}$, $\sqrt[3]{11}$ are surds, but $\sqrt{\pi}$, $\sqrt{e}$ are not surds.

Remember!

Every surd is an irrational number, but every irrational number is not a surd, e.g., π is not a surd.

The different types of surds are as follow:

(i) A surd that contains a single term is called a **monomial surd** e.g., $\sqrt{5}$, $\sqrt{7}$ etc.

(ii) A surd that contains the sum of two monomial surds is called a **binomial surd** e.g., $\sqrt{3} + \sqrt{5}$, $\sqrt{2} + \sqrt{7}$ etc.

(iii) $\sqrt{a} + \sqrt{b}$ and $\sqrt{a} - \sqrt{b}$ are called **conjugate surds** of each other.

Remember!

The product of two conjugate surds is a rational number.