Unit No. 6

Trigonometry

Basic Concepts

Trigonometry:

Trigonometry is a branch of mathematics that deals with the relationships between the angles and sides of a triangle, especially right-angled triangles.

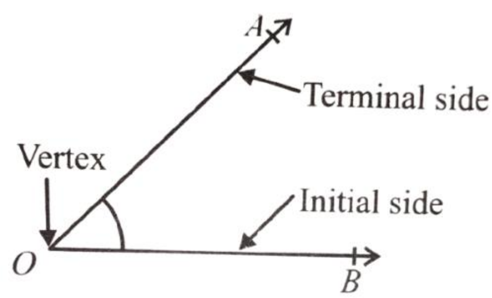
Role of Trigonometry:

It plays a vital role in various fields such as physics, engineering, architecture, and astronomy. The trigonometric concepts can solve problems involving angles and distances that appear in real-life situations such as calculating the height of buildings, distance between objects, and angle measurements in navigation.

Angle:

A plane figure which is formed by two rays sharing a common end point is called an angle. The two rays are known as the sides of the angle. The common end point is known as ***vertex***.

Pictorial Representation:



Brain teaser!

The plane geometry is the study of two dimensional figures.

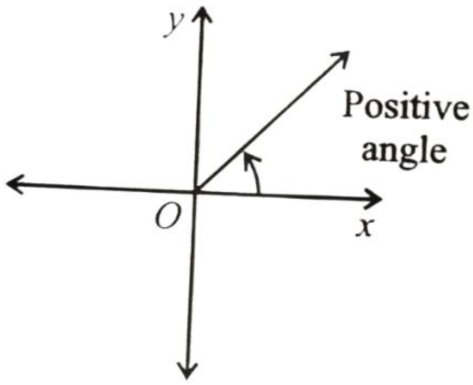
Types of Angles:

- Acute angle $0° < \theta < 90°$
- Obtuse angle $90° < \theta < 180°$
- Right angle $\theta = 90°$
- Straight angle $\theta = 180°$
- Reflex angle $180° < \theta < 360°$
- Full rotation $\theta = 360°$

Positive Angles:

The angle will be positive if the terminal side is rotated counterclockwise from the initial side. The given angle is in the 1^{st} quadrant.

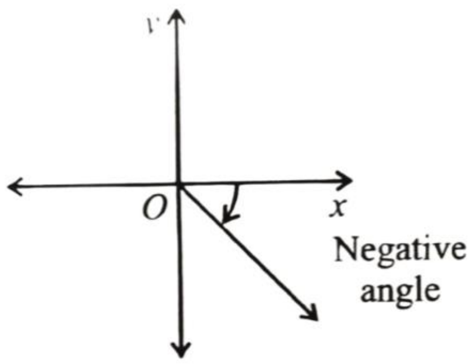
Pictorial Representation:



Negative Angles:

The angle will be negative if the terminal side is rotated clockwise from the initial side. The given angle is in the 4th quadrant.

Pictorial Representation:

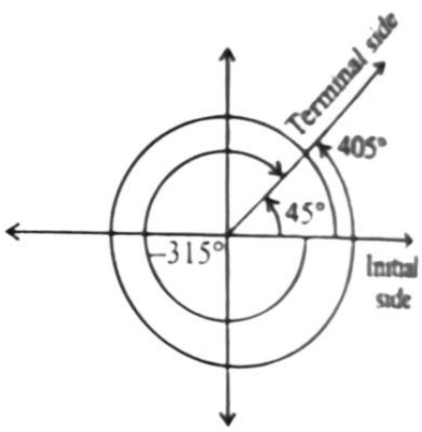


Co-Terminal Angles:

Co-terminal angles are angles that share the same initial side and terminal side in standard position, but they may have different measures. These angles differ by a multiple of 360∘ or 2π rad. For example, 45∘, 405∘, and −315∘ are co-terminal angles because:

$405^\circ = 45^\circ + 360^\circ$ and $-315^\circ = 45^\circ - 360^\circ$.

Pictorial Representation:



Degree Measurement:

A degree (∘) is a unit of measurement of angles. It represents $\frac{1}{360}$ of a full rotation around a point. In simpler terms, a degree is the measure of an angle with a complete circle being 360∘.

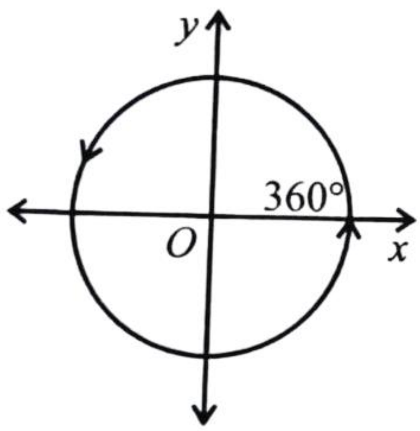
Why 360∘ Historically?

The choice of 360∘ to divide a circle dates back to the Babylonians, who used a base-60 number system (sexagesimal system). They were among the first to formalize the concept of angle measurement, and 360 was chosen likely because it is a highly composite number (it can be divided by 2, 3, 4, 5, 6, 9, 10, 12, 15, and more), making calculations easier. This system persisted throughout ancient times and degrees became entrenched in various cultures and mathematical traditions.

Full Circle:

A full rotation around a central point forms an angle of 360°.

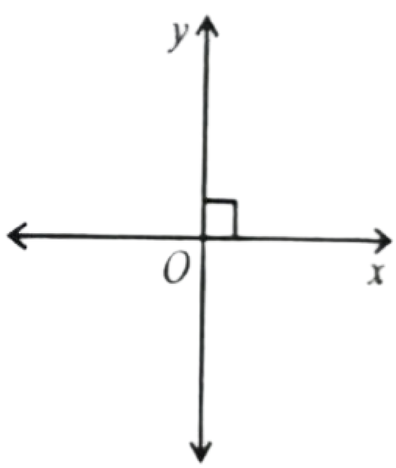
Pictorial Representation:



Right Angle:

One-quarter of a full rotation, or a 90° angle, is called a right angle.

Pictorial Representation:



Half Circle:

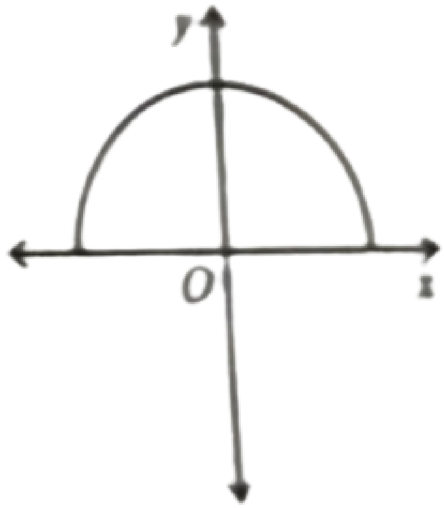
A straight angle, or half of a full rotation, measures 180°. The degree measure is further divided into minutes (′) and seconds (″).

1° = 60′ (60 minutes)

1′ = 60″ (60 seconds)

1° = 3600″ (60×60 seconds)

Pictorial Representation:



Converting Degrees to Minutes and Seconds:

To convert decimal degrees to degrees, minutes and seconds (DMS), follow the steps:

- Separate the whole number part (degrees) of the decimal.
- Multiply the decimal part by 60 to get the minutes.
- The whole number part of the result is the minutes. Multiply the decimal part of the minutes by 60 to get the seconds.

Converting from Degrees, Minutes and Seconds to Decimal Degrees:

To convert from degrees, minutes and seconds (DMS) to decimal degrees, follow the steps:

- Keep the degrees as they are.
- Convert minutes to decimal degrees: Divide the number of minutes by 60.
- Convert seconds to decimal degrees: Divide the number of seconds by 3600.
- Add all the values together.

Historical Background of the Radian:

The concept of radian measure, was first formalized by mathematicians in the 18th century, but the principles behind it had been understood much earlier by Euclid and Archimedes.

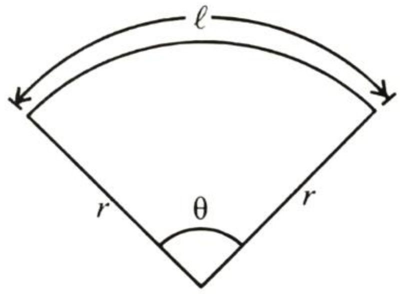
The word "radian" comes from the radius of a circle, as the radian is fundamentally related to the ratio between the arc length and the radius. The first known use of the term radian in the context of angular measurement was by Scottish mathematician James Thomson in 1873. His brother, William Thomson, also known as Lord Kelvin, was made a prominent physicist and both were influential in establishing radians as a standard unit.

Circular Measure (Radian):

The radian, denoted by the symbol "rad", is the unit of angle in the International System of Units (SI) and is the standard unit of angular measure used in many areas of mathematics.

A radian is a unit of angular measure in mathematics, particularly in trigonometry. It is defined as, "the angle subtended at the centre of a circle by an arc whose length is equal to the radius of the circle". Unlike degrees, which are based on dividing a circle into 360 parts, the radian is inherently related to the circle's geometry and arc length.

$$\theta = \frac{r}{l}$$



Conversion between degrees and radians:

Radians to degrees:

$$1 \text{ rad} = \frac{180}{\pi} \text{ degrees}$$

Degrees to radians:

$$1^{\circ} = \frac{\pi}{180} \text{ rad}$$

Turns	Radians	Degrees
0 turn	0 rad	0°
$\frac{1}{12}$ turn	$\frac{\pi}{6}$ rad	30°
$\frac{1}{8}$ turn	$\frac{\pi}{4}$ rad	45°
$\frac{1}{6}$ turn	$\frac{\pi}{3}$ rad	60°
$\frac{1}{4}$ turn	$\frac{\pi}{2}$ rad	90°
$\frac{1}{2}$ turn	π rad	90°
1 turn	2π rad	360°

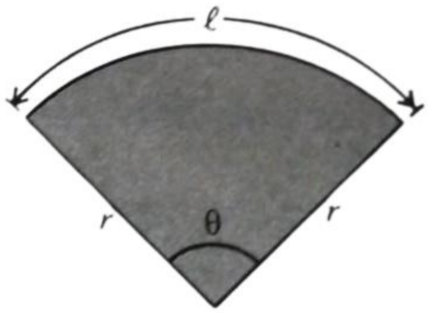
Arc Length and Area of Sector:

If r is radius and θ (rad) is the angle subtended by the arc of length 'l', then

Arc length of sector = $l = r\,\theta$

Area of sector = A = $\frac{1}{2}\,r^2\,\theta$

Pictorial Form:



Trigonometric Ratios:

The functions that relate angles to side in a right-angled triangle are known as trigonometric functions (sine, cosine, tangent etc.). Trigonometry has since been extensively used in various scientific disciplines such as physics (especially wave theory) engineering, and computer graphics.

History of Sine, Cosinenand and Tangent:

- **Hipparchus of Nicaea** (c. 190 – 120 BC) is considered the "father of trigonometry." He was the first to compile a trigonometric table for solving problems related to astronomy, utilizing chord functions.
  Hipparchus divided a circle into 360 degrees and used this system for measuring angles.
- In the Islamic golden age, **Al-Battani** (c. 858 – 929 AD) was among the first to replace chord functions with the modern sine function and calculated tables of sines and tangents.
- **Al-Khwarizmi** (c. 780 – 850 CE), known for his work in algebra, and **Omar Khayyam** (c. 1048 – 1131) worked on spherical trigonometry, which has applications in astronomy.
- **Isaac Newton** and **Gottfried Wilhelm Leibniz** (17th century) developed calculus, which further expanded the use of trigonometric functions beyond geometry into more abstract fields of mathematics.

Trigonometric Ratios of an Acute Angle:

$$\text{Sin}\theta = \frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{a}{c}$$

$$\text{Cosec}\theta = \frac{\text{Hypotenuse}}{\text{Perpendicular}} = \frac{c}{a}$$

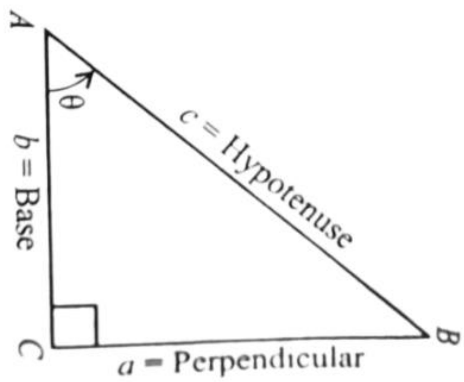
$$\text{Cos}\theta = \frac{\text{Base}}{\text{Hypotenuse}} = \frac{b}{c}$$

$$\text{Sec}\theta = \frac{\text{Hypotenuse}}{\text{Base}} = \frac{c}{b}$$

$$\text{Tan}\theta = \frac{\text{Perpendicular}}{\text{Base}} = \frac{a}{b}$$

$$\text{Cot}\theta = \frac{\text{Base}}{\text{Perpendicular}} = \frac{b}{a}$$

Pictorial Form:



$\mathrm{Tan}\theta = \frac{\mathrm{Sin}\theta}{\mathrm{Cos}\theta}$

$\mathrm{Cot}\theta = \frac{\mathrm{Cos}\theta}{\mathrm{Sin}\theta}$

$\mathrm{Cosec}\theta = \frac{1}{\mathrm{Sin}\theta}$

$\mathrm{Sec}\theta = \frac{1}{\mathrm{Cos}\theta}$

$\mathrm{Cot}\theta = \frac{1}{\mathrm{Tan}\theta}$

$Sin^2\theta + Cos^2\theta = 1$

$tan^2\theta + 1 = sec^2\theta$

$cot^2\theta + 1 = cosec^2\theta$

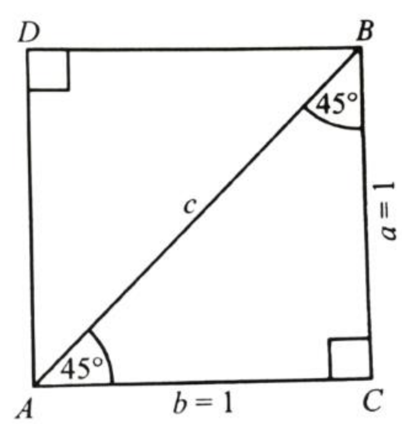
Pythagoras Theorem:

The square of the length of the hypotenuse of a right triangle is equal to the sum of the squares of the lengths of the other two sides. Mathematically;

$$c^2 = a^2 + b^2$$

Values of Trigonometric Ratios of Special Angles:

Trigonometric Ratios of 45°[$\frac{\pi}{4}$ rad]:



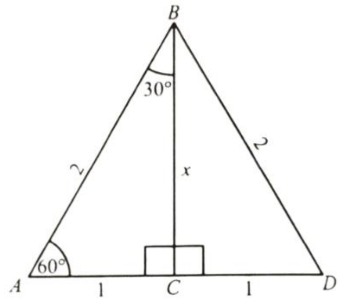
The trigonometric ratios are:

$\sin 45° = \frac{a}{c} = \frac{1}{\sqrt{2}}$; $\mathrm{cosec}45° = \frac{c}{a} = \sqrt{2}$

$\cos 45° = \frac{b}{c} = \frac{1}{\sqrt{2}}$; $\sec 45° = \frac{c}{b} = \sqrt{2}$

$\tan 45° = \frac{a}{b} = 1$; $\cot 45° = \frac{b}{a} = 1$

Trigonometric Ratios of 30° [$\frac{\pi}{6}$ rad]:



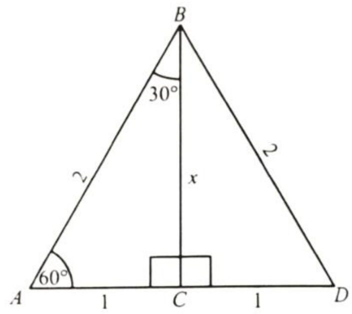
The trigonometric ratios are:

Sin30° $= \frac{a}{c} = \frac{1}{2}$; cosec30° $= \frac{c}{a} = 2$

Cos30° $= \frac{b}{c} = \frac{\sqrt{3}}{2}$; sec30° $= \frac{c}{b} = \frac{2}{\sqrt{3}}$

Tan30° $= \frac{a}{b} = \frac{1}{\sqrt{3}}$; cot30° $= \frac{b}{a} = \sqrt{3}$

Trigonometric Ratios of 60°[$\frac{\pi}{3}$ rad]:



The trigonometric ratios are:

Sin60° $= \frac{a}{c} = \frac{\sqrt{3}}{2}$; cosec60° $= \frac{c}{a} = \frac{2}{\sqrt{3}}$

Cos60° $= \frac{b}{c} = \frac{1}{2}$; sec60° $= \frac{c}{b} = 2$

Tan60° $= \frac{a}{b} = \sqrt{3}$; cot60° $= \frac{b}{a} = \frac{1}{\sqrt{3}}$

Θ	0	30°	45°	60°	90°
sin θ	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan θ	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

Solution of a Triangle:

There are three sides and three angles in a triangle. Out of these six elements, if we know three of them including at least one side, then we can find the measures of the remaining elements. Finding the measures of the remaining elements is called the solution of a triangle.