Unit No. 1

Real Numbers

Exercise No. 1.2

Question No. 1

Rationalize the denominator of following:

(i). $\frac{13}{4+\sqrt{3}}$

Solution:

$$\frac{13}{4+\sqrt{3}}$$

By rationalizing:

$$= \frac{13}{4+\sqrt{3}} \times \frac{4-\sqrt{3}}{4-\sqrt{3}}$$

$$= \frac{13\,(4-\sqrt{3})}{(4)^2 - (\sqrt{3})^2}$$

$$= \frac{13\,(4-\sqrt{3})}{16-3}$$

$$= \frac{13\,(4-\sqrt{3})}{13}$$

$$= (4-\sqrt{3})$$

(ii). $\frac{\sqrt{2}+\sqrt{5}}{\sqrt{3}}$

Solution:

$$\frac{\sqrt{2}+\sqrt{5}}{\sqrt{3}}$$

By rationalizing:

$$= \frac{\sqrt{2}+\sqrt{5}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$= \frac{(\sqrt{2}+\sqrt{5})\sqrt{3}}{(\sqrt{3})^2}$$

$$= \frac{(\sqrt{2 \times 3}+\sqrt{5 \times 3})}{3}$$

$$= \frac{\sqrt{6}+\sqrt{15}}{3}$$

(iii). $\frac{\sqrt{2}-1}{\sqrt{5}}$

Solution:

$$\frac{\sqrt{2}-1}{\sqrt{5}}$$

By rationalizing:

$$= \frac{\sqrt{2}-1}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$$

$$= \frac{(\sqrt{2}-1)\sqrt{5}}{(\sqrt{5})^2}$$

$$= \frac{(\sqrt{2 \text{ x } 5} - 1 \text{ x } \sqrt{5})}{5}$$

$$= \frac{\sqrt{10}-\sqrt{5}}{5}$$

(iv). $\mathbf{\frac{6-4\sqrt{2}}{6+4\sqrt{2}}}$

Solution:

$$\frac{6 - 4\sqrt{2}}{6 + 4\sqrt{2}}$$

By rationalizing:

$$= \frac{6-4\sqrt{2}}{6+4\sqrt{2}} \times \frac{6-4\sqrt{2}}{6-4\sqrt{2}}$$

$$= \frac{(6-4\sqrt{2})^2}{(6)^2-(4\sqrt{2})^2}$$

$$= \frac{(6)^2-2(6)(4\sqrt{2})+(4\sqrt{2})^2}{(6)^2-(4\sqrt{2})^2}$$

$$= \frac{36-48\sqrt{2}+(4)^2(\sqrt{2})^2}{36-(4)^2(\sqrt{2})^2}$$

$$= \frac{36-48\sqrt{2}+(16)(2)}{36-(16)(2)}$$

$$= \frac{36-48\sqrt{2}+32}{36-32}$$

$$= \frac{68-48\sqrt{2}}{4}$$

$$= \frac{4(17-12\sqrt{2})}{4}$$

$$= (17 - 12\sqrt{2})$$

(v). $\mathbf{\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}}$

Solution:

$$\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$$

By rationalizing:

$$= \frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}} \times \frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}}$$

$$= \frac{(\sqrt{3}-\sqrt{2})^2}{(\sqrt{3})^2-(\sqrt{2})^2}$$

$$= \frac{(\sqrt{3})^2-2(\sqrt{3})(\sqrt{2})+(\sqrt{2})^2}{(\sqrt{3})^2-(\sqrt{2})^2}$$

$= \frac{3-2\sqrt{6}+2}{3-2}$

$= \frac{5-2\sqrt{6}}{1}$

$= 5 - 2\sqrt{6}$

(vi). $\frac{4\sqrt{3}}{\sqrt{7}+\sqrt{5}}$

Solution:

$$\frac{4\sqrt{3}}{\sqrt{7}+\sqrt{5}}$$

By rationalizing:

$= \frac{4\sqrt{3}}{\sqrt{7}+\sqrt{5}} \times \frac{\sqrt{7}-\sqrt{5}}{\sqrt{7}-\sqrt{5}}$

$= \frac{4\sqrt{3}(\sqrt{7}-\sqrt{5})}{(\sqrt{7})^2-(\sqrt{5})^2}$

$= \frac{4\sqrt{3}(\sqrt{7}-\sqrt{5})}{7-5}$

$= \frac{4\sqrt{3}(\sqrt{7}-\sqrt{5})}{2}$

$= \frac{2\sqrt{3}(\sqrt{7}-\sqrt{5})}{1}$

$= 2\sqrt{3}(\sqrt{7}-\sqrt{5})$

Question No. 2

Simplify the following:

(i). $(\frac{81}{16})^{-\frac{3}{4}}$

Solution:

$(\frac{81}{16})^{-\frac{3}{4}}$

$= (\frac{3\times3\times3\times3}{2\times2\times2\times2})^{-\frac{3}{4}}$

$= (\frac{3^4}{2^4})^{-\frac{3}{4}}$

$= (\frac{3}{2})^{-3}$

$= (\frac{2}{3})^{3}$

$= \frac{8}{27}$

2nd Method:

Solution:

$(\frac{81}{16})^{-\frac{3}{4}}$

$= (\frac{16}{81})^{\frac{3}{4}}$

$= (\frac{2 \times 2 \times 2 \times 2}{3 \times 3 \times 3 \times 3})^{\frac{3}{4}}$

$= (\frac{2^4}{3^4})^{\frac{3}{4}}$

$= (\frac{2}{3})^3$

$= \frac{8}{27}$

Any one from both methods can be used to solve question.

(ii). $(\frac{3}{4})^{-2} \div (\frac{4}{9})^3 \times \frac{16}{27}$

Solution:

$(\frac{3}{4})^{-2} \div (\frac{4}{9})^3 \times \frac{16}{27}$

$=(\frac{4}{3})^2 \times (\frac{9}{4})^3 \times \frac{16}{27}$

$=(\frac{2^2}{3})^2 \times (\frac{3^2}{2^2})^3 \times \frac{2^4}{3^3}$

$=\frac{2^4}{3^2} \times \frac{3^6}{2^6} \times \frac{2^4}{3^3}$

$= \frac{2^{4-6+4}}{3^{2-6+3}}$

$= \frac{2^2}{3^{-1}}$

$= 4 \times 3$

$= 12$

(iii). $(0.027)^{-\frac{1}{3}}$

Solution:

$(0.027)^{-\frac{1}{3}}$

$= (\frac{27}{1000})^{-\frac{1}{3}}$

$= (\frac{3^3}{10^3})^{-\frac{1}{3}}$

$= (\frac{10^3}{3^3})^{\frac{1}{3}}$

$= \frac{10}{3}$ or $3\frac{1}{3}$

(iv). $\sqrt[7]{\frac{x^{14} \text{ x } y^{21} \text{ x } z^{35}}{y^{14}\ z^7}}$

Solution:

$\sqrt[7]{\frac{x^{14} \times y^{21} \times z^{35}}{y^{14}\ z^7}}$

$= \sqrt[7]{x^{14} \text{ x } y^{21-14} \text{ x } z^{35-7}}$

$= \sqrt[7]{x^{14} \text{ x } y^{7} \text{ x } z^{28}}$

$= x^{\frac{14}{7}} \times y^{\frac{7}{7}} \times z^{\frac{28}{7}}$

$= x^2 \times y^1 \times z^4$

$= x^2\, y\, z^4$

(v). $\frac{\mathbf{5.(25)^{n+1}-25.(5)^{2n}}}{\mathbf{5.(5)^{2n+3}-(25)^{n+1}}}$

Solution:

$$\frac{5.(25)^{n+1} - 25.(5)^{2n}}{5.(5)^{2n+3} - (25)^{n+1}}$$

$= \frac{5.(5^2)^{n+1}-5^2.(5)^{2n}}{5.(5)^{2n+3}-(5^2)^{n+1}}$

$= \frac{5.5^{2n+2}-5^2.5^{2n}}{5.5^{2n+3}-5^{2n+2}}$

$= \frac{5.5^{2n}.5^2-5^2.5^{2n}}{5.5^{2n}.5^3-5^{2n}.5^2}$

$= \frac{5^{2n}(5.5^2-5^2)}{5^{2n}(5.5^3-5^2)}$

$= \frac{(5.5^2-5^2)}{(5.5^3-5^2)}$

$= \frac{5^2(5-1)}{5^2(5.5^1-1)}$

$= \frac{(5-1)}{(5.5^1-1)}$

$= \frac{4}{25-1}$

$= \frac{4}{24}$

$= \frac{1}{6}$

(vi). $\frac{\mathbf{(16)^{x+1}+20.(4^{2x})}}{\mathbf{2^{x-3}\times 8^{x+2}}}$

Solution:

$$\frac{(16)^{x+1} + 20.(4^{2x})}{2^{x-3} \times 8^{x+2}}$$

$= \frac{(2^4)^{x+1}+2^2.5.(2^{2\text{x}2x})}{2^{x-3}\times(2^3)^{x+2}}$

$= \frac{2^{4x+4}+2^2.5.2^{4x}}{2^{x-3}\times 2^{3x+6}}$

$= \frac{2^{4x+4}+2^{2+4x}.5}{2^{x-3+3x+6}}$

$= \frac{2^{4x}(2^4+2^2.5)}{2^{4x+3}}$

$= \frac{2^{4x}(2^4+2^2.5)}{2^{4x}.2^3}$

$= \frac{(2^4+2^2.5)}{2^{2+1}}$

$= \frac{2^2(2^2+1.5)}{2^2.2^1}$

$= \frac{4+5}{2}$

$= \frac{9}{2}$

$= \mathbf{4\ 1/2}$

(vii). $(\mathbf{64})^{-\frac{2}{3}} \div (\mathbf{9})^{-\frac{3}{2}}$

Solution:

$(64)^{-\frac{2}{3}} \div (9)^{-\frac{3}{2}}$

$= (2^6)^{-\frac{2}{3}} \div (3^2)^{-\frac{3}{2}}$

$= (2^2)^{-2} \div (3)^{-3}$

$= 2^{-4} \div (3)^{-3}$

$= \frac{2^{-4}}{3^{-3}}$

$= \frac{3^3}{2^4}$

$= \frac{27}{16}$

$= 1\frac{11}{16}$

(viii). $\frac{\mathbf{3^n \text{ x } 9^{n+1}}}{\mathbf{3^{n-1} \text{ x } 9^{n-1}}}$

Solution:

$$\frac{3^n \times 9^{n+1}}{3^{n-1} \times 9^{n-1}}$$

$= \frac{3^n \times (3^2)^{n+1}}{3^{n-1} \times (3^2)^{n-1}}$

$= \frac{3^n \times 3^{2n+2}}{3^{n-1} \times 3^{2n-2}}$

$= \frac{3^{n+2n+2}}{3^{n-1+2n-2}}$

$= \frac{3^{3n+2}}{3^{3n-3}}$

$= 3^{3n+2-(3n-3)}$

$= 3^{3n+2-3n+3}$

$= 3^5$

$= 243$

(ix). $\frac{\mathbf{5^{n+3}-6.5^{n+1}}}{\mathbf{9 \text{ x } 5^n - 4 \text{ x } 5^n}}$

Solution:

$$\frac{5^{n+3} - 6.5^{n+1}}{9 \text{ x } 5^{n} - 4 \text{ x } 5^{n}}$$

$$= \frac{5^{n+3} - 6.5^{n+1}}{3^2 \text{ x } 5^{n} - 2^2 \text{ x } 5^{n}}$$

$$= \frac{5^{n}(5^3 - 6.5^1)}{5^{n}(3^2 - 2^2)}$$

$$= \frac{(5^3 - 6.5^1)}{(3^2 - 2^2)}$$

$$= \frac{125 - 30}{9 - 4}$$

$$= \frac{95}{5}$$

$$= 19$$

Question No. 3

If x = 3 + $\sqrt{8}$ then find the value of:

(i). $x + \frac{1}{x}$

Solution:

$$x = 3 + \sqrt{8}$$

$$\frac{1}{x} = \frac{1}{3 + \sqrt{8}}$$

By rationalizing:

$$\frac{1}{x} = \frac{1}{3 + \sqrt{8}} \times \frac{3 - \sqrt{8}}{3 - \sqrt{8}}$$

$$\frac{1}{x} = \frac{1(3 - \sqrt{8})}{3^2 - \sqrt{8}^2}$$

$$\frac{1}{x} = \frac{3 - \sqrt{8}}{9 - 8}$$

$$\frac{1}{x} = \frac{3 - \sqrt{8}}{1}$$

$$\frac{1}{x} = 3 - \sqrt{8}$$

By adding x & $\frac{1}{x}$:

$$x + \frac{1}{x} = (3 + \sqrt{8}) + (3 - \sqrt{8})$$

$$x + \frac{1}{x} = 3 + \sqrt{8} + 3 - \sqrt{8}$$

$$x + \frac{1}{x} = 6$$

(ii). $x - \frac{1}{x}$

Solution:

By subtracting x & $\frac{1}{x}$:

$$x - \frac{1}{x} = \left(3 + \sqrt{8}\right) - \left(3 - \sqrt{8}\right)$$

$$x - \frac{1}{x} = 3 + \sqrt{8} - 3 + \sqrt{8}$$

$$x - \frac{1}{x} = 2\sqrt{8}$$

(iii). $x^2 + \frac{1}{x^2}$

Solution:

From (i) where:

$$x + \frac{1}{x} = 6$$

Squaring on both sides:

$$(x + \frac{1}{x})^2 = (6)^2$$

$$x^2 + \frac{1}{x^2} + 2(x)(\frac{1}{x}) = 36$$

$$x^2 + \frac{1}{x^2} + 2 = 36$$

$$x^2 + \frac{1}{x^2} = 36 - 2$$

$$x^2 + \frac{1}{x^2} = 34$$

(iv). $x^2 - \frac{1}{x^2}$

Solution:

From (i) & (ii) where:

$$x + \frac{1}{x} = 6$$

$$x - \frac{1}{x} = 2\sqrt{8}$$

Using Formula:

$$x^2 - \frac{1}{x^2} = \left(x + \frac{1}{x}\right)\left(x - \frac{1}{x}\right)$$

By putting values:

$$x^2 - \frac{1}{x^2} = (6)\left(2\sqrt{8}\right)$$

$$x^2 - \frac{1}{x^2} = 12\sqrt{8}$$

(v). $x^4 + \frac{1}{x^4}$

Solution:

From (iii) where:

$$x^2 + \frac{1}{x^2} = 34$$

Squaring on both sides:

$$(x^2 + \frac{1}{x^2})^2 = (34)^2$$

$$x^4 + \frac{1}{x^4} + 2(x^2)(\frac{1}{x^2}) = 1156$$

$$x^4 + \frac{1}{x^4} + 2 = 1156$$

$$x^4 + \frac{1}{x^4} = 1156 - 2$$

$$x^4 + \frac{1}{x^4} = 1154$$

(vi). $(\mathbf{x} - \frac{1}{x})^2$

Solution:

From (ii) where:

$$x - \frac{1}{x} = 2\sqrt{8}$$

Squaring on both sides:

$$(x - \frac{1}{x})^2 = (2\sqrt{8})^2$$

$$(x - \frac{1}{x})^2 = 32$$

Question No. 4

Find the rational numbers p and q such that

$$\frac{\mathbf{8 - 3\sqrt{2}}}{\mathbf{4 + 3\sqrt{2}}} = \boldsymbol{p + q\sqrt{2}}$$

Solution:

$$p + q\sqrt{2} = \frac{8 - 3\sqrt{2}}{4 + 3\sqrt{2}}$$

By rationalizing:

$$p + q\sqrt{2} = \frac{8 - 3\sqrt{2}}{4 + 3\sqrt{2}} \times \frac{4 - 3\sqrt{2}}{4 - 3\sqrt{2}}$$

$$p + q\sqrt{2} = \frac{(8 - 3\sqrt{2})(4 - 3\sqrt{2})}{(4)^2 - (3\sqrt{2})^2}$$

$$p + q\sqrt{2} = \frac{8\left(4 - 3\sqrt{2}\right) - 3\sqrt{2}(4 - 3\sqrt{2})}{16 - 18}$$

$$p + q\sqrt{2} = \frac{32 - 24\sqrt{2} - 12\sqrt{2} + 18}{16 - 18}$$

$$p + q\sqrt{2} = \frac{50 - 36\sqrt{2}}{-2}$$

$$p + q\sqrt{2} = \frac{-2(-25 + 18\sqrt{2})}{-2}$$

$$p + q\sqrt{2} = -25 + 18\sqrt{2}$$

By comparing:

$p = -25$; $q = 18$

Question No. 5

Simplify the following:

(i). $\dfrac{(25)^{\frac{3}{2}} \times (243)^{\frac{3}{5}}}{(16)^{\frac{5}{4}} \times (8)^{\frac{4}{3}}}$

Solution:

$$\frac{(25)^{\frac{3}{2}} \times (243)^{\frac{3}{5}}}{(16)^{\frac{5}{4}} \times (8)^{\frac{4}{3}}}$$

$$= \frac{(5^2)^{\frac{3}{2}} \times (3^5)^{\frac{3}{5}}}{(2^4)^{\frac{5}{4}} \times (2^3)^{\frac{4}{3}}}$$

$$= \frac{5^3 \times 3^3}{2^5 \times 2^4}$$

$$= \frac{125 \times 27}{32 \times 16}$$

$$= \frac{3375}{512}$$

(ii). $\dfrac{54 \times \sqrt[3]{(27)^{2x}}}{9^{x+1} + 216(3^{2x-1})}$

Solution:

$$\frac{54 \times \sqrt[3]{(27)^{2x}}}{9^{x+1} + 216(3^{2x-1})}$$

$$= \frac{2\times3\times3\times3 \times \sqrt[3]{(3^3)^{2x}}}{(3^2)^{x+1} + 6^3(3^{2x-1})}$$

$$= \frac{2\times3^3 \times \sqrt[3]{3^{6x}}}{3^{2x+2} + (2\times3)^3(3^{2x-1})}$$

$$= \frac{2\times3^3 \times 3^{\frac{6x}{3}}}{3^{2x+2} + 2^3\times3^3\times3^{2x-1}}$$

$$= \frac{2\times3^1\times3^2 \times 3^{2x}}{3^{2x+2} + 2^3\times 3^{3+2x-1}}$$

$$= \frac{2\times3\times3^{2x+2}}{3^{2x+2} + 2^3\times 3^{2x+2}}$$

$$= \frac{6\times3^{2x+2}}{3^{2x+2}(1+2^3)}$$

$$= \frac{6}{(1+8)}$$

$= \frac{6}{9}$

$= \frac{2}{3}$

(iii). $\sqrt{\frac{(216)^{\frac{2}{3}} \times (25)^{\frac{1}{2}}}{(0.04)^{\frac{-3}{2}}}}$

Solution:

$$\sqrt{\frac{(216)^{\frac{2}{3}} \times \ (25)^{\frac{1}{2}}}{(0.04)^{\frac{-3}{2}}}}$$

$$= \sqrt{\frac{(6^3)^{\frac{2}{3}} \times (5^2)^{\frac{1}{2}}}{(\frac{4}{100})^{\frac{-3}{2}}}}$$

$$= \sqrt{6^2 \ \times \ 5 \ \times (\frac{1}{25})^{\frac{3}{2}}}$$

$$= 6\sqrt{5 \ \times (\frac{1}{5^2})^{\frac{3}{2}}}$$

$$= 6\sqrt{5 \ \times (\frac{1}{5})^3}$$

$$= 6\sqrt{(\frac{1}{5})^2}$$

$$= 6 \ \times \frac{1}{5}$$

$$= \frac{6}{5}$$

(iv). $\left[\boldsymbol{a^{\frac{1}{3}} + \ b^{\frac{2}{3}}}\right] \times \left[\boldsymbol{a^{\frac{2}{3}} - \ a^{\frac{1}{3}}b^{\frac{2}{3}} + \ b^{\frac{4}{3}}}\right]$

Solution:

$$\left[a^{\frac{1}{3}} + \ b^{\frac{2}{3}}\right] \times \left[a^{\frac{2}{3}} - \ a^{\frac{1}{3}}b^{\frac{2}{3}} + \ b^{\frac{4}{3}}\right]$$

$$= \left[a^{\frac{1}{3}} + \ (b^2)^{\frac{1}{3}}\right] \times \left[(a^{\frac{1}{3}})^2 - \ ab^2 + \ ((b^2)^{\frac{1}{3}})^2\right]$$

Using Formula:

$$(a + b)(a^2 - 2ab + \ b^2) = a^3 + \ b^3$$

$$= (a^{\frac{1}{3}})^3 + \ ((b^2)^{\frac{1}{3}})^3$$

$$= a + \ b^2$$