Unit No. 8

Logic

Basic Concepts

Logic:

Logic is a systematic method of reasoning that enables one to interpret the meanings of statements, examine their truth, and deduce new information from existing facts. Logic plays a key role in problem-solving and decision-making.

History:

Aristotle:

The history of logic began with **Aristotle**, who is considered the father of formal logic. He developed a system of deductive reasoning known as syllogistic logic, which became the foundation of logical thought.

Stoics:

The **Stoics** followed, contributing to propositional logic and exploring paradoxes such as the Liar Paradox.

Medieval Period:

During the **medieval period**, scholars like Peter Abelard and William of Ockham expanded **Aristotle's** work, introducing theories of semantics and consequences. In the 19th century, logic advanced through the works of **George Boole**, who developed Boolean algebra, and **Gottlob Frege**, who formalized modern predicate logic. **Bertrand Russell** and **Alfred North Whitehead** attempted to reduce mathematics to logic in their seminal work, **Principia Mathematica**. The 20th century saw significant progress with **Kurt Gödel**, who introduced his incompleteness theorems, reshaping our understanding of mathematical logic.

Induction:

The way of drawing conclusions is called **induction**. Inductive reasoning is helpful in natural sciences, where we must depend upon repeated experiments or observations. In fact, a greater part of our knowledge is based on induction.

Example; You observe several swans in a park, and every single swan you see is white. Through induction, you might conclude that "all swans are white." This is a generalization based on your limited observations.

Deduction:

The way of reasoning i.e., drawing conclusions from premises believed to be true, is called **deduction**. One usual example of deduction is: All men are mortal. We are men. Therefore, we are also mortal. To study logic, we start with a statement.

Statement:

A sentence or mathematical expression which may be true or false but not both is called a statement.

This is correct so far as mathematics and other sciences are concerned. For instance, the statement a=b can be either true or false.

Logical Operators:

The letters p, q, etc., will be used to denote the statements. A brief list of the symbols which will be used is given below:

Symbols	How to be read	Symbolic Expression	How to be read
$\sim$	Not	$\sim p$	Not *p*, negation of *p*
$\wedge$	And	$p \wedge q$	p and q
$\vee$	Or	$p \vee q$	P or q
$\rightarrow$	If…then, implies	$p \rightarrow q$	If p then q, P implies q
$\leftrightarrow$	Is equivalent to, if and only if	$p \leftrightarrow q$	p if and only if q, p is equivalent to q

Negation:

If p is any statement, its negation is denoted by $\sim p$, read 'not p'. It follows from this definition that if p is true, $\sim p$ is false, and if p is false, $\sim p$ is true. The possible truth values of p and $\sim p$ are given in Table, which is called a **truth table**, where the true value is denoted by T and the false value is denoted by F.

Table:

p	**~q**
T	F
F	T

Conjunction:

The conjunction of two statements p and q is symbolically written as $p \wedge q$ (p and q). A conjunction is considered to be true only if both statements are true.

Table:

p	**q**	**p ∧ q**
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction:

The disjunction of p and q is symbolically written as $p \vee q$ (p or q). The disjunction $p \vee q$ is considered to be true when at least one of the statements is true. It is false when both of them are false.

Table:

p	**q**	**p ∨ q**
T	T	T
T	F	T
F	T	T
F	F	F

Implication or conditional:

A compound statement of the form *if p then q* (p→q) also written as *p implies q* is called a **conditional** or an **implication**. p is called the **antecedent** or **hypothesis** and q is called the **consequent** or the **conclusion**. A conditional is regarded as false only when the antecedent is true and the consequent is false.

Table:

p	q	p → q
T	T	T
T	F	F
F	T	T
F	F	T

Biconditional p ↔ q:

The statement (p → q)∧(q → p) is shortly written as p ↔ q and is called the **biconditional** or **equivalence**. It is read p iff q (iff stands for "if and only if"). p ↔ q is true only when both statements p and q are true or both statements p and q are false.

Table:

P	q	p → q	q → p	p ↔ q
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Conditionals related with a given conditional:

Let p and q be the statements and

p → q be a given conditional, then

(i) q → p is called the **converse** of p → q;

(ii) ~p → ~q is called the **inverse** of p → q;

(iii) ~q → ~p is called the **contrapositive** of

p → q.

Table:

p	q	~p	~q	Given conditional p → q	Converse q → p	Inverse ~p → ~q	Contrapositive ~q →~p
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

Role:

Many claims and statements in our daily routine are disproved by it. In mathematics, the evidence that a statement is correct is provided by proofs, demonstrating a logical sequence of steps by which the final conclusion is led to.

Note:

I f x is odd, then x can be expressed in the form: $x = 2k + 1$ for some $k \in Z$

Note:

If x is an even integer, then x can be expressed in the form: $x = 2k$ for some $k \in Z$

Note:

A set B is a subset of a set A if every element of set B is also an element of a set A.

Mathematically, we write it as:

$B \subseteq A$ if $\forall\, x \in B \Longrightarrow x \in A$

The concepts of theorems, conjectures, and axioms in mathematics.

1. Theorem:

The sum of the interior angles of a quadrilateral is 360 degrees.

2. The Fundamental Theorem of Arithmetic:

Every integer greater than 1 can be uniquely expressed as a product of prime numbers up to the order of the factors.

3. Fermat's Last Theorem:

There are no three positive integers a, b, c which satisfy the equation an + bn = cn, where $n \in N$ and $n > 2$.

Conjecture:

A conjecture is a mathematical statement or hypothesis that is believed to be true based on observations but has not yet been proved. In mathematics, conjectures often serve as hypotheses, and if a conjecture is proven to be true, it becomes a theorem.

Conversely: If evidence is found that disproves it, the conjecture is shown to be false. Here, is another well-known statement that has gained enough recognition to be named. First proposed in the 18th century by the German mathematician Christian Goldbach, it is known as the Goldbach Conjecture. The Goldbach Conjecture states that: **Statement:** Every even integer greater than 2 is a sum of two prime numbers.

Axiom:

An axiom is a mathematical statement that we believe to be true without any evidence or requiring any proof. For example, the following are the statements of axioms.

Axiom: Through a given point, infinitely many lines can pass.

Euclid Axioms: A straight line can be drawn between any two points.

Peano Axioms: Every natural number has a successor, which is also a natural number.

Axiom of Extensionality: Two sets are equal if they have the same elements.

Axiom of Power Set: Any set has a set of all its subsets.

Deductive Proof:

Deductive reasoning is a way of drawing conclusions from premises believed to be true. If the premises are true, then the conclusion must also be true. For example: All human beings need to breathe to live. Ahmad is a human. Therefore, Ahmad is also breathing to live.

In mathematics, deductive proof in AN algebraic expression is a technique to show the validity of a mathematical statement through logical reasoning based on known rules, theorems, axioms, or previously proven statements.