Unit No. 6

Trigonometry

Exercise No. 6.3

Question No. 1

If θ lies in the first quadrant, find the remaining trigonometric ratios of θ.

(i) $\sin\theta = \frac{2}{3}$

Solution:

$\sin\theta = \frac{a}{c} = \frac{2}{3}$

So, $a = 2$, $c = 3$, $b = ?$

By using Pythagoras theorem:

$c^2 = a^2 + b^2$

$3^2 = 2^2 + b^2$

$9 = 4 + b^2$

$b^2 = 9 - 4$

$b^2 = 5$

$b = \sqrt{5}$

$a = 2, \quad b = \sqrt{5}, \quad c = 3$

$\sin\theta = \frac{a}{c} = \frac{2}{3}$, $\operatorname{cosec}\theta = \frac{c}{a} = \frac{3}{2}$

$\cos\theta = \frac{b}{c} = \frac{\sqrt{5}}{3}$, $\sec\theta = \frac{c}{b} = \frac{3}{\sqrt{5}}$

$\tan\theta = \frac{a}{b} = \frac{2}{\sqrt{5}}$, $\cot\theta = \frac{b}{a} = \frac{\sqrt{5}}{2}$

(ii) $\cos\theta = \frac{3}{4}$

Solution:

$\cos\theta = \frac{b}{c} = \frac{3}{4}$

So, $b = 3$, $c = 4$, $a = ?$

By using Pythagoras theorem:

$c^2 = a^2 + b^2$

$4^2 = a^2 + 3^2$

$16 = a^2 + 9$

$a^2 = 16 - 9$

$a^2 = 7$

$a = \sqrt{7}$

$a = \sqrt{7}, \quad b = 3, \quad c = 4$

$\sin\theta = \frac{a}{c} = \frac{\sqrt{7}}{4}$, $\quad \operatorname{cosec}\theta = \frac{c}{a} = \frac{4}{\sqrt{7}}$

$\cos\theta = \frac{b}{c} = \frac{3}{4}$, $\quad \sec\theta = \frac{c}{b} = \frac{4}{3}$

$\tan\theta = \frac{a}{b} = \frac{\sqrt{7}}{3}$, $\quad \cot\theta = \frac{b}{a} = \frac{3}{\sqrt{7}}$

(iii) $\tan\theta = \frac{1}{2}$

Solution:

$\tan\theta = \frac{a}{b} = \frac{1}{2}$

So, $a = 1, \quad b = 2, \quad c = ?$

By using Pythagoras theorem:

$c^2 = a^2 + b^2$

$c^2 = 1^2 + 2^2$

$c^2 = 1 + 4$

$c^2 = 5$

$c = \sqrt{5}$

$a = 1, \quad b = 2, \quad c = \sqrt{5}$

$\sin\theta = \frac{a}{c} = \frac{1}{\sqrt{5}}$, $\quad \operatorname{cosec}\theta = \frac{c}{a} = \frac{\sqrt{5}}{1} = \sqrt{5}$

$\cos\theta = \frac{b}{c} = \frac{2}{\sqrt{5}}$, $\quad \sec\theta = \frac{c}{b} = \frac{\sqrt{5}}{2}$

$\tan\theta = \frac{a}{b} = \frac{1}{2}$, $\quad \cot\theta = \frac{b}{a} = \frac{2}{1} = 2$

(iv) $\sec\theta = 3$

Solution:

$\sec\theta = \frac{c}{b} = 3 = \frac{3}{1}$

So, $b = 1, \quad c = 3, \quad a = ?$

By using Pythagoras theorem:

$c^2 = a^2 + b^2$

$3^2 = a^2 + 1^2$

$9 = a^2 + 1$

$a^2 = 9 - 1$

$a^2 = 8$

$a = 2\sqrt{2}$

$a = 2\sqrt{2}$, $b = 1$, $c = 3$

$\sin\theta = \frac{a}{c} = \frac{2\sqrt{2}}{3}$, $\operatorname{cosec}\theta = \frac{c}{a} = \frac{3}{2\sqrt{2}}$

$\cos\theta = \frac{b}{c} = \frac{1}{3}$, $\sec\theta = \frac{c}{b} = \frac{3}{1} = 3$

$\tan\theta = \frac{a}{b} = \frac{2\sqrt{2}}{1} = 2\sqrt{2}$, $\cot\theta = \frac{b}{a} = \frac{1}{2\sqrt{2}}$

(v) $\cot\theta = \sqrt{\frac{3}{2}}$

Solution:

$$\cot\theta = \frac{b}{a} = \sqrt{\frac{3}{2}}$$

So, $b = \sqrt{3}$, $a = \sqrt{2}$, $c = ?$

By using Pythagoras theorem:

$c^2 = a^2 + b^2$

$c^2 = (\sqrt{2})^2 + (\sqrt{3})^2$

$c^2 = 2 + 3$

$c^2 = 5$

$c = \sqrt{5}$

$a = \sqrt{2}$, $b = \sqrt{3}$, $c = \sqrt{5}$

$\sin\theta = \frac{a}{c} = \sqrt{\frac{2}{5}}$, $\operatorname{cosec}\theta = \frac{c}{a} = \sqrt{\frac{5}{2}}$

$\cos\theta = \frac{b}{c} = \sqrt{\frac{3}{5}}$, $\sec\theta = \frac{c}{b} = \sqrt{\frac{5}{3}}$

$\tan\theta = \frac{a}{b} = \sqrt{\frac{2}{3}}$, $\cot\theta = \frac{b}{a} = \sqrt{\frac{3}{2}}$

Prove the following trigonometric identities:

Question No. 2

$(\sin\theta + \cos\theta)^2 = 1 + 2\sin\theta\cos\theta$

Solution:

$(\sin\theta + \cos\theta)^2 = 1 + 2\sin\theta\cos\theta$

Using formula: $(a + b)^2 = a^2 + 2ab + b^2$

$(\sin\theta)^2 + 2\sin\theta\cos\theta + (\cos\theta)^2 = 1 + 2\sin\theta\cos\theta$

$\sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta = 1 + 2\sin\theta\cos\theta$

as: $\sin^2\theta + \cos^2\theta = 1$

so, we can write:

$1 + 2\sin\theta\cos\theta = 1 + 2\sin\theta\cos\theta$

Hence proved given trigonometric identity.

Question No. 3

$\frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta}$

Solution:

$\frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta}$

As: $\frac{\sin\theta}{\cos\theta} = \tan\theta$

And its reciprocal is:

$\frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta}$

Hence proved given trigonometric identity.

Question No. 4

$\frac{\sin\theta}{\operatorname{cosec}\theta} + \frac{\cos\theta}{\sec\theta} = 1$

Solution:

$\frac{\sin\theta}{\operatorname{cosec}\theta} + \frac{\cos\theta}{\sec\theta} = 1$

As: $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$, and $\sec\theta = \frac{1}{\cos\theta}$

$\frac{\sin\theta}{\frac{1}{\sin\theta}} + \frac{\cos\theta}{\frac{1}{\cos\theta}} = 1$

$\sin^2\theta + \cos^2\theta = 1$

$1 = 1$

Hence proved given trigonometric identity.

Question No. 5

$\cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$

Solution:

$\cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$

as: $\sin^2\theta + \cos^2\theta = 1$

$\sin^2\theta = 1 - \cos^2\theta$

$\cos^2\theta - (1 - \cos^2\theta) = 2\cos^2\theta - 1$

$\cos^2\theta - 1 + \cos^2\theta = 2\cos^2\theta - 1$

$2\cos^2\theta - 1 = 2\cos^2\theta - 1$

Hence proved given trigonometric identity.

Question No. 6

$\mathbf{\cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta}$

Solution:

$\cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$

as: $\sin^2\theta + \cos^2\theta = 1$

$\cos^2\theta = 1 - \sin^2\theta$

$1 - \sin^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$

$1 - 2\sin^2\theta = 1 - 2\sin^2\theta$

Hence proved given trigonometric identity.

Question No. 7

$$\frac{1-\sin\theta}{\cos\theta} = \frac{\cos\theta}{1+\sin\theta}$$

Solution:

$$\frac{1-\sin\theta}{\cos\theta} = \frac{\cos\theta}{1+\sin\theta}$$

Multiply & Divide L.H.S by $(\mathbf{1 + \sin\theta})$:

$$\frac{1-\sin\theta}{\cos\theta} \times \frac{1+\sin\theta}{1+\sin\theta} = \frac{\cos\theta}{1+\sin\theta}$$

$$\frac{1-\sin^2\theta}{\cos\theta(1+\sin\theta)} = \frac{\cos\theta}{1+\sin\theta}$$

As: $1 - \sin^2\theta = \cos^2\theta$, So;

$$\frac{\cos^2\theta}{\cos\theta(1+\sin\theta)} = \frac{\cos\theta}{1+\sin\theta}$$

$$\frac{\cos\theta}{\cos\theta} \times \frac{\cos\theta}{1+\sin\theta} = \frac{\cos\theta}{1+\sin\theta}$$

$$\frac{\cos\theta}{1+\sin\theta} = \frac{\cos\theta}{1+\sin\theta}$$

Hence proved given trigonometric identity.

Question No. 8

$$\mathbf{(\sec\theta - \tan\theta)^2 = \frac{1-\sin\theta}{1+\sin\theta}}$$

Solution:

$$(\sec\theta - \tan\theta)^2 = \frac{1-\sin\theta}{1+\sin\theta}$$

As: $\sec\theta = \frac{1}{\cos\theta}$ and $\tan\theta = \frac{\sin\theta}{\cos\theta}$;

$$\left(\frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta}\right)^2 = \frac{1-\sin\theta}{1+\sin\theta}$$

$$\left(\frac{1-\sin\theta}{\cos\theta}\right)^2 = \frac{1-\sin\theta}{1+\sin\theta}$$

As: $1 - \sin^2\theta = \cos^2\theta$, So;

$$\frac{(1-\sin\theta)(1-\sin\theta)}{1-\sin^2\theta}=\frac{1-\sin\theta}{1+\sin\theta}$$

$$\frac{(1-\sin\theta)(1-\sin\theta)}{(1-\sin\theta)(1+\sin\theta)}=\frac{1-\sin\theta}{1+\sin\theta}$$

$$\frac{(1-\sin\theta)}{(1+\sin\theta)}=\frac{1-\sin\theta}{1+\sin\theta}$$

Hence proved given trigonometric identity.

Question No. 9

$(\tan\theta + \cot\theta)^2 = \sec^2\theta + \csc^2\theta$

Solution:

$(\tan\theta + \cot\theta)^2 = \sec^2\theta + \csc^2\theta$

$$\left(\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}\right)^2 = \sec^2\theta + \csc^2\theta$$

$$\left(\frac{\sin^2\theta+\cos^2\theta}{\cos\theta.\sin\theta}\right)^2 = \sec^2\theta + \csc^2\theta$$

As: $\sin^2\theta + \cos^2\theta = 1$, So;

$$\left(\frac{1}{\cos\theta.\sin\theta}\right)^2 = \sec^2\theta + \csc^2\theta$$

$$\frac{1}{\cos^2\theta}\cdot\frac{1}{\sin^2\theta} = \sec^2\theta + \csc^2\theta$$

$\sec^2\theta + \csc^2\theta = \sec^2\theta + \csc^2\theta$

Hence proved given trigonometric identity.

Question No. 10

$$\boldsymbol{\frac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1} = \tan\theta + \sec\theta}$$

Solution:

$$\frac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1} = \tan\theta + \sec\theta$$

As: $\sec^2\theta - \tan^2\theta = 1$, So;

$$\frac{\tan\theta + \sec\theta - (\sec^2\theta - \tan^2\theta)}{\tan\theta - \sec\theta + 1} = \tan\theta + \sec\theta$$

$$\frac{\tan\theta + \sec\theta - \sec^2\theta + \tan^2\theta}{\tan\theta - \sec\theta + 1} = \tan\theta + \sec\theta$$

Re-arranged:

$$\frac{\tan\theta + \sec\theta + \tan^2\theta - \sec^2\theta}{\tan\theta - \sec\theta + 1} = \tan\theta + \sec\theta$$

Using formula where $a^2 - b^2 = (a + b)(a - b)$

$$\frac{\tan\theta + \sec\theta + (\tan\theta + \sec\theta)(\tan\theta - \sec\theta)}{\tan\theta - \sec\theta + 1} = \tan\theta + \sec\theta$$

$$\frac{(\tan\theta + \sec\theta)(1 + \tan\theta - \sec\theta)}{\tan\theta - \sec\theta + 1} = \tan\theta + \sec\theta$$

$$\frac{(\tan\theta + \sec\theta)(1 + \tan\theta - \sec\theta)}{1 + \tan\theta - \sec\theta} = \tan\theta + \sec\theta$$

$\tan\theta + \sec\theta = \tan\theta + \sec\theta$

Hence proved given trigonometric identity.

Question No. 11

$\sin^3\theta - \cos^3\theta = (\sin\theta - \cos\theta)(1 + \sin\theta\cos\theta)$

Solution:

$\sin^3\theta - \cos^3\theta = (\sin\theta - \cos\theta)(1 + \sin\theta\cos\theta)$

Using Formula:

$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

$(\sin\theta - \cos\theta)(\sin^2\theta + \sin\theta\cos\theta + \cos^2\theta) = (\sin\theta - \cos\theta)(1 + \sin\theta\cos\theta)$

As: $\sin^2\theta + \cos^2\theta = 1$, So;

$(\sin\theta - \cos\theta)(1 + \sin\theta\cos\theta) = (\sin\theta - \cos\theta)(1 + \sin\theta\cos\theta)$

Hence proved given trigonometric identity.

Question No. 12

$\sin^6\theta - \cos^6\theta = (\sin^2\theta - \cos^2\theta)(1 - \sin^2\theta\cos^2\theta)$

Solution:

$\sin^6\theta - \cos^6\theta = (\sin^2\theta - \cos^2\theta)(1 - \sin^2\theta\cos^2\theta)$

By solving L.H.S:

$\sin^6\theta - \cos^6\theta$

$= (\sin^3\theta)^2 - (\cos^3\theta)^2$

As: $a^2 - b^2 = (a + b)(a - b)$, So;

$= [\sin^3\theta - \cos^3\theta][\sin^3\theta - \cos^3\theta]$

As: $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$, So;

$= [(\sin\theta - \cos\theta)(\sin^2\theta + \sin\theta\cos\theta + \cos^2\theta)]\ [(\sin\theta + \cos\theta)(\sin^2\theta - \sin\theta\cos\theta + \cos^2\theta)]$

Re-arranging:

$= (\sin\theta - \cos\theta)(\sin\theta + \cos\theta)(\sin^2\theta + \cos^2\theta + \sin\theta\cos\theta)\ (\sin^2\theta + \cos^2\theta - \sin\theta\cos\theta)$

As: $\sin^2\theta + \cos^2\theta = 1$

$= (\sin^2\theta - \cos^2\theta)(1 + \sin\theta\cos\theta)\ (1 - \sin\theta\cos\theta)$

$= (\sin^2\theta - \cos^2\theta)[(1)^2 - (\sin\theta\cos\theta)^2]$

$= (\sin^2\theta - \cos^2\theta)(1 - \sin^2\theta\cos^2\theta)$

$=$ R.H.S

Hence proved given trigonometric identity.

$\sin^6\theta - \cos^6\theta = (\sin^2\theta - \cos^2\theta)(1 - \sin^2\theta\cos^2\theta)$