

Unit No. 4

Factorization and Algebraic Manipulation

Exercise No. 4.2

Question No. 1

Factorize each of the following expressions:

(i). $4x^4 + 81y^4$

Solution:

$$\begin{aligned} 4x^4 + 81y^4 \\ = (2x^2)^2 + (9y^2)^2 \end{aligned}$$

By adding and subtracting $2(2x^2)(9y^2)$ for completing square:

$$\begin{aligned} &= [(2x^2)^2 + (9y^2)^2 + 2(2x^2)(9y^2)] - 2(2x^2)(9y^2) \\ &= (2x^2 + 9y^2)^2 - 36x^2y^2 \\ &= (2x^2 + 9y^2)^2 - (6xy)^2 \end{aligned}$$

By using formula: $a^2 - b^2 = (a + b)(a - b)$

$$= (2x^2 + 9y^2 + 6xy)(2x^2 + 9y^2 - 6xy)$$

By arranging:

$$= (2x^2 + 6xy + 9y^2)(2x^2 - 6xy + 9y^2)$$

(ii). $a^4 + 64b^4$

Solution:

$$\begin{aligned} a^4 + 64b^4 \\ = (a^2)^2 + (8b^2)^2 \end{aligned}$$

By adding and subtracting $2(a^2)(8b^2)$ for completing square:

$$\begin{aligned} &= [(a^2)^2 + (8b^2)^2 + 2(a^2)(8b^2)] - 2(a^2)(8b^2) \\ &= (a^2 + 8b^2)^2 - 16a^2b^2 \\ &= (a^2 + 8b^2)^2 - (4ab)^2 \end{aligned}$$

By using formula: $a^2 - b^2 = (a + b)(a - b)$

$$= (a^2 + 8b^2 + 4ab)(a^2 + 8b^2 - 4ab)$$

By arranging:

$$= (a^2 + 4ab + 8b^2)(a^2 - 4ab + 8b^2)$$

(iii). $x^4 + 4x^2 + 16$

Solution:

$$\begin{aligned} x^4 + 4x^2 + 16 \\ = (x^2)^2 + (4)^2 + 4x^2 \end{aligned}$$

By adding and subtracting $2(x^2)(4)$ for completing square:

$$\begin{aligned}
 &= [(x^2)^2 + (4)^2 + 2(x^2)(4)] - 2(x^2)(4) + 4x^2 \\
 &= (x^2 + 4)^2 - 8x^2 + 4x^2 \\
 &= (x^2 + 4)^2 - 4x^2 \\
 &= (x^2 + 4)^2 - (2x)^2
 \end{aligned}$$

By using formula: $a^2 - b^2 = (a + b)(a - b)$

$$= (x^2 + 4 + 2x)(x^2 + 4 - 2x)$$

By arranging:

$$= (x^2 + 2x + 4)(x^2 - 2x + 4)$$

(iv). $x^4 - 14x^2 + 1$

Solution:

$$\begin{aligned}
 &x^4 - 14x^2 + 1 \\
 &= x^4 + 1 - 14x^2 \\
 &= (x^2)^2 + (1)^2 - 14x^2
 \end{aligned}$$

By adding and subtracting $2(x^2)(1)$ for completing square:

$$\begin{aligned}
 &= (x^2)^2 + (1)^2 + 2(x^2)(1) - 2(x^2)(1) - 14x^2 \\
 &= (x^2 + 1)^2 - 2x^2 - 14x^2 \\
 &= (x^2 + 1)^2 - 16x^2 \\
 &= (x^2 + 1)^2 - (4x)^2
 \end{aligned}$$

By using formula: $a^2 - b^2 = (a + b)(a - b)$

$$= (x^2 + 1 + 4x)(x^2 + 1 - 4x)$$

By arranging:

$$= (x^2 + 4x + 1)(x^2 - 4x + 1)$$

(v). $x^4 - 30x^2y^2 + 9y^4$

Solution:

$$\begin{aligned}
 &x^4 - 30x^2y^2 + 9y^4 \\
 &= x^4 + 9y^4 - 30x^2y^2 \\
 &= (x^2)^2 + (3y^2)^2 - 30x^2y^2
 \end{aligned}$$

By adding and subtracting $2(x^2)(3y^2)$ for completing square:

$$\begin{aligned}
 &= (x^2)^2 + (3y^2)^2 + 2(x^2)(3y^2) - 2(x^2)(3y^2) - 30x^2y^2 \\
 &= (x^2 + 3y^2)^2 - 6x^2y^2 - 30x^2y^2 \\
 &= (x^2 + 3y^2)^2 - 36x^2y^2 \\
 &= (x^2 + 3y^2)^2 - (6xy)^2
 \end{aligned}$$

By using formula: $a^2 - b^2 = (a + b)(a - b)$

$$= (x^2 + 3y^2 + 6xy)(x^2 + 3y^2 - 6xy)$$

By arranging:

$$= (x^2 + 6xy + 3y^2)(x^2 - 6xy + 3y^2)$$

(vi). $x^4 - 11x^2y^2 + y^4$ (Wrong value in book)

Solution:

$$\begin{aligned} & x^4 - 11x^2y^2 + y^4 \\ &= x^4 + y^4 - 11x^2y^2 \\ &= (x^2)^2 + (y^2)^2 - 11x^2y^2 \end{aligned}$$

By adding and subtracting $2(x^2)(y^2)$ for completing square:

$$\begin{aligned} &= (x^2)^2 + (y^2)^2 - 2(x^2)(y^2) + 2(x^2)(y^2) - 11x^2y^2 \\ &= (x^2 - y^2)^2 + 2x^2y^2 - 11x^2y^2 \\ &= (x^2 - y^2)^2 - 9x^2y^2 \\ &= (x^2 - y^2)^2 - (3xy)^2 \end{aligned}$$

By using formula: $a^2 - b^2 = (a + b)(a - b)$

$$= (x^2 - y^2 + 3xy)(x^2 - y^2 - 3xy)$$

By arranging:

$$= (x^2 + 3xy - y^2)(x^2 - 3xy - y^2)$$

Question No. 2

Factorize each of the following expressions:

(i). $(x + 1)(x + 2)(x + 3)(x + 4) + 1$

Solution:

$$(x + 1)(x + 2)(x + 3)(x + 4) + 1$$

By arranging:

$$\begin{aligned} &= (x + 1)(x + 4)(x + 2)(x + 3) + 1 \\ &= (x^2 + 4x + x + 4)(x^2 + 3x + 2x + 6) + 1 \\ &= (x^2 + 5x + 4)(x^2 + 5x + 6) + 1 \end{aligned}$$

Let: $x^2 + 5x = y$

$$\begin{aligned} &= (y + 4)(y + 6) + 1 \\ &= y^2 + 6y + 4y + 24 + 1 \\ &= y^2 + 10y + 25 \\ &= y^2 + 5y + 5y + 25 \\ &= y(y + 5) + 5(y + 5) \\ &= (y + 5)(y + 5) \\ &= (y + 5)^2 \end{aligned}$$

Recal “y”:

$$= (x^2 + 5x + 5)^2$$

(ii). $(x + 2)(x - 7)(x - 4)(x - 1) + 17$

Solution:

$$\begin{aligned} & (x + 2)(x - 7)(x - 4)(x - 1) + 17 \\ &= (x^2 - 7x + 2x - 14)(x^2 - x - 4x + 4) + 17 \\ &= (x^2 - 5x - 14)(x^2 - 5x + 4) + 17 \end{aligned}$$

Let: $x^2 - 5x = y$

$$\begin{aligned} &= (y - 14)(y + 4) + 17 \\ &= y^2 + 4y - 14y - 56 + 17 \\ &= y^2 - 10y - 39 \\ &= y^2 + 3y - 13y - 39 \\ &= y(y + 3) - 13(y + 3) \\ &= (y + 3)(y - 13) \end{aligned}$$

Recal “y”:

$$= (x^2 - 5x + 3)(x^2 - 5x - 13)$$

(iii). $(2x^2 + 7x + 3)(2x^2 + 7x + 5) + 1$

Solution:

$$\begin{aligned} & (2x^2 + 7x + 3)(2x^2 + 7x + 5) + 1 \\ \text{Let: } 2x^2 + 7x &= y \\ &= (y + 3)(y + 5) + 1 \\ &= y^2 + 5y + 3y + 15 + 1 \\ &= y^2 + 8y + 16 \\ &= y^2 + 4y + 4y + 16 \\ &= y(y + 4) + 4(y + 4) \\ &= (y + 4)(y + 4) \\ &= (y + 4)^2 \end{aligned}$$

Recal “y”:

$$= (2x^2 + 7x + 4)^2$$

(iv). $(3x^2 + 5x + 3)(3x^2 + 5x + 5) - 3$

Solution:

$$\begin{aligned} & (3x^2 + 5x + 3)(3x^2 + 5x + 5) - 3 \\ \text{Let: } 3x^2 + 5x &= y \\ &= (y + 3)(y + 5) - 3 \end{aligned}$$

$$= y^2 + 5y + 3y + 15 - 3$$

$$= y^2 + 8y + 12$$

$$= y^2 + 2y + 6y + 12$$

$$= y(y + 2) + 6(y + 2)$$

$$= (y + 2)(y + 6)$$

Recal “y”:

$$= (3x^2 + 5x + 2)(3x^2 + 5x + 6)$$

(v). $(x + 1)(x + 2)(x + 3)(x + 6) - 3x^2$

Solution:

$$(x + 1)(x + 2)(x + 3)(x + 6) - 3x^2$$

By arranging:

$$= (x + 1)(x + 6)(x + 2)(x + 3) - 3x^2$$

$$= (x^2 + 6x + x + 6)(x^2 + 3x + 2x + 6) - 3x^2$$

$$= (x^2 + 7x + 6)(x^2 + 5x + 6) - 3x^2$$

Let: $x^2 + 6 = y$

$$= (y + 7x)(y + 5x) - 3x^2$$

$$= y^2 + 5xy + 7xy + 35x^2 - 3x^2$$

$$= y^2 + 12xy + 32x^2$$

$$= y^2 + 4xy + 8xy + 32x^2$$

$$= y(y + 4x) + 8x(y + 4x)$$

$$= (y + 4x)(y + 8x)$$

Recal “y”:

$$= (x^2 + 6 + 4x)(x^2 + 6 + 8x)$$

Re-arranging:

$$= (x^2 + 4x + 6)(x^2 + 8x + 6)$$

(vi). $(x + 1)(x - 1)(x + 2)(x - 2) + 13x^2$

Solution:

$$(x + 1)(x - 1)(x + 2)(x - 2) + 13x^2$$

By arranging:

$$= [(x + 1)(x + 2)][(x - 1)(x - 2)] + 13x^2$$

$$= (x^2 + 2x + x + 2)(x^2 - 2x - x + 2) + 13x^2$$

$$= (x^2 + 3x + 2)(x^2 - 3x + 2) + 13x^2$$

$$= (x^2 + 2 + 3x)(x^2 + 2 - 3x) + 13x^2$$

Let: $x^2 + 2 = y$

$$\begin{aligned}
 &= (y + 3x)(y - 3x) + 13x^2 \\
 &= y^2 - 3xy + 3xy - 9x^2 + 13x^2 \\
 &= y^2 + 4x^2
 \end{aligned}$$

Recal “y”:

$$\begin{aligned}
 &= (x^2 + 2)^2 + 4x^2 \\
 &= (x^2)^2 + 2(x^2)(2) + (2)^2 + 4x^2 \\
 &= x^4 + 4x^2 + 4 + 4x^2 \\
 &= x^4 + 8x^2 + 4
 \end{aligned}$$

(Wrong value in book or wrong answer)

Question No. 3

Factorize:

(i). $8x^3 + 12x^2 + 6x + 1$

Solution:

$$\begin{aligned}
 &8x^3 + 12x^2 + 6x + 1 \\
 &= (2x)^3 + 3(2x)^2(1) + 3(2x)(1)^2 + (1)^3
 \end{aligned}$$

By using formula:

$$\begin{aligned}
 a^3 + 3a^2b + 3ab^2 + b^3 &= (a + b)^3 \\
 &= (2x + 1)^3
 \end{aligned}$$

(ii). $27a^3 + 108a^2b + 144ab^2 + 64b^3$

Solution:

$$\begin{aligned}
 &27a^3 + 108a^2b + 144ab^2 + 64b^3 \\
 &= (3a)^3 + 3(3a)^2(4b) + 3(3a)(4b)^2 + (4b)^3
 \end{aligned}$$

By using formula:

$$\begin{aligned}
 a^3 + 3a^2b + 3ab^2 + b^3 &= (a + b)^3 \\
 &= (3a + 4b)^3
 \end{aligned}$$

(iii). $x^3 + 48x^2y + 108xy^2 + 216y^3$

Solution:

$$\begin{aligned}
 &x^3 + 48x^2y + 108xy^2 + 216y^3 \\
 &= (x)^3 + 3(x)^2(6y) + 3(x)(6y)^2 + (6y)^3
 \end{aligned}$$

By using formula:

$$\begin{aligned}
 a^3 + 3a^2b + 3ab^2 + b^3 &= (a + b)^3 \\
 &= (x + 6y)^3
 \end{aligned}$$

(iv). $8x^3 - 125y^3 + 150xy^2 - 60x^2y$

Solution:

$$8x^3 - 125y^3 + 150xy^2 - 60x^2y$$

By arranging:

$$\begin{aligned} & 8x^3 - 60x^2y + 150xy^2 - 125y^3 \\ &= (2x)^3 - 3(2x)^2(5y) + 3(2x)(5y)^2 - (5y)^3 \end{aligned}$$

By using formula:

$$\begin{aligned} & a^3 - 3a^2b + 3ab^2 - b^3 = (a - b)^3 \\ &= (2x - 5y)^3 \end{aligned}$$

Question No. 4

Factorize:

(i). $125a^3 - 1$

Solution:

$$\begin{aligned} & 125a^3 - 1 \\ &= (5a)^3 - (1)^3 \end{aligned}$$

By using formula:

$$\begin{aligned} & a^3 - b^3 = (a - b)(a^2 + ab + b^2) \\ &= (5a - 1)[(5a)^2 + (5a)(1) + (1)^2] \\ &= (5a - 1)(25a^2 + 5a + 1) \end{aligned}$$

(ii). $64x^3 + 125$

Solution:

$$\begin{aligned} & 64x^3 + 125 \\ &= (4x)^3 + (5)^3 \end{aligned}$$

By using formula:

$$\begin{aligned} & a^3 + b^3 = (a + b)(a^2 - ab + b^2) \\ &= (4x + 5)[(4x)^2 - (4x)(5) + (5)^2] \\ &= (4x + 5)(16x^2 - 20x + 25) \end{aligned}$$

(iii). $x^6 - 27$

Solution:

$$\begin{aligned} & x^6 - 27 \\ &= (x^2)^3 - (3)^3 \end{aligned}$$

By using formula:

$$\begin{aligned} & a^3 - b^3 = (a - b)(a^2 + ab + b^2) \\ &= (x^2 - 3)[(x^2)^2 + (x^2)(3) + (3)^2] \\ &= (x^2 - 3)(x^4 + 3x^2 + 9) \end{aligned}$$

(iv). $1000a^3 + 1$

Solution:

$$\begin{aligned} &1000a^3 + 1 \\ &= (10a)^3 + (1)^3 \end{aligned}$$

By using formula:

$$\begin{aligned} a^3 + b^3 &= (a + b)(a^2 - ab + b^2) \\ &= (10a + 1)[(10a)^2 - (10a)(1) + (1)^2] \\ &= (10a + 1)(100a^2 - 10a + 1) \end{aligned}$$

(v). $343x^3 + 216$

Solution:

$$\begin{aligned} &343x^3 + 216 \\ &= (7x)^3 + (6)^3 \end{aligned}$$

By using formula:

$$\begin{aligned} a^3 + b^3 &= (a + b)(a^2 - ab + b^2) \\ &= (7x + 6)[(7x)^2 - (7x)(6) + (6)^2] \\ &= (7x + 6)(49x^2 - 42x + 36) \end{aligned}$$

(vi). $27 - 512y^3$

Solution:

$$\begin{aligned} &27 - 512y^3 \\ &= (3)^3 - (8y)^3 \end{aligned}$$

By using formula:

$$\begin{aligned} a^3 - b^3 &= (a - b)(a^2 + ab + b^2) \\ &= (3 - 8y)[(3)^2 + (3)(8y) + (8y)^2] \\ &= (3 - 8y)(9 + 24y + 64y^2) \end{aligned}$$