Unit No. 8

Logic

Review Exercise No. 8

Question No. 1

Four options are given against each statement. Encircle the correct option.

(i). Which of the following expressions is often related to inductive reasoning?

(a) based on repeated experiments

(b) if and only if statements

(c) Statement is proven by a theorem

(d) based on general principles

(ii) Which of the following sentences describe deductive reasoning?

(a) general conclusions from a limited number of observations

(b) based on repeated experiments

(c) based on units of information that are accurate

(d) draw conclusion from well-known facts

(iii) Which one of the following statements is true?

(a) The set of integers is finite.

(b) The sum of the interior angles of any quadrilateral is always 180°.

(c) $22/7 \notin Q$

(d) All isosceles triangles are equilateral triangles.

(iv) Which of the following statements is the best to represent the negation of the statement "The stove is burning"?

(a) the stove is not burning.

(b) the stove is dim.

(c) the stove is turned to low heat.

(d) it is both burning and not burning.

(v) The conjunction of two statements p and q is true when:

(a) both p and q are false.

(b) both p and q are true.

(c) only q is true.

(d) only p is true.

(vi) A conditional is regarded as false only when:

(a) antecedent is true and consequent is false.

(b) consequent is true and antecedent is false.

(c) antecedent is true only.

(d) consequent is false only.

(vii) Contrapositive of $p \Rightarrow q$ is:

(a) $q \Rightarrow p$

(b) $\sim q \Rightarrow p$

(c) $\sim p \Rightarrow \sim q$

(d) $\sim q \Rightarrow \sim p$

(viii) The statement "Every integer greater than 2 is a sum of two prime numbers" is:

(a) theorem

(b) conjecture

(c) axiom

(d) postulates

(ix) The statement "A straight line can be drawn between any two points" is:

(a) theorem

(b) conjecture

(c) axiom

(d) logic

(x) The statement "The sum of the interior angle of a triangle is 180°" is:

(a) converse

(b) theorem

(c) axiom

(d) conditional

Question No. 2

Write the converse, inverse and contrapositive of the following conditionals:

(i) $p \Rightarrow q$

Solution:

- Converse: $q \rightarrow \sim p$
- Inverse: $p \rightarrow \sim q$
- Contrapositive: $\sim q \rightarrow p$

(ii) $q \Rightarrow p$

Solution:

- Converse: $p \rightarrow q$
- Inverse: $\sim q \rightarrow \neg p$
- Contrapositive: $\sim p \rightarrow \sim q$

(iii) $\sim p \Rightarrow q$

Solution:

- Converse: $\sim q \rightarrow \sim p$
- Inverse: $p \rightarrow q$
- Contrapositive: $q \rightarrow p$

(iv) $\sim q \Rightarrow p$

Solution:

- Converse: $\sim p \rightarrow \sim q$
- Inverse: $q \rightarrow p$
- Contrapositive: $p \rightarrow q$

Question No. 3

Write the truth table of the following:

(i) $\sim(p \vee q) \vee (\sim q)$

Solution:

p	q	$p \vee q$	$\sim(p \vee q)$	$\sim q$	$\sim(p \vee q) \vee (\sim q)$
T	T	T	F	F	F
T	F	T	F	T	T
F	T	T	F	F	F
F	F	F	T	T	T

(ii) ~(q∨~p)

Solution:

p	q	$\sim p$	$\sim q$	$\sim q \vee \sim p$	$\sim(q \vee \sim p)$
T	T	F	F	F	T
T	F	F	T	T	F
F	T	T	F	T	F
F	F	T	T	T	F

(iii) (p∨q)⟺(p∧q)

Solution:

p	q	$p \vee q$	$p \wedge q$	$(p \vee q) \leftrightarrow (p \wedge q)$
T	T	T	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	F	T

Question No. 4

Differentiate between a mathematical statement and its proof. Given two examples.

Solution:

Mathematical Statement:

A sentence or mathematical expression which may be true or false but not both is called a statement.

This is correct so far as mathematics and other sciences are concerned. For instance, the statement a = b can be either true or false.

Proof: A logical argument that demonstrates the truth of a mathematical statement. It uses definitions, axioms, previously proven theorems, and logical rules to establish the validity of the statement.

Example 1: "The sum of the interior angles of a triangle is 180 degrees."

Proof of Triangle Angle Sum: Using geometric constructions and properties of parallel lines.

Example 2: "For any real numbers a and b, a + b = b + a."

Proof of Commutative Property of Addition: Using diagrams or algebraic manipulations.

Question No. 5

What is the difference between an axiom and a theorem? Give examples of each.

Solution:

Axiom:

A fundamental assumption or self-evident truth that is accepted without proof. Axioms form the basis of a mathematical system.

Example 1: Euclid's Postulates (e.g., "A straight line can be drawn between any two points.")

Example 2: Peano Axioms (e.g., "Every natural number has a successor.")

Theorem:

A mathematical statement that has been proven to be true using logical reasoning and previously established facts (axioms, definitions, other proven theorems).

Example 1: Pythagorean Theorem

Example 2: Fundamental Theorem of Arithmetic

Question No. 6

What is the importance of logical reasoning in mathematical proofs? Give an example to illustrate your point.

Solution:

Importance: Logical reasoning is essential for constructing valid mathematical proofs. It ensures that each step in the proof follows logically from previous statements and established principles.

Example: In proving the Pythagorean Theorem, we use logical reasoning, geometric constructions, and previously established properties of similar triangles.

Question No. 7

Indicate whether it is an axiom, conjecture or theorem and explain your reasoning.

(i) There is exactly one straight line through any two points.

Solution:

Axiom:

This is a fundamental assumption in geometry, often considered a postulate or axiom. It is accepted without proof.

(ii) Every even number greater than 2 can be written as the sum of two prime numbers."

Solution:

Conjecture:

This is the Goldbach Conjecture, which is a famous unproven statement in number theory. It is believed to be true but has not yet been proven mathematically.

(iii) The sum of the angles in a triangle is 180°.

Solution:

Theorem:

This statement has been rigorously proven in Euclidean geometry. It is not an axiom because it requires a proof. It is not a conjecture because it has been proven true.

Question No. 8

Formulate simple deductive proofs for each of the following algebraic expressions, prove that the L.H.S is equal to the R.H.S:

(i) prove that $(\boldsymbol{x-4})^2+\mathbf{9}=\boldsymbol{x}^2-\mathbf{8}\boldsymbol{x}+\mathbf{25}$

Solution:

$(x-4)^2+9= x^2-8x+25$

Solving L.H.S:

$(x-4)^2+9$

$=(x)^2-2\,(x)(4)+\,(4)^2+9$

$=x^2-8x+\,16+9$

$=x^2-8x+\,25=\,R.H.S$

Hence proved that:

$(x-4)^2+9= x^2-8x+25$

(ii) prove that $(\boldsymbol{x+1})^2-\,(\boldsymbol{x-1})^2=\,\mathbf{4}\boldsymbol{x}$

Solution:

$(x+1)^2-\,(x-1)^2=\,4x$

Solving L.H.S:

$(x+1)^2-\,(x-1)^2$

$=[(x)^2+2\,(x)(1)+\,(1)^2]-[(x)^2-2\,(x)(1)+\,(1)^2]$

$= (x^2 + 2x + 1) - (x^2 - 2x + 1)$

$= x^2 + 2x + 1 - x^2 + 2x - 1$

$= 4x = R.H.S$

Hence proved that:

$$(x+1)^2 - (x-1)^2 = 4x$$

(iii) prove that $(x+5)^2 - (x-5)^2 = 20x$

Solution:

$(x+5)^2 - (x-5)^2 = 20x$

Solving L.H.S:

$(x+5)^2 - (x-5)^2$

$= [(x)^2 + 2\,(x)(5) + (5)^2] - [(x)^2 - 2\,(x)(5) + (5)^2]$

$= (x^2 + 10x + 25) - (x^2 - 10x + 25)$

$= x^2 + 10x + 25 - x^2 + 10x - 25$

$= 20x = R.H.S$

Hence proved that:

$$(x+5)^2 - (x-5)^2 = 20x$$

Question No. 9

Prove the following by justifying each step:

(i) $\frac{4+16x}{4} = 1 + 4x$

Solution:

$$\frac{4+16x}{4} = 1 + 4x$$

By solving L.H.S:

$$\frac{4+16x}{4}$$

$= \frac{4}{4} + \frac{16x}{4}$ (∵ Distributive law)

$= 1 + 4x$ (∵ Cancellation property of real numbers)

Hence proved that:

$\frac{4+16x}{4} = 1 + 4x$

(ii) $\frac{6x^2+18x}{3x^2-27} = \frac{2x}{x-3}$

Solution:

$\frac{6x^2+18x}{3x^2-27} = \frac{2x}{x-3}$

By solving L.H.S:

$$\frac{6x^2 + 18x}{3x^2 - 27}$$

$= \frac{6x(x+3)}{3(x^2-9)}$ (∵ factorization of numerator and denominator)

$= \frac{2x(x+3)}{(x^2-3^2)}$ (∵ Cancellation property of real numbers and factorization of denominator)

$= \frac{2x}{(x-3)}$ (∵ Cancellation property of real numbers)

Hence proved that:

$\frac{6x^2+18x}{3x^2-27} = \frac{2x}{x-3}$

(iii) $\frac{x^2+7x+10}{x^2-3x-10} = \frac{x+5}{x-5}$

Solution:

$\frac{x^2+7x+10}{x^2-3x-10} = \frac{x+5}{x-5}$

By solving L.H.S:

$$\frac{x^2 + 7x + 10}{x^2 - 3x - 10}$$

$= \frac{x^2+5x+2x+10}{x^2-5x+2x-10}$ (∵ factorization of numerator and denominator)

$= \frac{x(x+5)+2(x+5)}{x(x-5)+2(x-5)}$ (∵ taking common)

$= \frac{(x+5)(x+2)}{(x-5)(x+2)}$

$= \frac{(x+5)}{(x-5)}$ (∵ Cancellation property of real numbers)

Hence proved that:

$\frac{x^2+7x+10}{x^2-3x-10} = \frac{x+5}{x-5}$

Question No. 10

Suppose x is an integer. If x is odd, then 9x + 4 is odd.

Solution:

Part 1: If x is odd, then 9x + 4 is odd.

If x is odd, we can express it as;

$x = 2k + 1$, where k is an integer. Substitute x into the expression $9x + 4$:

$9(2k + 1) + 4$

$= 18k + 9 + 4$

$= 18k + 13$

$= 2(9k + 6) + 1$

Since $9k + 6$ is an integer, $2(9k + 6) + 1$ is an odd integer.

Part 2: If 9x + 4 is odd, then x is odd.

Solution:

If $9x + 4$ is odd, we can express it as;

$9x + 4 = 2m + 1$, where m is an integer.

$9x = 2m – 4 + 1$

$9x = 2m – 3$

$$x = \frac{(2m - 3)}{9}$$

Since $2m - 3$ is always odd, and 9 is odd,

$(2m - 3) / 9$ will be an odd integer. Therefore, x is odd if and only if $9x + 4$ is odd.

Question No. 11

Suppose x is an integer. If x is odd, then 7x + 5 is even.

Solution:

If x is an odd integer, we can represent it as;

$x = 2k + 1$, where k is any integer.

Substitute this value of x into the expression $7x + 5$:

$= 7(2k + 1) + 5$

$= 14k + 7 + 5$

= 14k + 12

= 2(7k + 6)

Since k is an integer, 7k + 6 is also an integer.

Therefore, 7x + 5 can be expressed in the form;

2m, where m = 7k + 6 is an integer.

Hence, if x is an odd integer, then 7x + 5 is even.

Question No. 12

Prove the following statements:

(a) If x is an odd integer, then show that $x^2 - 4x + 6$ is odd.

Solution:

Proof:

If x is an odd integer,

we can represent it as x = 2k + 1, where k is any integer.

Substitute this value of x into the expression $x^2 - 4x + 6$:

$= (2k + 1)^2 - 4(2k + 1) + 6$

$= (2k)^2 + 2(2k)(1) + (1)^2 - 8k - 4 + 6$

$= 4k^2 + 4k + 1 - 8k + 2$

$= 4k^2 - 4k + 3$

$= 4k(k - 1) + 3$

Since k and k - 1 are consecutive integers, one of them must be even.

Therefore, 4k(k − 1) is always divisible by 2. Thus, 4k(k − 1) is divisible by 4 × 2 = 8.

So, 4k(k − 1) can be expressed in the form 2m + 1, where m is an integer.

Hence, if x is an odd integer, then

$x^2 - 4x + 6$ is odd.

(b) If x is an even integer, then show that $x^2 + 2x + 4$ is even.

Solution:

Proof:

If x is an even integer,

we can represent it as x = 2k, where k is any integer.

Substitute this value of x into the expression $x^2 + 2x + 4$:

$= (2k)^2 + 2(2k) + 4$

$= 4k^2 + 4k + 4$

$= 2(2k^2 + 2k + 2)$

Since k is an integer, $2(2k^2 + 2k + 2)$

is also an integer.

Hence, if x is an even integer, then

$x^2 + 2x + 4$ is even.

Question No. 13

Prove that for any two non-empty sets A and B, (A∩B)′=A′∪B′.

Solution:

L.H.S. = (A∩B)′

Let $x \in (A \cap B)'$

By definition;

$\Rightarrow x \notin A \cap B$

$\Rightarrow x \notin A$ and $x \notin B$

$\Rightarrow x \in A'$ or $x \in B'$

$\Rightarrow x \in A' \cup B'$

$\Rightarrow (A \cap B)' \subseteq A' \cup B'$ ...(1)

Now R.H.S;

Let $x \in A' \cup B'$

$\Rightarrow x \in A'$ or $x \in B'$

$\Rightarrow x \notin A$ and $x \notin B$

$\Rightarrow x \notin A \cap B$

$\Rightarrow x \in (A \cap B)'$

$\Rightarrow A' \cup B' \subseteq (A \cap B)'$ ...(2)

Conclusion:

From (1) and (2) we get;

(A ∩ B)′ = A′ ∪ B′

Similarly, we can prove that;

(A ∪ B)′ = A′ ∩ B′

Question No. 14

If x and y are positive real numbers and $x^2 < y^2$ then x < y.

Solution:

Since x and y are positive real numbers, we can take the square root of both sides of the inequality $x^2 < y^2$ without changing the direction of the inequality.

This gives us:

$\sqrt{x^2} < \sqrt{y^2}$

|x| < |y|

As x and y are positive,

|x| = x and |y| = y.

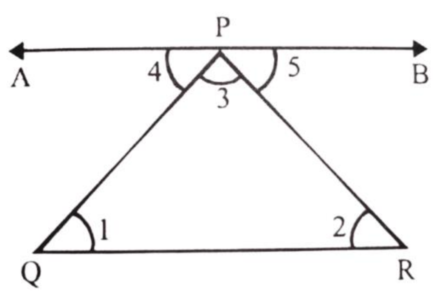
Therefore, x < y.

Question No. 15

The sum of the interior angles of a triangle is 180°.

Solution:

Draw a triangle PQR.



Draw a line AB ∥ QR (construction)

∠3 + ∠4 + ∠5 = 180° (straight angle) ∠4 = ∠1 (alternate interior angles) ∠2 = ∠5 (alternate interior angles) So,

∠1 + ∠2 + ∠3 = 180°

Conclusion:

So proved that The sum of the interior angles of a triangle is 180°.

Question No. 16

If a, b and c are non-zero real numbers, prove that:

(a) $\boldsymbol{\frac{a}{b} = \frac{c}{d} \Leftrightarrow ad = \mathrm{bc}}$

Proof:

$\frac{a}{b} = \frac{c}{d} \Leftrightarrow ad = \mathrm{bc}$

$\frac{a}{b} \times (d \times \frac{1}{d}) = \frac{c}{d} \times (b \times \frac{1}{b})$

(∵ Multiplicative inverse)

$$\frac{\mathrm{a} \times \mathrm{d}}{\mathrm{b} \times \mathrm{d}} = \frac{\mathrm{c} \times \mathrm{b}}{\mathrm{d} \times \mathrm{b}}$$

$$\frac{\mathrm{ad}}{\mathrm{bd}} = \frac{\mathrm{cb}}{\mathrm{db}}$$

(∵ Rule of multiplication of fraction)

$$\frac{\mathrm{ad}}{\mathrm{bd}} \times \mathrm{bd} = \frac{\mathrm{cb}}{\mathrm{db}} \times \mathrm{bd}$$

(∵ Multiplicative property of real numbers)

ad = cb (∵ Cancellation property)

ad = bc (∵ Commutative property)

Conclusion:

So proved that $\frac{a}{b} = \frac{c}{d} \Leftrightarrow ad = \mathrm{bc}$

(b) $\boldsymbol{\frac{a}{b} . \frac{c}{d} \Leftrightarrow \frac{ac}{bd}}$

Proof:

$\frac{a}{b} . \frac{c}{d} \Leftrightarrow \frac{ac}{bd}$

By solving L.H.S:

$= \frac{a}{b} . \frac{c}{d}$

$= \frac{ac}{bd}$

(∵ Rule of multiplication of fractions $\frac{a}{b} . \frac{c}{d} = \frac{ac}{bd}$)

Conclusion:

So proved that $\frac{a}{b}.\frac{c}{d} \Leftrightarrow \frac{ac}{bd}$

(c) $\boldsymbol{\frac{a}{b} + \frac{c}{b} \Leftrightarrow \frac{a+c}{b}}$

Proof:

$\frac{a}{b} + \frac{c}{b} \Leftrightarrow \frac{a+c}{b}$

By solving L.H.S:

$\frac{a}{b} + \frac{c}{b}$

$= a \times \frac{1}{b} + c \times \frac{1}{b}$ $\quad$ ($\because$ $\mathrm{a} \times \frac{1}{b} = \frac{\mathrm{a}}{b}$)

$= (\mathrm{a} + \mathrm{c}) \times \frac{1}{b}$ $\quad$ ($\because$ Distributive property)

$= \frac{\mathrm{a} + \mathrm{c}}{b}$

Thus;

$= \frac{\mathrm{a}}{b} + \frac{\mathrm{c}}{b}$

Conclusion:

So proved that $\frac{a}{b} + \frac{c}{b} \Leftrightarrow \frac{a+c}{b}$