Unit No. 3

Set and Functions

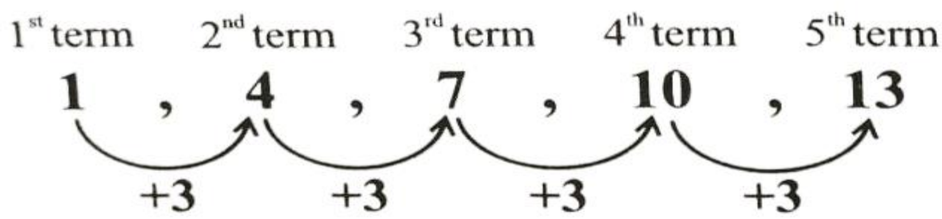
Basic Concepts

Mathematics:

Mathematics is the science of patterns, structures, and relationships, comprising various branches that explore and analyze our world's logical and quantitative aspects.

Mathematical Pattern:

A mathematical pattern is a predictable arrangement of numbers, shapes, or symbols that follows a specific rule or relationship.



Set:

A set is described as a well-defined collection of distinct objects, numbers or elements, so that we may be able to decide whether the object belongs to the collection or not.

Capital letters A, B, C, X, Y, Z etc., are generally used as names of sets and small letters a, b, c, x, y, z etc., are used as members or elements of sets.

Georg Cantor (1845-1918):

Georg Cantor (1845-1918) was a German mathematician who significantly contributed to the development of set theory, a key area in mathematics. He showed how to compare two sets by matching their members one-to-one. Cantor defined different types of infinite sets and proved that there are more real numbers than natural numbers. His proof revealed that there are many sizes of infinity. Additionally, he introduced the concepts of cardinal and ordinal numbers, along with their arithmetic operations.

Methods of Describing a Set:

There are three distinct ways to describe a set in mathematics:

1. Descriptive Form:

A set can be described using words.
Example: The set of all vowels in the English alphabet.

2. Tabular Form (Roster Form):

A set can be defined by explicitly listing its elements within curly braces.
Example: If A is the set of vowels, it can be written as:

A = {a,e,i,o,u}

3. Set-Builder Notation:

This method specifies a set by describing the common property shared by all its members. It uses a variable and a condition.
Example: The set of vowels can be written in set-builder notation as:

A = {x | x is a vowel in the English alphabets}

Some Common Sets:

N = The set of natural numbers

= {1, 2, 3, ...}

W = The set of whole numbers

= {0, 1, 2, ...}

Z = The set of integers

= {0, ±1, ±2, ...}

O = The set of odd integers

= {±1, ±3, ±5, ...}

E = The set of even integers

= {0, ±2, ±4, ...}

P = The set of prime numbers

= {2, 3, 5, 7, 11, 13, 17, ...}

Q = The set of all rational numbers

$= \{ x \mid x = \frac{p}{q}, \text{ where } p, q \in Z \text{ and } q \neq 0 \}$

Q' = The set of all irrational numbers

$= \{ x \mid x \neq \frac{p}{q}, \text{ where } p, q \in Z \text{ and } q \neq 0 \}$

R = The set of all real numbers

$= Q \cup Q'$

Singleton Set:

A set with only one element is called a singleton set. For example, {3}, {a}, and {Saturday} are singleton sets.

Empty Set:

The set with no elements (zero number of elements) is called an empty set, null set, or Void set. The empty set is denoted by the symbol Ø or { }.

Key Takeaway:

$\{0\} \neq \emptyset$. The set {0} includes the element 0, while the empty set has no element.

Equal Sets:

Two sets A and B are equal if they have exactly the same elements. If every element of set A is an element of set B and every element of set B is an element of A. Thus, the sets {1, 2, 3} and {2, 1, 3} are equal, we write A = B.

Equivalent Sets:

Two sets A and B are equivalent if they have the same number of elements. For example, if A = {a, b, c, d, e} and

$B = \{1, 2, 3, 4, 5\}$, then A and B are equivalent sets. The symbol '↔' is used to represent equivalent sets. Thus, we can write $A \leftrightarrow B$.

Subset / Superset:

If every element of a set A is an element of set B, then A is a subset of B. Symbolically this is written as $A \subseteq B$ (A is a subset of B). In such a case, we say B is a Superset of A. Symbolically this is written as: $B \supseteq A$ (B is a superset of A).

Proper Subset:

If A is a subset of B and B contains at least one element that is not an element of A, then A is said to be a proper subset of B. In such a case, we write: $A \subset B$ (A is a proper subset of B).

Improper Subset:

If A is a subset of B and A = B, then we say that A is an improper subset of B. From this definition, it also follows that every set A is a subset of itself and is called an improper subset. For example,

Let A = {a, b, c}, B = {c, a, b}, and C = {a, b, c, d}, then clearly $A \subseteq C$, $B \subseteq C$ but A = B.

Notice that each of sets A and B is an improper subset of the other because A = B.

Universal Set:

The set that contains all objects or elements under consideration is called the universal set or the universe of discourse. It is denoted by U.

Power Set:

The power set of a set S denoted by P(S) is the set containing all the possible subsets of S.

For Example:

(i) If C = {a, b, c}
P(C) = {∅, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c}}
(ii) If D = {a}
P(D) = {∅, {a}}

Union of Two Sets:

The union of two sets A and B, denoted by A∪B, is the set containing all elements that belong to either set A, set B, or both. Symbolically, it is defined as:

$A \cup B = \{x \mid x \in A \lor x \in B\}$

The notation ∪ represents the union operation.

Intersection of Two Sets:

The intersection of two sets A and B, denoted by A ∩ B, is the set of all elements that belong to both A and B. Symbolically:

$A \cap B = \{x \mid x \in A \land x \in B\}$.

Remember!

The symbol ∨ means or. The symbol ∧ means and.

Disjoint Sets:

If the intersection of two sets is the empty set, the sets are said to be disjoint. For example, if

A = The set of odd natural numbers

B = The set of even natural numbers,

then A and B are disjoint sets. Similarly, the set of arts students and the set of science students of a college are disjoint sets.

Overlapping Sets:

If the intersection of two sets is non-empty but neither is a subset of the other, the sets are called overlapping sets, e.g., if

L = {2, 3, 4, 5, 6}, M = {5, 6, 7, 8, 9, 10}, then L and M are overlapping sets.

Difference of Two Sets:

The difference between the sets A and B denoted by A – B, consists of all the elements that belong to A but do not belong to B. Symbolically, $A - B = \{x \mid x \in A \land x \notin B\}$ and

$B - A = \{x \mid x \in B \land x \notin A\}$

For example, if A = {1, 2, 3, 4, 5} and B = {4, 5, 6, 7, 8, 9, 10}, then

A – B = {1, 2, 3} and

B – A = {6, 7, 8, 9, 10}

Notice that: $A - B \neq B - A$.

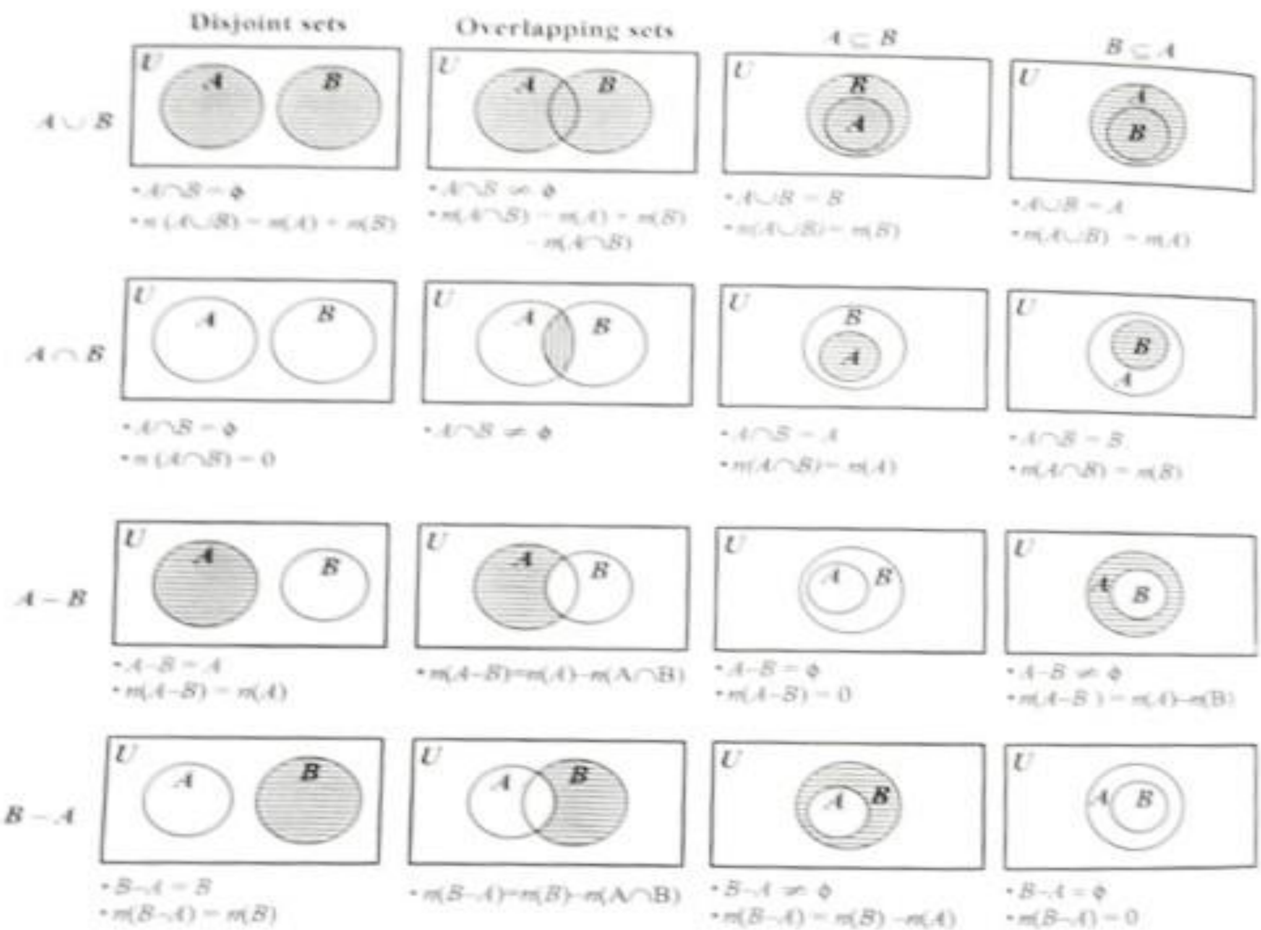
<u>Complement of a Set:</u>

The complement of a set A, denoted by A' or A^c, relative to the universal set U is the set of all elements of U, which do not belong to A.

Symbolically: $A' = \{x \mid x \in U \wedge x \in / A\}$

<u>Identification of Sets Using Venn Diagram:</u>

Venn diagrams are very useful in depicting visually the basic concepts of sets and relationships between sets. These diagrams were first used by an English logician and mathematician John Venn (1834 to 1883 A.D.).



<u>Properties of union and intersection:</u>

(i) $A \cup B = B \cup A$

(Commutative property of Union)

(ii) $A \cap B = B \cap A$

(Commutative property of Intersection)

(iii) $A \cup (B \cup C) = (A \cup B) \cup C$

(Associative property of Union)

(iv) $A \cap (B \cap C) = (A \cap B) \cap C$

(Associative property of Intersection)

(v) A ∪ (B ∩ C) = (A ∪ B) ∩ (A ∪ C)

(Distributivity of Union over intersection)

(vi) A ∩ (B ∪ C) = (A ∩ B) ∪ (A ∩ C)

(Distributivity of intersection over Union)

(vii) (A ∪ B)' = A' ∩ B'

(De Morgan's 1st Law)

(viii) (A ∩ B)' = A' ∪ B'

(De Morgan's 2nd Law)

Principle of Inclusion and Exclusion for Two Sets:

Let A and B be finite sets, then n(A ∪ B) = n(A) + n(B) – n(A ∩ B) and A ∪ B and A ∩ B are also finite.

Principle of Inclusion and Exclusion for Three Sets:

If A, B and C are finite sets, then n(A ∪ B ∪ C) = n(A) + n(B) + n(C) – n(A ∩ B) – n(A ∩ C) – n(B ∩ C) + n(A ∩ B ∩ C) and A ∪ B ∪ C, A ∩ B, A ∩ C, B ∩ C and A ∩ B ∩ C are also finite.

Binary Relations:

In everyday use, relation means an abstract type of connection between two persons or objects, for instance, (teacher, pupil), (mother, son), (husband, wife), (brother, sister), (friend, friend), (house, owner). In mathematics also 1 some operations determine the relationship between two numbers, for example:

> (5, 4); square: (25, 5);

Square root : (2, 4) ; Equal: (2 × 2, 4)

In the above examples >, square, square root and equal are examples of relations.

Mathematically, a relation is any set of ordered pairs. The relationship between the components of an ordered pair may or may not be mentioned.

(i) Let A and B be two non-empty sets, then the *Cartesian Product* is the set of all ordered pairs (x, y) such that x ∈ A and y ∈ B and is denoted by A × B. Symbolically we can write it as A × B = {(x, y) | x ∈ A and y ∈ B}.

(ii) Any subset of A × B is called a *Binary Relation*, or simply a relation, from A to B. Ordinarily a relation will be denoted by the letter r.

(iii) The set of the first elements of the ordered pairs forming a relation is called its *Domain*. The domain of any relation r is denoted as Dom r.

(iv) The set of the second elements of the ordered pairs forming a relation is called its *Range*. The range of any relation r is denoted as Ran r.

(v) If A is a non-empty set, any subset of A × A is called a *Relation* in A.

(vi) Each ordered pair consists of two coordinates, x and y. The x coordinate is called *Abscissa*, and the y coordinate is *Ordinate*, often representing an input and an output.

Ordered Pairs:

A relation can be represented by a set of ordered pairs. For example, consider a water tank that starts with 1 litre of water already inside. Each minute, 1 additional litre of water is added to the tank. The situation can be represented by the relation r = {(x, y) | y = x + 1}, where x is the number of minutes (time) that have passed since the filling started and y is the total amount of water (in litres) in the tank.

When x = 0, y = 1 and x = 1, y = 2

In order pair this relation is represented as:

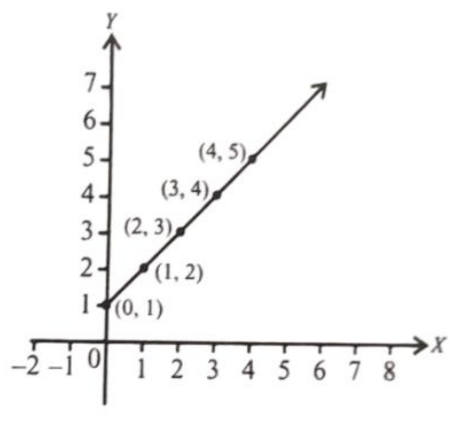
{(0, 1), (1, 2), (2, 3), (3, 4), (4, 5), (5, 6)}

Graph:

We can also represent the relations visually by drawing a graph. To draw the diagram, we use ordered pairs. Each ordered pair (x, y) is plotted as a point in the coordinate plane, where x is the first element and y is the second element of the ordered pair.

The relation is represented graphically by the line passing through the points,

{(0, 1), (1, 2), (2, 3), (3, 4), (4, 5)} as shown in the adjacent Figure.



Function and its Domain and Range:

Functions:

A very important particular type of relation is a function defined as below:

Let A and B be two non-empty sets such that:

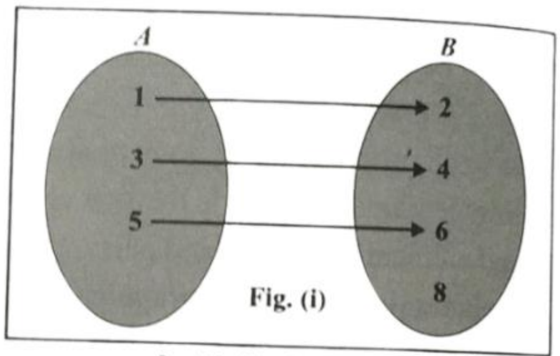
(i) f is a relation from A to B, that is, f is a subset of $A \times B$
(ii) Domain $f = A$
(iii) First element of no two pairs of f are equal, then f is said to be a function from A to B.

The function f is also written as:

$$f : A \rightarrow B$$

<u>(i) Into Function:</u>

If a function $f: A \rightarrow B$ is such that Range $f \subset B$ i.e., Range $f \neq B$, then f is said to be a function from A into B. In Fig. (i), f is clearly a function. But Range $f \neq B$. Therefore, f is a function from A into B.

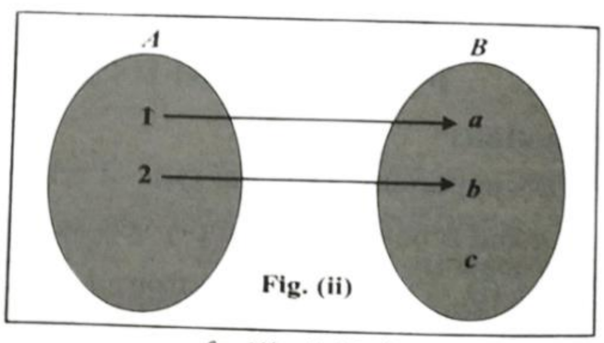


$f = \{(1, 2), (3, 4), (5, 6)\}$

<u>(ii) (One - One) Function (or Injective Function):</u>

If a function f from A into B is such that second elements of no two of its ordered pairs are same, then it is a (one-one) or injective function. In Fig. (ii),

$f = \{(1, a), (2, b)\}$ is a one-one function because the second elements 'a' and 'b' are different.



$f = \{(1, a), (2, b)\}$

(iii) Onto Function (or Surjective function):

If a function $f: A \rightarrow B$ is such that Range $f = B$ i.e., every element of B is the image of some element of A, then f is called an onto function or a surjective function. The function shown in Fig. (iii) is such a function.

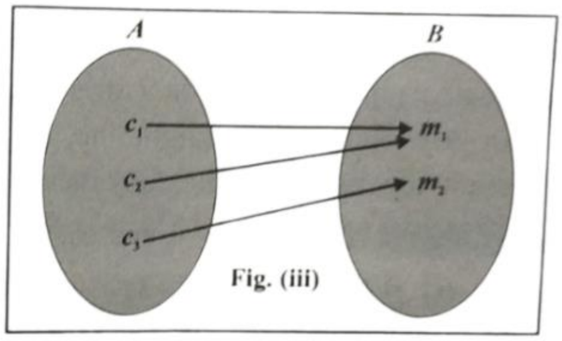


Fig. (iii)

$f = \{(c_1, m_1), (c_2, m_1), (c_3, m_2)\}$

<u>(iv) (One - One) and onto Function (or Bijective Function):</u>

A function *f* from A to B is said to be a Bijective function if it is both one-one and onto. Such a function is also called (1 - 1) correspondence between the sets A and B.

(a, z), (b, x) and (c, y) are the pairs of corresponding elements i.e., in this case

f = {(a, z), (b, x), (c, y)} which is a bijective function or (1 - 1) correspondence between the sets A and B.

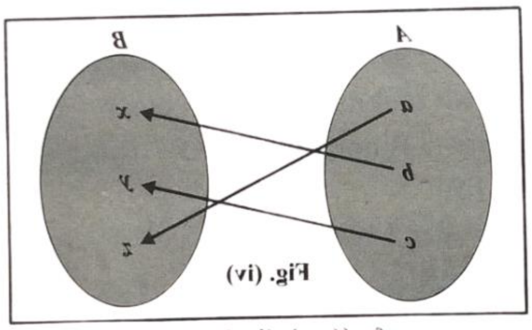


Fig. (iv)

$f = \{(a, z), (b, x), (c, y)\}$

<u>Linear and Quadratic Functions:</u>

The function {(x, y) | y = mx + c} is called a *linear function* because its graph (geometric representation) is a straight line.

An equation of the form y = mx + c represents a straight line. The function

{(x, y) | y = ax² + b x + c} is called a *quadratic function*.