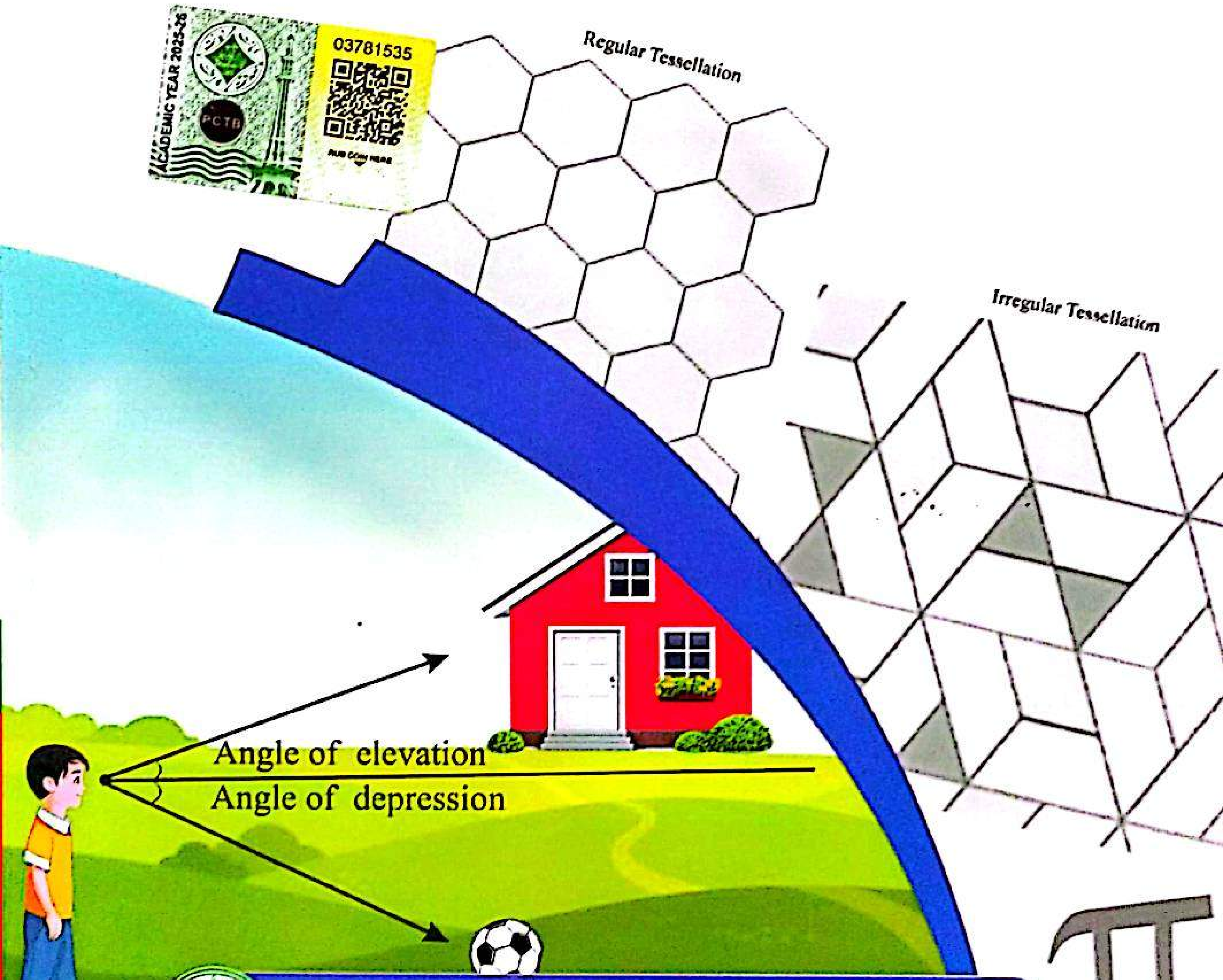
MATHEMATICS 9



PUNJAB CURRICULUM AND TEXTBOOK BOARD, LAHORE

UNIT 8

LOGIC

EXERCISE 8

.1. **Four options are given against each statement. Encircle the correct option.**

) Which of the following expressions is often related to inductive reasoning?

(a) based on repeated experiments

(b) if and only if statements

(c) statement is proven by a theorem

(d) based on general principles

ii) Which of the following sentences describe deductive reasoning?

(a) general conclusions from a limited number of observations

(b) based on repeated experiments

(c) based on units of information that are accurate

(d) draw conclusion from well-known facts

(iii) Which one of the following statements is true?

(a) The set of integers is finite

(b) The sum of the interior angles of any quadrilateral is always 180°

(c) $\frac{22}{7} \notin Q'$

(d) All isosceles triangles are equilateral triangles

(iv) Which of the following statements is the best to represent the negation of the statement "The stove is burning"?

(a) the stove is not burning.

(b) the stove is dim

(c) the stove is turned to low heat

(d) it is both burning and not burning.

(v) The conjunction of two statements p and q is true when:

(a) both p and q are false.

(b) both p and q are true.

(c) only q is true.

(d) only p is true

(vi) A conditional is regarded as false only when:

(a) antecedent is true and consequent is false.

(b) consequent is true and antecedent is false.

(c) antecedent is true only.

(d) consequent is false only.

(vii) Contrapositive of $q \rightarrow p$ is:

(a) $q \rightarrow \sim p$ (b) $\sim q \rightarrow p$

(c) $\sim p \rightarrow \sim q$ (d) $\sim q \rightarrow \sim p$

(viii) The statement "Every integer greater than 2 is a sum of two prime numbers" is:

(a) theorem (b) conjecture

(c) axiom (d) postulates

(ix) The statement "A straight line can be drawn between any two points" is:

(a) theorem (b) conjecture

(c) axiom (d) logic

(x) The statement "The sum of the interior angle of a triangle is 180^{o}" is:

(a) converse (b) theorem

(c) axiom (d) conditional

Answers:

(i)	(a)	(ii)	(d)	(iii)	(c)	(iv)	(a)	(v)	(b)
(vi)	(a)	(vii)	(c)	(viii)	(b)	(ix)	(c)	(x)	(b)

Q.2. Write the converse, inverse and contrapositive of the following conditionals:

(i) $\sim p \to q$

Sol. Converse is $q \to \sim p$

Inverse is $p \to \sim q$

Contrapositive is $\sim q \to p$

(ii) $q \to p$

Sol. Converse is $p \to q$

Inverse is $\sim q \to \sim p$

Contrapositive is $\sim p \to \sim q$

(iii) $\sim p \to \sim q$

Sol. Converse is $\sim q \to \sim p$

Inverse is $p \to q$

Contrapositive is $q \to p$

(iv) $\sim q \to \sim p$

Sol. Converse is $\sim p \to \sim q$

Inverse is $q \to p$

Contrapositive is $p \to q$

Q.3. Write the truth table of the following:

(i) $\sim(p \vee q) \vee (\sim q)$

Sol.

p	q	$\sim q$	$p \vee q$	$\sim(p \vee q)$	$\sim(p \vee q) \vee (\sim q)$
T	T	F	T	F	F
T	F	T	T	F	T
F	T	F	T	F	F
F	F	T	F	T	T

(ii) $\sim(\sim q \vee \sim p)$

Sol.

p	q	~q	~q	~q ∨ ~p	~(~q ∨ ~p)
T	T	F	F	F	T
T	F	F	T	T	F
F	T	T	F	T	F
F	F	T	T	T	F

(iii) $(p \vee q) \leftrightarrow (p \wedge q)$

Sol.

p	q	p ∨ q	p ∧ q	(p ∨ q) ↔ (p ∧ q)
T	T	T	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	F	T

Q.4. Differentiate between a mathematical statement and its proof. Given two examples.

Sol. A mathematical statement is a claim that can be true or false while a proof is the method used to verify whether the statement is true.

A mathematical statement is a sentence which is either true or false. It may contain words and symbols. For example, "the square root of '4 is 5'. Is a mathematical statement which is of course false.

Q.5. What is the difference between an axiom and a theorem? Give examples of each.

Sol. **Axiom:** A mathematical statement which we believe to be true without any evidence or requiring any proof.

Theorem: A theorem is a mathematical statement that has been proved true based on previously known facts. For example, the following statements are theorem.

- The sum of the interior angles of a quadrilateral is 360°.
- The sum of interior angles of a polygon is $(n - 2) \times 180^\circ$

Examples of Axiom:

- Through a given point, there pass infinitely many lines.
- A straight line can be drawn between any two points.

Q.6. What is the important of logical reasoning in mathematical proofs? Give an example to illustrate your point.

Sol. Logical reasoning is a fundamental part of mathematical thinking and is essential for proving theorem and other mathematical facts.

Logical reasoning is used to prove mathematical theorems based on a set of premises called axioms.

For example, Peano arithmetic uses a small set of axioms to infer the essential properties of natural numbers.

Q.7. Indicate whether it is an axiom, conjecture or theorem and explain your reasoning.

(i) There is exactly one straight line through any two points.

Sol. "Through any two points, there is exactly one straight line". The is Euclid Axiom.

(ii) Every even number greater than 2 can be written as the sum of two prime numbers."

Sol. Every even number greater than 2 can be written as the sum of two prime numbers. This is conjecture.

(iii) The sum of the angles in a triangle is 180°.

Sol. The sum of angles in a triangle is 180°.

This is theorem.

Q.8. Formulate simple deductive proofs for each of the following algebraic expressions, prove that the L.H.S is equal to the R.H.S:

(i) Prove that $(x - 4)^2 + 9 = x^2 - 8x + 25$

Sol. L.H.S $= (x - 4)^2 + 9$

Expand $(x-4)^2$ as

$$= (x-4)(x-4)+9$$
$$= x(x-4)-4(x-4)+9$$
$$= x^2-4x-4x+16+9$$

Add like terms

$$= x^2-8x+25$$

Conclusion:

The L.H.S is exactly the same as the R.H.S thus, we have shown that:

$$(x-4)^2+9 = x^2-8x+25$$

This completes the proof.

(ii) Prove that $(x+1)^2-(x-1)^2 = 4x$

Sol. L.H.S $= (x+1)^2-(x-1)^2$

$$= (x+1)(x+1)-(x-1)(x-1)$$
$$= x(x+1)+1(x+1)-[x(x-1)-1(x-1)]$$
$$= x^2+x+x+1-(x^2-x-x+1)$$

Adding like terms

$$= x^2+2x+1-(x^2-2x+1)$$
$$= x^2+2x+1-x^2+2x-1$$

Adding like terms

$$= x^2-x^2+2x+2x+1-1$$
$$= 0x^2+4x+0$$
$$= 4x$$

Conclusion:

The L.H.S is exactly the same as the R.H.S

Thus we have shown that

$(x+1)^2-(x-1)^2 = 4x$

This completes the proof.

(iii) Prove that $(x+5)^2-(x-5)^2 = 20x$

Sol. L.H.S $= (x+5)^2-(x-5)^2$

Expand both terms

$$= (x + 5)(x + 5) - (x - 5)(x - 5)$$
$$= x(x + 5) + 5(x + 5) - [x(x - 5) - 5(x - 5)]$$
$$= x^2 + 5x + 5x + 25 - (x^2 - 5x - 5x + 25)$$

Adding like terms

$$= x^2 + 10x + 25 - (x^2 - 10x + 25)$$
$$= x^2 + 10x + 25 - x^2 + 10x - 25$$

Adding like terms

$$= x^2 - x^2 + 10x + 10x + 25 - 25$$
$$= 0x^2 + 20x + 0$$
$$= 20x$$

Conclusion:

The L.H.S is exactly the same as the R.H.S.

Thus we have shown that

$$(x + 5)^2 - (x - 5)^2 = 20x$$

This completes the proof.

Q.9. Prove the following by justifying each step:

(i) $\mathbf{\dfrac{4 + 16x}{4} = 1 + 4x}$

Sol. L.H.S $= \dfrac{4 + 16x}{4}$

$= \dfrac{1}{4} . (4 + 16x)$ (Multiplication of fractions)

$= \dfrac{1}{4} . 4(1 + 4x)$ (Distributive property)

$= 1 . (1 + 4x)$ (Multiplicative inverse)

$= 1 + 4x$ (Multiplicative identity)

$=$ R.H.S

(ii) $\mathbf{\dfrac{6x^2 + 18x}{3x^2 - 9} = \dfrac{2x}{x - 3}}$

Sol. L.H.S $= \dfrac{6x^2 + 18x}{3x^2 - 9}$

$$= \frac{6x(x+3)}{3(x^2-3)} \quad \text{(6x is common)}$$

$$= \frac{2x(x+3)}{x^2-3} \neq \frac{2x}{x-3}$$

(iii) $\mathbf{\dfrac{x^2+7x+10}{x^2-3x-9} = \dfrac{x+5}{x-5}}$

Sol. L.H.S $= \dfrac{x^2+7x+10}{x^2-3x-9}$ (Factorizing)

$$= \frac{x(x+2)+5(x+2)}{x^2-3x-9} \quad \text{(Distributive property)}$$

$$= \frac{(x+2)(x+5)}{x^2-3x-9} \neq \frac{x+5}{x-5}$$

Q.10. Suppose x is an integer. Then x is odd if and only if $9x+4$ is odd.

Sol. Let x be odd.

By definition, an odd integer can be written as

$x = 2k+1$ (for some integer)

Replace x by $2k+1$ in $9x+4$

$$\begin{aligned} 9x+4 &= 9(2k+1)+4 \\ &= 18k+9+4 \\ &= 18k+13 \\ &= 18k+12+1 \\ &= 2(9k+6)+1 \quad \ldots(1) \end{aligned}$$

Since $2(9k+6)+1$ is in the form $2m+1$. So $9x+4$ is odd.

Conversely,

If $9x+4$ is odd, then x is odd.

$$\begin{aligned} 9x+4 &= 2m+1 \\ 9x &= 2m+1-4 \\ 9x &= 2m-3 \\ 9x &= 2\left(m-\frac{3}{2}\right) \end{aligned}$$

Since 9x is even and 9 is odd. x must be odd (because even divided by odd yields an odd).

Conclusion:

x is odd if and only if $9x + 4$ is odd.

Q.11. Suppose x is an integer. If x is odd, then $7x + 5$ is even.

Sol. Given that x is an odd integer.

As we know that

Odd is written by definition as

$x = 2k + 1$ (for some integer k)

Replace x by $2k + 1$ in $7x + 5$.

So, $$\begin{aligned} 7x + 5 &= 7(2k + 1) + 5 \\ &= 14k + 7 + 5 \\ &= 14k + 12 \\ &= 2(7k + 6) \end{aligned}$$

As $2(7k + 6)$ is divisible by 2.

So, $7x + 5$ is even.

Conclusion:

If x is odd, then $7x + 5$ is even.

Q.12. Prove the following statements

(a) If x is an odd integer, then show that $x^2 - 4x + 6$ is odd.

Sol. Let x is odd (given)

By definition

$$x = 2k + 1 \text{ (for some integer)}$$

Replace x by $2k + 1$ in

$$x^2 - 4x + 6$$

So,

$$\begin{aligned} x^2 - 4x + 6 &= (2k + 1)^2 - 4(2k + 1) + 6 \\ &= 4k^2 + 4k + 1 - 8k - 4 + 6 \\ &= 4k^2 - 4k + 3 \\ &= 2(2k^2 - 2) + 3 \end{aligned}$$

As $2(2k^2 - 2)$ is even.

If we add 3 to it, it will become an odd.

Conclusion:

If x is an odd integer, then $x^2 - 4x + 6$ is odd.

(b) If x is an even integer then show that $x^2 + 2x + 4$ is even.

Sol. Let x be an even integer (given) then

By definition

$$x = 2k \text{ (for some integer k)}$$

Replace x by 2k in $x^2 + 2x + 4$

$$\begin{aligned} x^2 + 2x + 4 &= (2k)^2 + 2(2k) + 4 \\ &= 4k^2 + 4k + 4 \\ &= 2(2k^2 + 2k + 2) \end{aligned}$$

As $2(2k^2 + 2k + 2)$ is divisible by 2.

So, expression is even.

Conclusion:

If x is an even integer, then $x^2 + 2x + 4$ is even.

Q.13. Prove that for any two non-empty sets A and B, $(A \cap B)' = A' \cup B'$.

Sol. L.H.S $= (A \cap B)'$

Let $x \in (A \cap B)'$

By definition

$$x \notin A \cap B$$

$\Rightarrow \quad x \notin A$ and $x \notin B$

$\Rightarrow \quad x \in A'$ or $x \in B'$

$\Rightarrow \quad x \in A' \cup B'$

$\Rightarrow \quad (A \cap B)' \subseteq A' \cup B' \qquad \ldots(1)$

Now R.H.S

Let $x \in A' \cup B'$

$\Rightarrow \quad x \in A' \quad$ or $\quad x \in B'$

$$x \notin A \quad \text{and} \quad x \notin B$$

$$\Rightarrow \quad x \notin A \cap B$$

$$\Rightarrow \quad x \in (A \cap B)'$$

$$\Rightarrow \quad A' \cup B' \subseteq (A \cap B)' \qquad \ldots(2)$$

From (1) and (2) we get

$$(A \cap B)' = A' \cup B'$$

Similarly, we can prove that

$$(A \cup B)' = A' \cap B'$$

Q.14. If x and y are positive real numbers and $x^2 < y^2$ then $x < y$.

Sol. Since x and y are positive real numbers and $x^2 < y^2$

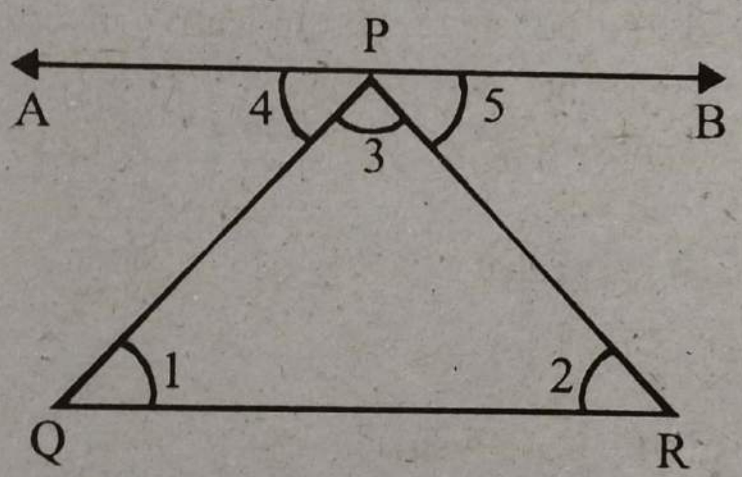
Taking square root of both sides.

$$\sqrt{x^2} < \sqrt{y^2}$$

$$x < y$$

Q.15. The sum of the interior angles of a triangle is 180°.

Sol. Draw a triangle PQR



Draw a line $\overleftrightarrow{AB} \parallel QR$ (construction)

$$\angle 3 + \angle 4 + \angle 5 = 180^\circ \qquad \text{(straight angle)}$$

$$\angle 4 = \angle 1 \qquad \text{(alternate interior angles)}$$

$$\angle 2 = \angle 5 \qquad \text{(alternate interior angles)}$$

So, $\angle 1 + \angle 2 + \angle 3 = 180^\circ$

So proved.

Q.16. If a, b and c are non-zero real numbers, prove that:

(a) $\frac{a}{b} = \frac{c}{d} \Leftrightarrow ad = bc$

Sol. Suppose $\frac{a}{b} = \frac{c}{d}$

We will prove that $ad = bc$

Then $\frac{a}{b} = \frac{c}{d}$ (Given)

By Golden rule of fraction.

$$\frac{ad}{bd} = \frac{bc}{bd}$$

$$ad \cdot \frac{1}{bd} = bc \cdot \frac{1}{bd}$$

$$\left(ad \cdot \frac{1}{bd}\right) \cdot bd = \left(bc \cdot \frac{1}{bd}\right) \cdot bd \quad (\because \text{ Multiplication property})$$

$$ad \cdot \left(\frac{1}{bd} \cdot bd\right) = bc \cdot \left(\frac{1}{bd} \cdot bd\right) \quad (\because \text{ associative property})$$

$$ad \cdot 1 = bc \cdot a \quad (\because \text{ Multiplicative inverse property})$$

$$ad = bc$$

Now suppose that

$$ad = bc$$

we shall prove that

$$\frac{a}{b} = \frac{c}{d}. \text{ Then}$$

$$ad = bc \text{ (Given)}$$

$$ad \cdot 1 = bc \cdot 1 \quad (\because \text{ Multiplicative inverse property})$$

$$ad \cdot \left(\frac{1}{bd} \cdot bd\right) = bc \cdot \left(\frac{1}{bd} \cdot bd\right)$$

$(\because$ Multiplicative inverse property)

$$ad \cdot \frac{1}{bc} = bc \cdot \frac{1}{bd}$$

$(\because$ Cancellation property of multiplication)

$$\frac{ad}{bd} = \frac{bc}{bd} \quad (\because \text{ Multiplication property})$$

$$\frac{a}{b} = \frac{c}{d} \quad (\text{Golden rule of fraction})$$

So $\quad \frac{a}{b} = \frac{c}{d} \Leftrightarrow ad = bc$

(b) $\quad \frac{\mathbf{a}}{\mathbf{b}} \cdot \frac{\mathbf{c}}{\mathbf{d}} = \frac{\mathbf{ac}}{\mathbf{bd}}$

Sol. L.H.S $= \left(a \cdot \frac{1}{b}\right) \cdot \left(\frac{1}{d} \cdot c\right) \quad (\because \text{ Multiplication property})$

$$= a \cdot \left[\frac{1}{b} \cdot \left(\frac{1}{d} \cdot c\right)\right] \quad (\because \text{ Associative property})$$

$$= a \cdot \left[\left(\frac{1}{b} \cdot \frac{1}{d}\right) \cdot c\right] \quad (\because \text{ Associative property})$$

$$= a \cdot \left[\frac{1}{bd} \cdot c\right] \quad (\because \text{ Multiplication property})$$

$$= a \cdot \frac{c}{bd} \quad (\because \text{ Multiplication property})$$

$$= \frac{ac}{bd} \quad (\because \text{ Multiplication property})$$

$=$ R.H.S

(c) $\quad \frac{\mathbf{a}}{\mathbf{b}} + \frac{\mathbf{c}}{\mathbf{d}} = \frac{\mathbf{a + c}}{\mathbf{b}}$

Sol. L.H.S $= \frac{a}{b} + \frac{c}{d}$

$$= a \cdot \frac{1}{b} + c \cdot \frac{1}{b} \quad (\because \text{ Multiplication property})$$

$$= (a + b) \cdot \frac{1}{b} \quad (\because \text{ Distributive property})$$

$$= \frac{a + c}{b} \quad (\because \text{ Multiplication property})$$

$=$ R.H.S